Research Article

Regular Generalized ω -Closed Sets

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In 1982 and 1970, Hdeib and Levine introduced the notions of ω -closed set and generalized closed set, respectively. The aim of this paper is to provide a relatively new notion of generalized closed set, namely, regular generalized ω -closed, regular generalized ω -continuous, a - ω -continuous, and regular generalized ω -irresolute maps and to study its fundamental properties.

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1. Introduction

All through this paper (X, τ) and (Y, σ) stand for topological spaces with no separation axioms assumed, unless otherwise stated. Let $A \subseteq X$, the closure of A and the interior of A will be denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. A is regular open if $A = \text{Int}(\text{Cl}(A))$ and A is regular closed if its complement is regular open; equivalently A is regular closed if $A = \text{Cl}(\text{Int}(A))$, see [1]. Let (X, τ) be a space and let A be a subset of X . A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is called ω -closed [2] if it contains all its condensation points. The complement of an ω -closed set is called ω -open. It is well known that a subset W of a space (X, τ) is ω -open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω -open subsets of a space (X, τ) , denoted by τ_ω or $\omega O(X)$, forms a topology on X finer than τ . The ω -closure and ω -interior, that can be defined in a manner similar to $\text{Cl}(A)$ and $\text{Int}(A)$, respectively, will be denoted by $\text{Cl}_\omega(A)$ and $\text{Int}_\omega(A)$, respectively. Several characterizations of ω -closed subsets were provided in [3, 2, 4]. Levine [5] introduced the notion of generalized closed sets and a class of topological spaces called $T_{1/2}$ -spaces. He defined a subset A of a space (X, τ) to be generalized closed

set (briefly g -closed) if $\text{Cl}(A) \subseteq U$ whenever $U \in \tau$ and $A \subseteq U$. Generalized semiclosed [6] (resp., α -generalized closed [7], θ -generalized closed [8], generalized semi-preclosed [9], δ -generalized closed [10], ω -generalized closed [3, 11]) sets are defined by replacing the closure operator in Levine's original definition by the semiclosure (resp., α -closure, θ -closure, semi-preclosure, δ -closure, ω -closure) operator.

2. Regular generalized ω -closed sets

A subset A of (X, τ) is called regular generalized closed (simply, rg -closed) (see [12]) if $\text{Cl}(A) \subset U$ whenever $A \subset U$ and U is regular open. Analogously, we begin this section by introducing the class of regular generalized ω -closed sets.

Definition 2.1. A subset A of (X, τ) is called regular generalized ω -closed (simply, $rg\omega$ -closed) if $\text{Cl}_\omega(A) \subset U$ whenever $A \subset U$ and U is regular open. A subset B of (X, τ) is called regular generalized ω -open (simply, $rg\omega$ -open) if the complement of B is $rg\omega$ -closed sets.

We have the following relation for $rg\omega$ -closed with the other known sets:

$$\begin{array}{ccccc}
 & \omega\text{-}c\text{-closed} & & & \\
 & \updownarrow & & & \\
 \text{closed} & \Longrightarrow & g\text{-closed} & \Longrightarrow & rg\text{-closed} \\
 & \downarrow & \downarrow & & \downarrow \\
 \omega\text{-closed} & \Longrightarrow & g\omega\text{-closed} & \Longrightarrow & rg\omega\text{-closed}
 \end{array} \tag{2.1}$$

Example 2.2. Let $\mathbb{R}$ be the set of all real numbers, let $\mathbb{Q}$ be the set of all rational numbers, with the topology $\tau = \{\mathbb{R}, \emptyset, \mathbb{R} - \mathbb{Q}\}$. Then $A = \mathbb{R} - \mathbb{Q}$ is not $g\omega$ -closed, since A is open, thus ω -open and $A \subseteq A$, $\text{Cl}_\omega(A) \not\subseteq A$ (because A is not ω -closed). Also the only regular open set containing A is X . Thus A is $rg\omega$ -closed.

Example 2.3. Let $X = \{a, b, c, d\}$, with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then the set $\{a\}$ is not rg -closed, see [13]. But $\{a\}$ is $rg\omega$ -closed set, since X is finite and τ_ω is discrete topology.

It is clear that if (X, τ) is a countable space, then $rg\omega(X, \tau) = \mathcal{P}(X)$, where $rg\omega(X, \tau)$ is the set of all $rg\omega$ -closed subsets of X and $\mathcal{P}(X)$ is the power set of X .

Since every closed set is ω -closed we have the following.

LEMMA 2.4. For every subset A of (X, τ) , $\text{Cl}_\omega(A) \subset \text{Cl}(A)$.

The proof of the following result follows from the fact that every regular open set is an open set together with Lemma 2.4.

THEOREM 2.5. Every $g\omega$ -closed set and rg -closed set are $rg\omega$ -closed.

THEOREM 2.6. *Let A be an $rg\omega$ -closed subset of (X, τ) . Then $Cl_\omega(A) - A$ does not contain any nonempty regular closed set.*

Proof. Let F be a regular closed subset of (X, τ) such that $F \subseteq Cl_\omega(A) - A$. Then $F \subseteq X - A$ and hence $A \subseteq X - F$. Since A is $rg\omega$ -closed set and $X - F$ is a regular open subset of (X, τ) , $Cl_\omega(A) \subseteq X - F$ and so $F \subseteq X - Cl_\omega(A)$. Therefore $F \subseteq Cl_\omega(A) \cap (X - Cl_\omega(A)) = \emptyset$. $\square$

THEOREM 2.7. *A subset A of (X, τ) is $rg\omega$ -open if and only if $F \subseteq Int_\omega(A)$ whenever F is a regular closed subset such that $F \subseteq A$.*

Proof. Let A be an $rg\omega$ -open subset of X and let F be a regular closed subset of X such that $F \subseteq A$. Then $X - A$ is an $rg\omega$ -closed set and $X - A \subseteq X - F$. Since $X - A$ is $rg\omega$ -closed, $X - Int_\omega(A) = Cl_\omega(X - A) \subseteq X - F$. Thus $F \subseteq Int_\omega(A)$. Conversely, if $F \subseteq Int_\omega(A)$ where F is a regular closed subset of (X, τ) such that $F \subseteq A$, then for any regular open subset U such that $X - A \subseteq U$, we have $X - U \subseteq A$ and thus $X - U \subseteq Int_\omega(A)$. That is, $X - Int_\omega(A) = Cl_\omega(X - A) \subseteq U$. Therefore $X - A$ is $rg\omega$ -closed. $\square$

LEMMA 2.8 [14]. *For every open U in a topological space X and every $A \subseteq X$, $Cl(U \cap A) = Cl(U \cap Cl(A))$.*

Recall that two nonempty sets A and B of X are said to be separated if $Cl(A) \cap B = \emptyset = A \cap Cl(B)$.

THEOREM 2.9. *If A and B are open, $rg\omega$ -open, and separated sets, then $A \cup B$ is $rg\omega$ -open.*

Proof. Let F be a regular closed subset of $A \cup B$. Then $F \cap Cl(A) \subseteq A$, since A is open and by Lemma 2.8 we have $F \cap Cl(A)$ is regular closed hence by Theorem 2.7 $F \cap Cl(A) \subseteq Int_\omega(A)$. Similarly, $F \cap Cl(B) \subseteq Int_\omega(B)$. Then we have $F \subseteq Int_\omega(A \cup B)$ and hence $A \cup B$ is $rg\omega$ -open. $\square$

The following example shows that the union of $rg\omega$ -open sets need not be $rg\omega$ -open.

Example 2.10. Let X be an uncountable set and let A, B, C, D be subsets of X , such that each of them is uncountable set and the family $\{A, B, C, D\}$ is a partition of X . We defined the topology $\tau = \{\emptyset, X, \{A\}, \{B\}, \{A, B\}, \{A, B, C\}\}$. Choose $x, y \notin A$ and $x \neq y$. Then $H = A \cup \{x\}$ and $G = A \cup \{y\}$ are $rg\omega$ -closed, since only regular open set containing H, G is X . But $H \cap G = \{A\}$ and $\{A\}$ is regular open in X and $Cl_\omega(A) \not\subseteq A$, since $\{A\}$ is not ω -closed. Thus $H \cap G$ is not $rg\omega$ -closed. Therefore the union of $rg\omega$ -open sets need not be $rg\omega$ -open.

The proof of the following result is straightforward since τ_ω is a topology on X and thus omitted.

THEOREM 2.11. *If A and B are $rg\omega$ -closed sets, then $A \cup B$ is $rg\omega$ -closed.*

THEOREM 2.12. *Let A be a $rg\omega$ -closed subset of (X, τ) . If $B \subseteq X$ such that $A \subseteq B \subseteq Cl_\omega(A)$, then B is also $rg\omega$ -closed. Let B be a subset of (X, τ) and let A be an $rg\omega$ -open subset such that $Int_\omega(A) \subseteq B \subseteq A$. Then B is also $rg\omega$ -open.*

The proof is obvious.

THEOREM 2.13. *If A be an $rg\omega$ -closed subset of (X, τ) , then $Cl_\omega(A) - A$ is $rg\omega$ -open set.*

Proof. Let A be an $rg\omega$ -closed subset of (X, τ) and let F be a regular closed subset such that $F \subseteq Cl_\omega(A) - A$. By Theorem 2.6, $F = \phi$ and thus $F \subseteq Int_\omega(Cl_\omega(A) - A)$. By Theorem 2.7, $Cl_\omega(A) - A$ is $rg\omega$ -open set. $\square$

We first recall the following lemmas to obtain further results for $rg\omega$ -closed sets.

LEMMA 2.14 [3]. *If Y is an open subspace of a space X and A is a subset of Y , then $Cl_{\omega|Y}(A) = Cl_\omega(A) \cap (Y)$.*

LEMMA 2.15. *If A is a regular open and $rg\omega$ -closed subset of a space X , then A is ω -closed in X .*

The proof is obvious.

THEOREM 2.16. *Let Y be an open subspace of a space X and $A \subseteq Y$. If A is $rg\omega$ -closed in X , then A is $rg\omega$ -closed in Y .*

Proof. Let U be a regular open set of Y such that $A \subseteq U$. Then $U = V \cap Y$ for some regular open set V of X . Since A is $rg\omega$ -closed in X , we have $Cl_\omega(A) \subseteq U$ and by Lemma 2.14, $Cl_{\omega|Y}(A) = Cl_\omega(A) \cap (Y) \subseteq V \cap Y = U$. Hence A is $rg\omega$ -closed in X . $\square$

COROLLARY 2.17. *If A is an $rg\omega$ -closed regular open set and B is an ω -closed set of a space X , then $A \cap B$ is $rg\omega$ -closed.*

THEOREM 2.18. *Let A be an $rg\omega$ -closed set. Then $A = Cl_\omega(Int_\omega(A))$ if and only if $Cl_\omega(Int_\omega(A)) - A$ is regular closed.*

Proof. If $A = Cl_\omega(Int_\omega(A))$, then $Cl_\omega(Int_\omega(A)) - A = \phi$ and hence $Cl_\omega(Int_\omega(A)) - A$ is regular closed. Conversely, let $Cl_\omega(Int_\omega(A)) - A$ be regular closed, since $Cl_\omega(A) - A$ contains the regular closed set $Cl_\omega(Int_\omega(A)) - A$. By Theorem 2.6 $Cl_\omega(Int_\omega(A)) - A = \phi$ and hence $A = Cl_\omega(Int_\omega(A))$. $\square$

LEMMA 2.19 [3]. *Let (A, τ_A) be an antilocally countable subspace of a space (X, τ) . Then $Cl(A) = Cl_\omega(A)$.*

We call (X, τ) an antilocally countable space if each nonempty open set is an uncountable set.

COROLLARY 2.20. *In an antilocally countable subspace of a space (X, τ) , the concepts of $rg\omega$ -closed set and rg -closed set coincide.*

LEMMA 2.21 [3]. *Let (X, τ) and (Y, σ) be two topological spaces. Then $(\tau \times \sigma)_\omega \subseteq \tau_\omega \times \sigma_\omega$.*

THEOREM 2.22. *If $A \times B$ is $rg\omega$ -open subset of $(X \times Y, \tau \times \sigma)$, then A is $rg\omega$ -open subset in (X, τ) and B is $rg\omega$ -open subset in (Y, σ) .*

Proof. Let F_A be a regular closed subset of (X, τ) and let F_B be a regular closed subset of (Y, σ) such that $F_A \subseteq A$ and $F_B \subseteq B$. Then $F_A \times F_B$ is regular closed in $(X \times Y, \tau \times \sigma)$ such that $F_A \times F_B \subseteq A \times B$. By assumption $A \times B$ is $rg\omega$ -open in $(X \times Y, \tau \times \sigma)$ and so

$F_A \times F_B \subseteq \text{Int}_\omega(A \times B) \subseteq \text{Int}_\omega(A) \times \text{Int}_\omega(B)$ by Lemma 2.21. Therefore $F_A \subseteq \text{Int}_\omega$, $F_B \subseteq \text{Int}_\omega(B)$. Hence A, B are $rg\omega$ -open. $\square$

The converse of the above need not be true in general.

Example 2.23. Let $X = Y = \mathbb{R}$ with the usual topology τ . Let $A = \{\{\mathbb{R} - \mathbb{Q}\} \cup [\sqrt{2}, 5]\}$ and $B = (1, 7)$. Then A and B are $rg\omega$ -open (ω -open) subsets of $(\mathbb{R}, \tau)$, while $A \times B$ is not $rg\omega$ -open in $(\mathbb{R} \times \mathbb{R}, \tau \times \tau)$, since the set $F = [\sqrt{2}, 3] \times [3, 5]$ is regular closed set contained in $A \times B$ and $F \not\subseteq \text{Int}_\omega(A \times B)$. The point $(\sqrt{2}, 4) \in F$ and $(\sqrt{2}, 4) \notin \text{Int}_\omega(A \times B)$, because if $(\sqrt{2}, 4) \in \text{Int}_\omega(A \times B)$, then there exist open set U containing $\sqrt{2}$ and open set V containing 4 such that $(U \times V) - (A \times B)$ is countable but $(U \times V) - (A \times B)$ is uncountable for any open set U containing $\sqrt{2}$ and open set V containing 4.

3. Regular generalized ω - $T_{1/2}$ space

Recall that a space (X, τ) is called $T_{1/2}$ [5] if every g -closed set is closed or equivalently if every singleton is open or closed, Dunham [15]. We introduce the following relatively new definition.

Definition 3.1. A space (X, τ) is a regular generalized ω - $T_{1/2}$ (simply, $rg\omega$ - $T_{1/2}$) if every $rg\omega$ -closed set in (X, τ) is ω -closed.

THEOREM 3.2. For a space (X, τ) , the following are equivalent.

- (1) X is a $rg\omega$ - $T_{1/2}$.
- (2) Every singleton is either regular closed or ω -open.

Proof. (1) $\Rightarrow$ (2) Suppose $\{x\}$ is not a regular closed subset for some $x \in X$. Then $X - \{x\}$ is not regular open and hence X is the only regular open set containing $X - \{x\}$. Therefore $X - \{x\}$ is $rg\omega$ -closed. Since (X, τ) is $rg\omega$ - $T_{1/2}$ space, $X - \{x\}$ is ω -closed and thus $\{x\}$ is ω -open.

(2) $\Rightarrow$ (1) Let A be an $rg\omega$ -closed subset of (X, τ) and $x \in \text{Cl}_\omega(A)$. We show that $x \in A$. If $\{x\}$ is regular closed and $x \notin A$, then $x \in (\text{Cl}_\omega(A) - A)$. Thus $\text{Cl}_\omega(A) - A$ contains a nonempty regular closed set $\{x\}$, a contradiction to Theorem 2.6. So $x \in A$. If $\{x\}$ is ω -open, since $x \in \text{Cl}_\omega(A)$, then for every ω -open set U containing x , we have $U \cap A \neq \emptyset$. But $\{x\}$ is ω -open then $\{x\} \cap A \neq \emptyset$. Hence $x \in A$. So in both cases we have $x \in A$. Therefore A is ω -closed. $\square$

THEOREM 3.3. Let (X, τ) be an antilocally countable space. Then (X, τ) is a T_1 -space if every $rg\omega$ -closed set is ω -closed.

Proof. Let $x \in X$, and suppose that $\{x\}$ is not closed. Then $A = X - \{x\}$ is not open, and thus A is $rg\omega$ -closed (the only regular open set containing A is X). Therefore, by assumption, A is ω -closed, and thus $\{x\}$ is ω -open. So there exists $U \in \tau$ such that $x \in U$ and $U - \{x\}$ is countable. It follows that U is a nonempty countable open subset of $x \in X$, a contradiction. $\square$

Definition 3.4. A map $f : X \rightarrow Y$ is said to be

- (i) approximately closed [16] (a -closed) provided that $f(F) \subseteq \text{Int}(A)$ whenever F is a closed subset of X , A is a g -open subset of Y , and $f(F) \subseteq A$;

- (ii) approximately continuous [16] (a -continuous) provided that $\text{Cl}(A) \subseteq f^{-1}(V)$ whenever V is an open subset of Y , A is a g -closed subset of X , and $A \subseteq f^{-1}(V)$.

Definition 3.5. A map $f : X \rightarrow Y$ is said to be approximately ω -closed (simply, a - ω -closed) provided that $f(F) \subseteq \text{Int}_\omega(A)$ whenever F is a regular closed subset of X , A is an $rg\omega$ -open of Y , and $f(F) \subseteq A$.

Definition 3.6. A map $f : X \rightarrow Y$ is said to be approximately ω -continuous (simply, a - ω -continuous) provided that $\text{Cl}_\omega(A) \subseteq f^{-1}(V)$ whenever V is a regular open subset of Y , A is an $rg\omega$ -closed subset of X , and $A \subseteq f^{-1}(V)$.

The notions of a -closed (resp.; a -continuous) and a - ω -closed (resp.; a - ω -continuous) are independent.

Example 3.7. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Let $f : (X, \tau) \rightarrow (X, \tau)$ be a function defined by $f(a) = a$, $f(b) = d$, $f(c) = b$, $f(d) = c$. Then f is a - ω -closed, since X is finite and thus τ_ω is a discrete topology, and f is not a -closed function. Because the set $A = \{b, c\}$ is g -open and $F = \{c, d\}$ is closed, $f(F) \subseteq A$, but $f(F) \not\subseteq \text{Int}(A)$.

Example 3.8. Let $X = \mathbb{R}$ with the topology $\tau = \{\phi, X, \mathbb{R} - \mathbb{Q}\}$. Let $f : (X, \tau) \rightarrow (X, \tau)$ be a function defined by $f(x) = 0$, for all $x \in X$. Then f is a -closed, since for any closed set F of X , the only g -open set containing $f(F)$ is X . And f is not a - ω -closed function. Because the set $A = \mathbb{Q}$ is $rg\omega$ -open and $F = \mathbb{R}$ is regular closed, $f(F) \subseteq A$, but $f(F) \not\subseteq \text{Int}_\omega(A) = \phi$.

THEOREM 3.9. A space X is $rg\omega$ - $T_{1/2}$ -space if and only if every space Y and every function $f : X \rightarrow Y$ are a - ω -continuous.

Proof. Let V be a regular open subset of Y and A is an $rg\omega$ -closed subset of X such that $A \subseteq f^{-1}(V)$, since X is $rg\omega$ - $T_{1/2}$ -space then A is ω -closed thus $A = \text{Cl}_\omega(A)$, hence $\text{Cl}_\omega(A) \subseteq f^{-1}(V)$ and f is a - ω -continuous. Let A be a nonempty $rg\omega$ -closed subset of X and let Y be the set X with the topology $\{Y, A, \phi\}$. Let $f : X \rightarrow Y$ be the identity mapping. By assumption f is a - ω -continuous. Since A is $rg\omega$ -closed subset in X and open in Y such that $A \subseteq f^{-1}(A)$, it follows that $\text{Cl}_\omega(A) \subseteq f^{-1}(A) = A$. Hence A is ω -closed in X and therefore X is $rg\omega$ - $T_{1/2}$ -space. $\square$

LEMMA 3.10. If the regular open and regular closed sets of X coincide, then all subsets of X are $rg\omega$ -closed (and hence all are $rg\omega$ -open).

Proof. Let A be any subset of X such that $A \subseteq U$ and U is regular open, then $\text{Cl}_\omega(A) \subseteq \text{Cl}_\omega(U) \subseteq \text{Cl}(U) = U$. Therefore A is $rg\omega$ -closed. $\square$

THEOREM 3.11. If the regular open and regular closed sets of Y coincide, then a function $f : X \rightarrow Y$ is a - ω -closed if and only if $f(F)$ is ω -open for every regular closed subset F of X .

Proof. Assume f is a - ω -closed by Lemma 3.10 all subsets of Y are $rg\omega$ -closed. So for any regular closed subset F of X , $f(F)$ is $rg\omega$ -closed in Y . Since f is a - ω -closed, $f(F) \subseteq \text{Int}_\omega(f(F))$, therefore $f(F) = \text{Int}_\omega(f(F))$ thus $f(F)$ is ω -open. Conversely if $f(F) \subseteq A$ where F is regular closed and A is $rg\omega$ -open, then $f(F) = \text{Int}_\omega(f(F)) \subseteq \text{Int}_\omega(A)$ hence f is a - ω -closed. $\square$

The proof of the following result for a - ω -continuous function is analogous and is omitted.

THEOREM 3.12. *If the regular open and regular closed sets of X coincide, then a function $f : X \rightarrow Y$ is a - ω -continuous if and only if $f^{-1}(V)$ is ω -closed for every regular open subset V of Y .*

4. $rg\omega$ -continuity

In this section, we will introduce some new classes of maps and study some of their characterizations. In [11, 3] a map $f : X \rightarrow Y$ is called ω -irresolute (resp., R -map [17]) if the inverse image of every ω -closed (resp., regular closed) subset of Y is ω -closed (resp., regular closed) in X . In [3], a map $f : X \rightarrow Y$ is called $g\omega$ -closed if the image of every closed subset of X is $g\omega$ -closed in Y . Relatively new definitions are given next.

Definition 4.1. A map $f : X \rightarrow Y$ is called $rg\omega$ -closed (resp., ro-preserving, pre- ω -closed) if $f(V)$ is $rg\omega$ -closed (resp., regular open, ω -closed) in Y for every closed (resp., regular open, ω -closed) subset V of X .

Example 4.2. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Let $f : (X, \tau) \rightarrow (X, \tau)$ be a function defined by $f(a) = a$, $f(b) = b$, $f(c) = d$, $f(d) = c$. Then f is ro-preserving, since the family of all regular open sets of X is $\{\phi, X, \{a\}, \{b\}\}$. But if we defined $g : (X, \tau) \rightarrow (X, \tau)$ as $g(a) = c$, $g(b) = d$, $g(c) = a$, $g(d) = b$, then g is not ro-preserving function.

Definition 4.3. A map $f : X \rightarrow Y$ is called $rg\omega$ -continuous (resp., $rg\omega$ -irresolute) if the inverse image of every ω -closed (resp., $rg\omega$ -closed) subset V of Y is $rg\omega$ -closed subset of X .

From the definition stated above we obtain the following diagram of implications:

$$\begin{array}{ccccc}
 \text{continuous} & & & & \\
 \Downarrow & & & & \\
 \omega\text{-continuous} & \implies & g\omega\text{-continuous} & \implies & rg\omega\text{-continuous} \\
 \Uparrow & & \Uparrow & & \Uparrow \\
 \omega\text{-irresolute} & \implies & g\omega\text{-irresolute} & \implies & rg\omega\text{-irresolute}
 \end{array} \tag{4.1}$$

THEOREM 4.4. *Let $f : X \rightarrow Y$ be a surjective, $rg\omega$ -irresolute, and pre- ω -closed map if X is $rg\omega$ - $T_{1/2}$ -space, then Y is also an $rg\omega$ - $T_{1/2}$ -space.*

Proof. Let A be $rg\omega$ -closed subset of Y . Since f is an $rg\omega$ -irresolute map, then $f^{-1}(A)$ is an $rg\omega$ -closed subset of X . Since X is $rg\omega$ - $T_{1/2}$ -space, then $f^{-1}(A)$ is an ω -closed subset of X . Since f is a pre- ω -closed map, then $f(f^{-1}(A)) = A$ is an ω -closed subset of Y . Therefore Y is also $rg\omega$ - $T_{1/2}$ -space. $\square$

Since every $g\omega$ -closed set is $rg\omega$ -closed, every $g\omega$ -closed map is $rg\omega$ -closed. Next we give new characterization of $g\omega$ -closed maps.

THEOREM 4.5. *A map $f : X \rightarrow Y$ is $g\omega$ -closed if and only if for each $A \subseteq Y$ and each open set U containing $f^{-1}(A)$, there exists a $g\omega$ -open subset V of Y such that $A \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. Let F be a $g\omega$ -closed map, $A \subseteq Y$, and let U be an open set containing $f^{-1}(A)$. Then $V = Y - f(X - U)$ is $g\omega$ -open subset of Y containing A and $f^{-1}(V) \subseteq U$. Conversely let F be closed subset of X and let H be an open subset of Y such that $f(F) \subseteq H$. Then $f^{-1}(Y - f(F)) \subseteq X - F$ and $X - F$ is open by hypothesis, there exists a $g\omega$ -open subset V of Y such that $Y - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Therefore, $F \subseteq X - f^{-1}(V)$ and hence $f(F) \subseteq Y - V$. Since $Y - H \subseteq Y - f(F)$, $f^{-1}(Y - H) \subseteq f^{-1}(Y - f(F)) \subseteq f^{-1}(V) \subseteq X - F$, by taking complement, we get $F \subseteq X - f^{-1}(V) \subseteq X - f^{-1}(Y - f(F)) \subseteq X - f^{-1}(Y - H)$. Therefore $f(F) \subseteq Y - V \subseteq H$. Since $Y - V$ is $g\omega$ -closed set and $\text{Cl}_\omega(f(F)) \subseteq \text{Cl}_\omega(Y - V) \subseteq H$, hence $f(F)$ is $g\omega$ -closed. Thus f is a $g\omega$ -closed map. $\square$

Since every ω -closed set is $rg\omega$ -closed, we have the following.

THEOREM 4.6. *Every $rg\omega$ -irresolute map is $rg\omega$ -continuous map.*

Definition 4.7. A subset $A \subseteq X$ is said to be ω - c -closed provided that there is a proper subset B for which $A = \text{Cl}_\omega(B)$. A map $f : X \rightarrow Y$ is said to be $g\omega$ - c -closed if $f(A)$ is $g\omega$ -closed in Y for every ω - c -closed subset $A \subseteq X$.

Since closed sets are obviously ω - c -closed, $g\omega$ -closed maps are $g\omega$ - c -closed. In a similar manner, we say a map $f : X \rightarrow Y$ is $rg\omega$ - c -closed if $f(A)$ is $rg\omega$ -closed in Y for every ω - c -closed subset $A \subseteq X$.

THEOREM 4.8. *Let $f : X \rightarrow Y$ be an R -map and $rg\omega$ - c -closed. Then $f(A)$ is $rg\omega$ -closed in Y for every $rg\omega$ -closed subset A of X .*

Proof. Let A be an $rg\omega$ -closed subset of X and let U be a regular open subset of Y such that $f(A) \subseteq U$. Since f is an R -map, $f^{-1}(U)$ is a regular open subset of X and $A \subseteq f^{-1}(U)$. As A is an $rg\omega$ -closed subset, $\text{Cl}_\omega(A) \subseteq f^{-1}(U)$. Hence $f(\text{Cl}_\omega(A)) \subseteq (U)$. Because $\text{Cl}_\omega(A)$ is ω - c -closed and F is $rg\omega$ - c -closed map, $f(\text{Cl}_\omega(A))$ is $rg\omega$ -closed. Therefore, $\text{Cl}_\omega(f(A)) \subseteq \text{Cl}_\omega(f(\text{Cl}_\omega(A))) \subseteq f(\text{Cl}_\omega(A)) \subseteq U$. Hence $f(A)$ is an $rg\omega$ -closed subset of Y . $\square$

THEOREM 4.9. *Let $f : X \rightarrow Y$ be ro -preserving and ω -irresolute function, if B is $rg\omega$ -closed in Y , then $f^{-1}(B)$ is $rg\omega$ -closed in X .*

Proof. Let G be a regular open subset of X such that $f^{-1}(B) \subseteq G$. Then $B \subseteq f(G)$ and $f(G)$ is regular open. Since B is $rg\omega$ -closed, then $\text{Cl}_\omega(B) \subseteq f(G)$ and $f^{-1}(\text{Cl}_\omega(B)) \subseteq G$. Since f is ω -irresolute then $f^{-1}(\text{Cl}_\omega(B))$ is ω -closed and $\text{Cl}_\omega(f^{-1}(\text{Cl}_\omega(B))) = f^{-1}(\text{Cl}_\omega(B))$, therefore $\text{Cl}_\omega(f^{-1}(B)) \subseteq \text{Cl}_\omega(f^{-1}(\text{Cl}_\omega(B))) \subseteq G$ thus $f^{-1}(B)$ is $rg\omega$ -closed in X . $\square$

THEOREM 4.10. *Let $f : X \rightarrow Y$ be a - ω -closed maps and ω -irresolute maps, if A is $rg\omega$ -closed in Y , then $f^{-1}(A)$ is $rg\omega$ -closed in X .*

Proof. Assume that A is an $rg\omega$ -closed in Y and $f^{-1}(A) \subseteq U$, where U is a regular open subset of X . Taking complements we obtain $X - U \subseteq X - f^{-1}(A) \subseteq f^{-1}(Y - A)$ and $f(X - U) \subseteq Y - A$. Since f is $a\omega$ -closed, $f(X - U) \subseteq \text{Int}_\omega(Y - A) = Y - \text{Cl}_\omega(A)$. It follows that $X - U \subseteq X - f^{-1}(\text{Cl}_\omega(A))$ and $f^{-1}(\text{Cl}_\omega(A)) \subseteq U$, since f is ω -irresolute, $f^{-1}(\text{Cl}_\omega(A))$ is ω -closed thus we have $f^{-1}(A) \subseteq f^{-1}(\text{Cl}_\omega(A)) \subseteq U$ and $\text{Cl}_\omega(f^{-1}(A)) \subseteq \text{Cl}_\omega(f^{-1}(\text{Cl}_\omega(A))) = f^{-1}(\text{Cl}_\omega(A)) \subseteq U$. Therefore $\text{Cl}_\omega(f^{-1}(A)) \subseteq U$ and $f^{-1}(A)$ is $rg\omega$ -closed in X . $\square$

THEOREM 4.11. *If $f : X \rightarrow Y$ is R -map and $rg\omega$ -closed and A is g -closed subset of X , then $f(A)$ is $rg\omega$ -closed.*

Proof. Let $f(A) \subseteq U$, where U is regular open subset of X then $f^{-1}(U)$ is regular open set containing A . Since A is g -closed, we have then $\text{Cl}(A) \subseteq f^{-1}(U)$ and $f(\text{Cl}(A)) \subseteq U$. Since f is $rg\omega$ -closed, $f(\text{Cl}(A))$ is $rg\omega$ -closed. Therefore $\text{Cl}_\omega(f(\text{Cl}(A))) \subseteq U$ which implies that $\text{Cl}_\omega(f(A)) \subseteq U$, hence $f(A)$ is $rg\omega$ -closed. $\square$

The proof of Theorem 4.8 can be easily modified to obtain the following result.

THEOREM 4.12. *Let $f : X \rightarrow Y$ be $a\omega$ -map and $rg\omega$ -c-closed. Then $f(A)$ is $rg\omega$ -closed subset of Y for every $rg\omega$ -closed subset A of X .*

THEOREM 4.13. *Let $f : X \rightarrow Y$ be R -map and pre- ω -closed. Then $f(A)$ is $rg\omega$ -closed in Y for every $rg\omega$ -closed subset A of X .*

Proof. Let A be any $rg\omega$ -closed subset of X and let U be any regular open subset of Y such that $f(A) \subseteq U$. Since f is R -map, $f^{-1}(U)$ is regular open and $A \subseteq f^{-1}(U)$. As A is $rg\omega$ -closed, $\text{Cl}_\omega(A) \subseteq f^{-1}(U)$. Hence $f(\text{Cl}_\omega(A)) \subseteq U$. Therefore $\text{Cl}_\omega(f(A)) \subseteq \text{Cl}_\omega(f(\text{Cl}_\omega(A))) = f(\text{Cl}_\omega(A)) \subseteq U$. Hence $f(A)$ is $rg\omega$ -closed in Y . $\square$

Definition 4.14. A map $f : X \rightarrow Y$ is said to be ω -contra- R -map if for every regular open subset V of Y , $f^{-1}(V)$ is ω -closed.

Example 4.15. Let $X = \mathbb{R}$ with the usual topology τ and let $Y = \{a, b, c, d\}$, with the topology $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then the function $f : (X, \tau) \rightarrow (Y, \sigma)$ defined by

$$f(x) = \begin{cases} a, & \text{if } x \in \mathbb{Q}, \\ c, & \text{if } x \notin \mathbb{Q}, \end{cases} \quad (4.2)$$

is ω -contra- R -map, since $\mathbb{Q}$ is ω -closed. But the function $f(x)$ defined by

$$f(x) = \begin{cases} a, & \text{if } x \in \mathbb{Q}, \\ b, & \text{if } x \notin \mathbb{Q}, \end{cases} \quad (4.3)$$

is not ω -contra- R -map, since the family of all regular open set in (Y, σ) is $\{\phi, Y, \{a\}, \{b\}\}$ and $f^{-1}(\{b\})$ is not ω -closed.

THEOREM 4.16. *Let $f : X \rightarrow Y$ be ω -contra- R -map and $rg\omega$ -c-closed. Then $f(A)$ is $rg\omega$ -closed in Y for every subset A of X .*

Proof. Let A be any subset of X and let U be any regular open subset of Y such that $f(A) \subseteq U$. Then $A \subseteq f^{-1}(U)$. Since f is ω -contra- R -map, $f^{-1}(U)$ is ω -closed and so $\text{Cl}_\omega(A) \subseteq \text{Cl}_\omega(f(U)) = f(U)$. Hence $f(\text{Cl}_\omega(A)) \subseteq U$. As $\text{Cl}_\omega(A)$ is ω - c -closed subset of X and f is $rg\omega$ - c -closed map, $f(\text{Cl}_\omega(A))$ is $rg\omega$ -closed. Therefore $\text{Cl}_\omega(f(A)) \subseteq \text{Cl}_\omega(f(\text{Cl}_\omega(A))) \subseteq f(\text{Cl}_\omega(A)) \subseteq U$. Thus $f(A)$ is $rg\omega$ -closed in Y . $\square$

THEOREM 4.17. *If map $f : X \rightarrow Y$ is $rg\omega$ -continuous (resp., $rg\omega$ -irresolute) and X is $rg\omega$ - $T_{1/2}$, then f is ω -continuous (resp., $rg\omega$ -irresolute).*

Proof. Let A be any closed (resp., ω -closed) subset of Y . Since f is an $rg\omega$ -continuous (resp., $rg\omega$ -irresolute) map, $f^{-1}(A)$ is an $rg\omega$ -closed subset of X . As (X, τ) is $rg\omega$ - $T_{1/2}$ space, $f^{-1}(A)$ is an ω -closed subset of X . Therefore, f is an ω -continuous (resp., $rg\omega$ -irresolute). $\square$

THEOREM 4.18. *Let $f : X \rightarrow Y$ be a bijective, ro-preserving, and $rg\omega$ -continuous map. Then f is $rg\omega$ -irresolute map.*

Proof. Let V be any $rg\omega$ -closed subset of X and let U be any regular open subset of Y such that $f^{-1}(V) \subseteq U$. Clearly $V \subseteq f(U)$. Since f is a ro-preserving map, $f(U)$ is regular open and, by assumption, V is $rg\omega$ -closed set. Hence $\text{Cl}_\omega(V) \subseteq f(U)$ and $f^{-1}(\text{Cl}_\omega(V)) \subseteq U$. Since f is $rg\omega$ -continuous and $\text{Cl}_\omega(V)$ is ω -closed in Y , then $f^{-1}(\text{Cl}_\omega(V))$ is a $rg\omega$ -closed subset of U and so $\text{Cl}_\omega(f^{-1}(\text{Cl}_\omega(V))) \subseteq U$. Since $\text{Cl}_\omega(f^{-1}(V)) \subseteq \text{Cl}_\omega(f^{-1}(\text{Cl}_\omega(V))) \subseteq U$, $\text{Cl}_\omega(f^{-1}(V)) \subseteq U$. Therefore $f^{-1}(V)$ is an $rg\omega$ -closed subset. Hence f is f $rg\omega$ -irresolute map. $\square$

THEOREM 4.19. *A map $f : X \rightarrow Y$ is f $rg\omega$ -closed if and only if for each subset B of Y and for each open set U containing $f^{-1}(B)$, there is an $rg\omega$ -open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. Suppose f is $rg\omega$ -closed, let B be a subset of Y , and U is an open set of X such that $f^{-1}(B) \subseteq U$. Then $f(X - U)$ is $rg\omega$ -closed in Y . Let $V = Y - f(X - U)$, then V is $rg\omega$ -open set and $f^{-1}(V) = f^{-1}(Y - f(X - U)) = X - (X - U) \subseteq U$ therefore V is an $rg\omega$ -open set containing B such that $f^{-1}(V) \subseteq U$. Conversely suppose that F is a closed set of X then $f^{-1}(Y - f(F)) \subseteq X - F$, and $X - F$ is open. By hypothesis, there is an $rg\omega$ -open set V of Y such that $Y - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$ therefore $F \subseteq X - f^{-1}(V)$. Hence $Y - V \subseteq f(F) \subseteq f(X - f^{-1}(V)) \subseteq Y - V$ implies that $f(F) = Y - V$, thus f is $rg\omega$ -closed. $\square$

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