Let $R$ be a ring with center $Z$, Jacobson radical $J$, and set $N$ of all nilpotent elements. Call $R$ generalized periodic-like if for all $x \in R \setminus (N \cup J \cup Z)$ there exist positive integers $m, n$ of opposite parity for which $x^m - x^n \in N \cap Z$. We identify some basic properties of such rings and prove some results on commutativity.
the set \([\{x, y \mid x \in X, y \in Y\}];\) and \(C(R)\) denotes the commutator ideal of \(R\). An element \(x \in R\) is called regular if it is not a zero divisor; it is called periodic if there exist distinct positive integers \(m, n\) for which \(x^m = x^n\); and it is called potent if there exists an integer \(n > 1\) for which \(x^n = x\). The set of all potent elements of \(R\) is denoted by \(P\) or \(P(R)\), and the prime radical by \(\Psi(R)\). Finally, \(R\) is called reduced if \(N(R) = \{0\}\).

**Lemma 2.1.** Let \(R\) be an arbitrary \(g\)–\(p\)–\(l\) ring.

(i) Every epimorphic image of \(R\) is a \(g\)–\(p\)–\(l\) ring.

(ii) \(N \subseteq J\).

(iii) If \([N, J] = \{0\}\), then \(N\) is an ideal.

(iv) \(C(R) \subseteq J\).

(v) If \(e\) is an idempotent, the additive order of which is not a power of 2, then \(e \in Z\).

**Proof.** (i) is clear, once we recall that if \(\sigma : R \rightarrow S\) is an epimorphism, then \(\sigma(J(R)) \subseteq J(S)\).

(ii) Let \(S = R/J(R)\). Then by (i), \(S\) is a \(g\)–\(p\)–\(l\) ring; and since \(J(S) = \{0\}\), \(S\) is a \(g\)–\(p\) ring. It follows from [1, Theorem 1] that \(N(S)\) is an ideal of \(S\), hence \(N(S) \subseteq J(S) = \{0\}\) and therefore \(N(R) \subseteq J(R)\).

(iii) Since \(N \subseteq J\), \(N\) is commutative and hence \((N, +)\) is an additive subgroup. Let \(a \in N\) and \(x \in R\). Then \(ax \in J\), so \([a, ax] = 0\), that is, \(a^2x = axa\). It follows that \((ax)^2 = a^2x^2\) and that \((ax)^n = a^n x^n\) for all positive integers \(n\). Therefore, \(ax \in N\).

(iv) As in (ii), \(R/J(R)\) is a \(g\)–\(p\) ring; hence, by [1, Lemma 2], \(C(R/J(R)) = \{0\}\). Therefore, \(C(R) \subseteq J(R)\).

(v) If \(e \notin Z\), then \(-e \notin J \cup Z\) and there exists \((m, n)\) such that \((-e)^m - (-e)^n \in N \cap Z\). Since \(m, n\) are of opposite parity, we get \(2e \in N\), so that \(2^ke = 0\) for some \(k\).

**Lemma 2.2.** Let \(R\) be an arbitrary \(g\)–\(p\)–\(l\) ring, and let \(x \in R\). Then either \(x \in J \cup Z\), or there exist a positive integer \(q\) and an idempotent \(e\) such that \(x^q = x^q e\).

**Proof.** If \(x \notin J \cup Z\), there exists \((m, n)\) such that \(x^m - x^n \in N \cap Z\). Therefore, there exist a positive integer \(q\) and \(g(t) \in Z[t]\) such that \(x^q = x^{q+1}g(x)\). It is now easy to verify that \(e = (xg(x))^q\) is an idempotent with \(x^q = x^q e\).

**Lemma 2.3.** Let \(R\) be a \(g\)–\(p\)–\(l\) ring and \(\sigma\) an epimorphism from \(R\) to \(S\). Then \(N(S) \subseteq \sigma(J(R)) \cup Z(S)\).

**Proof.** Let \(s \in N(S)\) with \(s^k = 0\) and let \(d \in R\) such that \(\sigma(d) = s\). If \(d \in J(R) \cup Z(R)\), then obviously \(s \in \sigma(J(R)) \cup Z(S)\); hence we may suppose that there exists \((m, n)\) with \(n > m\) such that \(d^m - d^n \in N(R) \cap Z(R)\). It is easy to show that \(d - d^h \in N\), where \(h = n - m + 1\); thus

\[
d - d^{k+1}d^{(k-1)} = d - d^h + d^{h-1}(d - d^h) + \cdots + (d^{h-1})^{k-1}(d - d^h)
\]

is a sum of commuting nilpotent elements, hence it is in \(N(R)\) and therefore in \(J(R)\). Consequently, \(s - s^{k+1}s^{(k-1)} \in \sigma(J(R))\); and since \(s^{k+1} = 0\), \(s \in \sigma(J(R))\).

We finish this section by stating two known results on periodic elements.
Lemma 2.4. Let $R$ be an arbitrary ring, and let $N^* = \{x \in R \mid x^2 = 0\}$.

(i) [2, Lemma 1] If $x \in R$ is periodic, then $x \in P + N$.

(ii) [3, Theorem 2] If $N^*$ is commutative and $N$ is multiplicatively closed, then $PN \subseteq N$.

3. Commutativity results

Theorem 3.1. If $R$ is a g–p–l ring with $J \subseteq Z$, then $R$ is commutative.

Proof. Suppose $x \notin Z$. Then by Lemma 2.1(ii), we have $((m,n))$ with $n > m$ such that $x^m - x^n \in N \cap Z$. Consequently $x^{n-m+1} - x \in N$; and since $N \subseteq Z$, commutativity of $R$ follows by a well-known theorem of Herstein [4].

Theorem 3.2. If $R$ is any g–p–l ring with 1, then $R$ is commutative.

Proof. We show that if $R$ is g–p–l with 1, then $J \subseteq Z$. Suppose that $x \in J \cap Z$. Then $-1 + x \notin J \cup Z$, so there exists $((m,n))$ such that $(-1 + x)^m - (1 + x)^n \in N \cap Z$; and we may assume that $m$ is even and $n$ is odd. Since $N \subseteq J$, it follows that $2 \in J$; thus for every integer $m$, $2m \in J$, and hence $2m + 1$ is invertible.

Now consider $((m_1,n_1))$ such that $(1 + x)^{m_1} - (1 + x)^{n_1} \in N \cap Z$. Then $(m_1 - n_1)x + x^2p(x) \in N \cap Z$ for some $p(t) \in Z[t]$; and since $m_1 - n_1$ is central and invertible, we get $x + x^2w \in N \cap Z$ for some $w \in R$ with $[x,w] = 0$. Thus, we have a positive integer $q$ and an element $y$ in $R$ such that $[x,y] = 0$ and $x^q = x^qy$. It follows that $e = (xy)^q$ is an idempotent such that $x^q = x^qe$; and since $J$ contains no nonzero idempotents, $x$ is in $N$.

Let $\alpha$ be the smallest positive integer for which $x^k \in Z$ for all $k \geq \alpha$, and note that, since $x \notin Z$, $\alpha \geq 2$. But $1 + x^{a-1} \notin J \cup Z$, so there exists $((m_2,n_2))$ such that $(1 + x^{a-1})^{m_2} - (1 + x^{a-1})^{n_2} \in N \cap Z$; hence $(m_2 - n_2)x^{a-1} \in Z$. But since $m_2 - n_2$ is invertible and central, we conclude that $x^{a-1} \in Z$—a contradiction.

Theorem 3.3. If $R$ is a reduced g–p–l ring with $R \notin I$, then $R$ is commutative.

Proof. If $R = J \cup Z$, then $R = Z$ and we are finished. Otherwise, if $x \in R \setminus (J \cup Z)$, there exists $((m,n))$ such that $x^m - x^n \in N \cap Z = \{0\}$; hence $x$ is periodic, and by Lemma 2.4(i), $x \notin P$. Thus, $R = P \cup J \cup Z$; and to complete the proof we need only to show that $P \subseteq Z$.

Let $y \in P$, and let $k > 1$ be such that $y^k = y$. Then $e = y^{k-1}$ is an idempotent for which $y = ye$, and $e \in Z$ since $N = \{0\}$. Now $eR$ is an ideal of $R$, so that $J(eR) = eR \cap J(R)$; hence $eR$ is a g–p–l ring with 1, which is commutative by Theorem 3.2. Therefore, $[ey,ew] = 0$ for all $w \in R$; and since $ey = y$ and $e \in Z$, we conclude that $[y,w] = 0$ for all $w \in R$, that is, $y \in Z$.

Theorem 3.4. If $R$ is a g–p–l ring in which $J$ is commutative and all idempotents are central, then $R$ is commutative.

Proof. We may express $R$ as a subdirect product of subdirectly irreducible rings, each of which is an epimorphic image of $R$. Let $R_a$ be such a subdirectly irreducible ring, and let $\sigma : R \to R_a$ be an epimorphism. Let $x_a \in R_a$ and let $x \in R$ such that $\sigma(x) = x_a$. By Lemma 2.2, $x \in J(R) \cup Z(R)$ or there exist an idempotent $e \in R$ and a positive integer $q$ such that $x^q = x^q e$. Thus, either $x_a \in \sigma(J(R)) \cup Z(R_a)$ or $x_a^q = x^q_a e_a$, where $e_a = \sigma(e)$ is a central idempotent of $R_a$. But $R_a$ is subdirectly irreducible, hence if $R_a$ has a nonzero central idempotent, then $R_a$ has 1 and is commutative by Theorem 3.2.
To complete the proof, we need only consider the case that for each $x_a \in R_a$, $x_a \in \sigma(f(R)) \cup Z(R_a) \cup N(R_a)$. Now by Lemma 2.3, $N(R_a) \subseteq \sigma(f(R)) \cup Z(R_a)$; hence $R_a = \sigma(f(R)) \cup Z(R_a)$, which is clearly commutative. Therefore, $R$ is commutative.

Theorem 3.4 has two corollaries, the first of which is immediate when we recall Lemma 2.1(v).

**Corollary 3.5.** If $R$ is a 2-torsion-free g–p–l ring with $J$ commutative, then $R$ is commutative.

**Corollary 3.6.** Let $R$ be a g–p–l ring containing a regular central element $c$. If $J$ is commutative, then $R$ is commutative.

**Proof.** It suffices to show that $N \subseteq Z$ since this condition implies that idempotents are central. Consider first the case $c \in J$. Then $cf \subseteq J^2$, which is central since $J$ is commutative. Since $c$ is regular and central, it is immediate that $J \subseteq Z$, so certainly $N \subseteq Z$.

Now assume that $c \not\in J$, and suppose that $a \in N \setminus Z$. Then $c + a \not\in J \cup Z$, and there exists $((m, n))$ such that $(c + a)^m - (c + a)^n \in N \cap Z$. It follows that $c^m - c^n$ is a sum of commuting nilpotent elements, hence $c^m - c^n \in N$ and there exists $q$ such that $c^q = c^{q + 1} p(c)$ for some $p(t) \in \mathbb{Z}[t]$. As before, we get an idempotent $e$ such that $c^q = c^qe$ and $[c, e] = 0$. Now $e$ cannot be a zero divisor, since that would force $c$ to be a zero divisor; therefore, $R$ has a regular idempotent, that is, $R$ has 1. We have contradicted Theorem 3.2, so $N \subseteq Z$ as claimed.

4. Nil-commutator-ideal theorems

**Theorem 4.1.** Let $R$ be a g–p–l ring. If $R \neq J$ and $N$ is an ideal, then $C(R)$ is nil.

**Proof.** We may assume $R \neq J \cup Z$, since otherwise $R$ is commutative. Let $\overline{R} = R/N$, and let the element $x + N$ of $\overline{R}$ be denoted by $\overline{x}$. We need to show that $\overline{R}$ is commutative—a conclusion that follows from Theorem 3.3 once we show that $J(\overline{R}) \neq \overline{R}$.

Suppose that $J(\overline{R}) = \overline{R}$, and let $x \in R \setminus (J \cup Z)$. By Lemma 2.2, there exists a positive integer $q$ and an idempotent $e \in R$ such that $x^q = x^qe$; and it follows that $\overline{e}$ is an idempotent of $\overline{R}$ such that $\overline{x}^q = \overline{x}^q \overline{e}$. But $\overline{R} = J(\overline{R})$ contains no nonzero idempotents, so that $\overline{x}^q = 0 = \overline{x}$ and hence $x \in N(R)$. This contradicts the fact that $x \not\in J \cup Z$, hence $\overline{R} \neq J(\overline{R})$ as required.

**Theorem 4.2.** If $R$ is a g–p–l ring and $J$ is commutative, then $C(R)$ is nil.

**Proof.** If $R = J$, then $R$ is commutative. If $R \neq J$, $N$ is an ideal by Lemma 2.1(iii) and $C(R)$ is nil by Theorem 4.1.

In fact, we can improve this result as follows.

**Theorem 4.3.** Let $R$ be a g–p–l ring with $R \neq J$. If $N$ is commutative, then $C(R)$ is nil.

This result follows from Theorem 4.1, once we prove our final theorem.

**Theorem 4.4.** Let $R$ be a g–p–l ring with $R \neq J$. If $N$ is commutative, then $N$ is an ideal.
Proof. Again we may assume that \( R \neq J \cup Z \). Since \( N \) is commutative, \( N \) is an additive subgroup of \( R \). To show that \( RN \subseteq N \), it is convenient to work with the ring \( \overline{R} = R/\mathfrak{P}(R) \). As in the proof of Theorem 4.1, we have \( J(\overline{R}) \neq \overline{R} \); and if \( \overline{R} = Z(\overline{R}) \), then \( C(\overline{R}) \subseteq \mathfrak{P}(\overline{R}) \subseteq N \). Therefore, we assume that \( \overline{R} \neq J(\overline{R}) \cup Z(\overline{R}) \). We note that if \( x + N = \overline{x} \in N(\overline{R}) \), then \( x \in N(R) \); consequently, \( N(\overline{R}) \) is commutative and hence is an additive subgroup of \( \overline{R} \).

Now \( \overline{R} \) is semiprime and therefore \( N(\overline{R}) \cap Z(\overline{R}) = \{0\} \). It follows that if \( \overline{x} \in \overline{R} \setminus (J(\overline{R}) \cup Z(\overline{R})) \), there exists \( (m, n) \) such that \( \overline{x}^m = \overline{x}^n \), that is, \( x \) is periodic. Thus \( x \in P(\overline{R}) + N(\overline{R}) \) by Lemma 2.4(i); and by commutativity of \( N(\overline{R}) \) and Lemma 2.4(ii) we get \( \overline{x}N(\overline{R}) \subseteq N(\overline{R}) \). Moreover, if \( \overline{y} \in Z(\overline{R}) \), \( \overline{y}N(\overline{R}) \subseteq N(\overline{R}) \). Now let \( \overline{y} \in J(\overline{R}) \setminus Z(\overline{R}) \), and let \( \overline{x} \in \overline{R} \setminus (J(\overline{R}) \cup Z(\overline{R})) \). Then \( \overline{x} + \overline{y} \notin J(\overline{R}) \), hence it is in \( \overline{R} \setminus (J(\overline{R}) \cup Z(\overline{R})) \) or in \( Z(\overline{R}) \); and in either case \( (\overline{x} + \overline{y})N(\overline{R}) \) and \( \overline{x}N(\overline{R}) \) are in \( N(\overline{R}) \), so that \( \overline{y}N(\overline{R}) \subseteq N(\overline{R}) \). We have shown that \( N(\overline{R}) \) is an ideal of \( \overline{R} \); therefore, if \( x \in R \) and \( a \in N(R) \), \( xa \in N(\overline{R}) \) and hence \( xa \in N(R) \). Thus, \( N(R) \) is an ideal of \( R \). \( \Box \)

Remark 4.5. There exist noncommutative g–p–l rings with \( J \) commutative. An accessible example is

\[
\left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in GF(2) \right\}.
\] (4.1)

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