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Research Article

Comparison of KP and BBM-KP Models

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It is shown that the solutions of the pure initial-value problem for the KP and regularized KP equations are the same, within the order of accuracy attributable to either, on the time scale $0 \le t \le \epsilon^{-3/2}$, during which nonlinear and dispersive effects may accumulate to make an order-one relative difference to the wave profiles.

1. Introduction

Bona et al. [1] compared the solutions of the Cauchy problem of the Korteweg-de Vries (KdV)

$$P_t + P_x + PP_x + P_{xxx} = 0 \tag{1.1}$$

and the Benjamin-Bona-Mahony (BBM) equation

$$Q_t + Q_x + QQ_x - Q_{xxt} = 0. \tag{1.2}$$

In (1.1) and (1.2), P and Q are functions of two real variables x and t.

Such equations have been derived as models for nonlinear dispersive waves in many different physical contexts, and in most cases where they arise, x is proportional to distance measured in the direction of wave propagation, while t is proportional to elapsed time. Interest is often focused on the pure initial-value problem for (1.1) and (1.2) in which P and Q are specified for all real x at some beginning value of t, say $t = 0$, and then the evolution equation is solved for $t \ge 0$ subject to the restriction that the solution respects the given initial condition. The thrust of their theory was that for suitably restricted

initial conditions, the solutions P and Q emanating therefrom are nearly identical at least for values of t in an interval $[0,T]$, where T is quite large.

This paper is concerned with mathematical models representing the unidirectional propagation of weakly nonlinear dispersive long waves with weak transverse effects. Interest will be directed toward two particular models. One is the Kadomtsev-Petviashvili (KP) equations

$$(\eta_t + \eta_x + \eta\eta_x + \eta_{xxx})_x + \alpha\eta_{yy} = 0 \tag{1.3}$$

and the other is a regularized version, the Benjamin-Bona-Mahony (BBM-KP) equation

$$(\xi_t + \xi_x + \xi\xi_x - \xi_{xxt})_x + \alpha\xi_{yy} = 0, \tag{1.4}$$

where $\alpha = \pm 1$. If $\alpha = +1$ in (1.3), the equation is known as the KP II equation, while for $\alpha = -1$, it is the KP I equation. The Kadomtsev-Petviashvili equations are two-dimensional extensions of the Korteweg-de-Vries equation. They occur naturally in many physical contexts as "universal" models for the propagation of weakly nonlinear dispersive long waves which are essentially one directional, with weak transverse effects. Observe that the linearized dispersion relation for (1.3) is

$$w(k,l) = k\left(1 - k^2 + \frac{\alpha l^2}{k^2}\right), \tag{1.5}$$

while that of (1.4) is

$$\widetilde{w}(k,l) = \frac{k^2 + \alpha l^2}{k(1+k^2)}, \tag{1.6}$$

within the scaling assumption that $k^2 = O(\delta)$, and $l = O(\delta)$. As $\delta \to 0$, $w(k,l)$ may be approximated by $\widetilde{w}(k,l)$ to the same order of δ (since $k^2 l^2$ is of higher order).

The Cauchy problem for these equations have been studied by a number of authors.

Bourgain [2], using Picard iteration, has proved that the pure initial-value problem for the KP II equation is locally well-posed, and hence in light of the conservation laws for the equation, globally well-posed for data in $L^2(\mathbb{R})$. The same method has been used to extend local well-posedness to some Sobolev spaces of negative indices.

A compactness method that uses only the divergence form of the nonlinearity and the skew-adjointness of the linear dispersion operator was employed by Iório and Nunes in [3] to establish local well-posedness for data in $H^s(\mathbb{R}^2)$, for $s > 2$, for the KP I equation. The Iório-Nunes approach applies equally well to KP II-type equation. Molinet et al. [4] using the local well-posedness of Iório and Nunes obtained a version of the classical energy method coupled with some of the known conserved quantities and delicate estimates of Strichartz type for the KP I equation to show global well-posedness for KP I equation in the space

$$\begin{aligned} Z = \Big\{ \varphi \in L^2(\mathbb{R}^2) : &\|\varphi\|_{L^2} + \|\varphi_{xxx}\|_{L^2} + \|\varphi_y\|_{L^2} \\ &+ \|\varphi_{xy}\|_{L^2} + \|\partial_x^{-1}\varphi_y\|_{L^2} + \|\partial_x^{-2}\varphi_{yy}\|_{L^2} < \infty \Big\}. \end{aligned} \tag{1.7}$$

Kenig [5] improved on this local well-posedness result given by the classical energy estimate by showing local well-posedness in the space

$$Y_s = \left\{\varphi \in L^2(\mathbb{R}^2) : \|\varphi\|_{L^2} + \|J^s\varphi\|_{L^2} + \|\partial_x^{-1}\varphi_y\|_{L^2} < \infty\right\} \tag{1.8}$$

for $s > 3/2$, where $\widehat{J^s f}(k,l) = (1+k^2)^{s/2}\hat{f}(k,l)$.

In [6], global well-posedness is established in

$$Z_0 = \left\{\varphi \in L^2(\mathbb{R}^2) : \|\varphi\|_{L^2} + \|\partial_x^{-1}\varphi_y\|_{L^2} + \|\varphi_{xx}\|_{L^2} + \|\partial_x^{-2}\varphi_{yy}\|_{L^2} < \infty\right\} \tag{1.9}$$

for the KP I equation. It is worth mentioning the result of Colliander et al. [7], dealing with the well-posedness of KP I equation when the initial data has low regularity. Bona et al. [8] have shown that (1.4) can be solved by Picard iteration yielding to local and global well-posedness results for the associated Cauchy problem. In particular, it is shown that the pure initial-value problem for (1.4), regardless of the sign of α, is globally well-posed in

$$W_1 = \left\{\varphi \in L^2(\mathbb{R}^2) : \|\varphi\|_{L^2} + \|\varphi_x\|_{L^2} + \|\varphi_{xx}\|_{L^2} + \|\partial_x^{-1}\varphi_y\|_{L^2} + \|\varphi_y\|_{L^2} < \infty\right\}. \tag{1.10}$$

Saut and Tzvetkov [9] improved this global well-posedness to the space

$$Y = \{\varphi \in L^2(\mathbb{R}^2) : \varphi_x \in L^2(\mathbb{R}^2)\}. \tag{1.11}$$

We remark that provided g satisfies an appropriate constraint, (1.3) and (1.4) are equivalent to the integrated forms

$$\begin{aligned} \eta_t + \eta_x + \eta\eta_x + \eta_{xxx} + \alpha\partial_x^{-1}\eta_{yy} &= 0, \\ \xi_t + \xi_x + \xi\xi_x - \xi_{xxt} + \alpha\partial_x^{-1}\xi_{yy} &= 0. \end{aligned} \tag{1.12}$$

To illustrate the kind of results we have in mind, we briefly outline below what Saut and Tzvetkov generally discussed concerning the relationship between the two models in [9]. Since both KP and BBM-KP equations model weakly nonlinear dispersive long waves which are valid to order ϵ^2, they can be considered as order ϵ perturbations of the linear transport equation, and hence can be written as

$$\eta_{\epsilon t} + \eta_{\epsilon x} + \epsilon\left(\eta_\epsilon\eta_{\epsilon x} + \eta_{\epsilon xxx} \pm \partial_x^{-1}\eta_{\epsilon yy}\right) = 0, \tag{1.13}$$

$$\xi_{\epsilon t} + \xi_{\epsilon x} + \epsilon\left(\xi_\epsilon\xi_{\epsilon x} - \xi_{\epsilon xxt} \pm \partial_x^{-1}\xi_{\epsilon yy}\right) = 0, \tag{1.14}$$

with initial data $\eta_{\epsilon 0}$, $\xi_{\epsilon 0}$, respectively, which are of order one. The neglected terms in the right-hand sides of (1.13) and (1.14) are of order ϵ^2. After performing the change of variables

$$M = \epsilon\eta, \qquad N = \epsilon\xi, \qquad x_1 = \epsilon^{-1/2}x, \qquad y_1 = \epsilon^{-1}y, \qquad \tau = \epsilon^{-1/2}t, \tag{1.15}$$

one can rewrite (1.12) and (1.13) as

$$\begin{aligned} M_\tau + M_{x_1} + MM_{x_1} + M_{x_1x_1x_1} \pm \partial_{x_1}^{-1} M_{y_1y_1} &= 0, \\ N_\tau + N_{x_1} + NN_{x_1} - N_{x_1x_1\tau} \pm \partial_{x_1}^{-1} N_{y_1y_1} &= 0, \end{aligned} \tag{1.16}$$

with initial data, respectively,

$$\begin{aligned} M(x_1, y_1, 0) &= \epsilon\eta_{\epsilon 0}(\epsilon^{1/2}x_1, \epsilon y_1), \\ N(x_1, y_1, 0) &= \epsilon\xi_{\epsilon 0}(\epsilon^{1/2}x_1, \epsilon y_1). \end{aligned} \tag{1.17}$$

The dispersive and nonlinear terms in (1.13) and (1.14) may have a significant influence on the structure of the waves on the time scale $t_1 = \epsilon^{-1}$ (corresponding to $\tau_1 = \epsilon^{-3/2}$), while the neglected order ϵ^2 terms might affect the solution at order one on time scale $t_2 = \epsilon^{-2}$ ($\tau_2 = \epsilon^{-5/2}$). It is therefore of interest to compare the solutions of (1.13) and (1.14) on time scales between t_1, and t_2. Such analysis was performed for the KdV and BBM models in [1].

To give an idea of the results one can expect that we consider the Cauchy problem for the linear versions of (1.13) and (1.14)

$$\begin{aligned} \eta_{\epsilon t} + \eta_{\epsilon x} + \epsilon\left(+ \eta_{\epsilon xxx} \pm \partial_x^{-1}\eta_{\epsilon yy}\right) &= 0, \\ \xi_{\epsilon t} + \xi_{\epsilon x} + \epsilon\left(- \xi_{\epsilon xxt} \pm \partial_x^{-1}\xi_{\epsilon yy}\right) &= 0 \end{aligned} \tag{1.18}$$

together with initial condition

$$\eta_\epsilon(x, y, 0) = \xi_\epsilon(x, y, 0) = f(x, y), \tag{1.19}$$

where f is of order one. Taking the Fourier transforms in both x and y variables, we have

$$\begin{aligned} \widehat{\eta_\epsilon}_t + ik\widehat{\eta_\epsilon} + \epsilon\left[(ik)^3\widehat{\eta_\epsilon} \pm \frac{-l^2}{ik}\widehat{\eta_\epsilon}\right] &= 0, \\ \hat{\xi}_{\epsilon t} + ik\hat{\xi}_\epsilon + \epsilon\left[k^2\hat{\xi}_{\epsilon t} \pm \frac{il^2}{k}\hat{\xi}_\epsilon\right] &= 0, \end{aligned} \tag{1.20}$$

with $\widehat{\eta_\epsilon}(k,l,0) = \widehat{\xi_\epsilon}(k,l,0) = \widehat{f}(k,l)$. Solving for η_ϵ and ξ_ϵ, there obtains

$$\begin{aligned} \widehat{\eta_\epsilon} &= e^{-it[k-\epsilon k^3 \pm \epsilon(l^2/k)]}\widehat{f}(k,l,t), \\ \widehat{\xi_\epsilon} &= e^{(-it/(1+\epsilon k^2))(k\pm\epsilon(l^2/k))}\widehat{f}(k,l,t). \end{aligned} \tag{1.21}$$

It is readily inferred that provided $k^5\widehat{f}(k,l)$, $l^2k\widehat{f}(k,l) \in L^1(\mathbb{R}^2)$, then

$$\left|\eta_\epsilon(\cdot,\cdot,t) - \xi_\epsilon(\cdot,\cdot,t)\right|_\infty \le \left|\widehat{\eta_\epsilon}(k,l,t) - \widehat{\xi_\epsilon}(k,l,t)\right|_{L^1(\mathbb{R}^2)} \le \epsilon^2 t C_f \le C \tag{1.22}$$

since η_ϵ and ξ_ϵ are of order one, the estimate (1.22) proves that up to time scale $t_1 = \epsilon^{-(1+\delta)}$, $0 < \delta < 1$, η_ϵ and ξ_ϵ are $\epsilon^{1-\delta}$ close to each other.

2. Notations

We will employ the following notations. We will let $|\cdot|_p$, $\|\cdot\|_s$ denote the norms in $L^p(\mathbb{R}^2)$ and the classical Sobolev spaces $H^s(\mathbb{R}^2)$, respectively, where

$$\|f\|_s^2 = \int_{\mathbb{R}^2} (1+k^2+l^2)^s \left|\widehat{f}(k,l)\right|^2 dk\,dl \tag{2.1}$$

and $\widehat{\ }$ connotes Fourier transformation. Thus, the norm in $L^2(\mathbb{R}^2)$ will simply be denoted by $\|\cdot\|_0$. By H^∞, we denote $\bigcap_{s\ge 0} H^s$. The elements of H^∞ are infinitely differentiable functions, all of whose derivatives lie in L^2. If X is an arbitrary Banach space and $T > 0$, the space $C(0,T,X)$ is the collection of continuous functions $u : [0,T] \to X$. This collection is a Banach space with the norm $\sup_{0\le t\le T}\|u(t)\|_X$, where $\|\cdot\|_X$ denotes the norm in X.

Define the space $H^s_{-1}(\mathbb{R}^2) = \{\eta \in S'(\mathbb{R}^2) : \|\eta\|_{H^s_{-1}(\mathbb{R}^2)} < \infty\}$ equipped with the norm

$$\|\eta\|_{H^s_{-1}(\mathbb{R}^2)} = \left(\int_{\mathbb{R}^2} (1+|k|^{-1})^2 (1+k^2+l^2)^s \left|\widehat{\eta}(k,l)\right|^2 dk\,dl\right)^{1/2}. \tag{2.2}$$

3. Summary of existence theory

As earlier mentioned, the pure initial-value problems for these model evolution equations have been studied. For our analysis, we will use the result of [6] for the Cauchy problem for KP I equation, and the result for the well-posedness of BBM-KP I equation is contained in [8].

We will first consider the initial-value problems for the KP I and BBM-KP I equations

$$(\eta_t + \eta_x + \eta\eta_x + \eta_{xxx})_x - \eta_{yy} = 0, \tag{3.1}$$

$$(\xi_t + \xi_x + \xi\xi_x - \xi_{xxt})_x - \xi_{yy} = 0, \tag{3.2}$$

with initial condition

$$\eta(x,y,0) = \xi(x,y,0) = g(x,y). \tag{3.3}$$

The theoretical results relating to the initial-value problems (3.1) and (3.2) are presented in the following theorems without proof.

Theorem 3.1. *Let $s \geq 2$. Then for any $g \in H^s_{-1}(\mathbb{R}^2)$, there exist a positive $T_0 = T_0(|\nabla g|_{L^\infty})$ ($\lim_{\rho\to 0} T_0(\rho) = \infty$) and a unique η of the integrated KP I equation (3.1) with initial data g on the time interval $[0,T_0]$ satisfying $\eta \in C([0,T_0];H^s_{-1}(\mathbb{R}^2))$, $\eta_t \in C([0,T_0];H^{s-3}(\mathbb{R}^2))$. Furthermore, the Map $g \mapsto \eta$ is continuous from $H^s_{-1}(\mathbb{R}^2)$ to $C([0,T_0];H^s_{-1}(\mathbb{R}^2))$.*

Theorem 3.2. *Let $g \in H^s_{-1}$ with $s > 3/2$. Then, there exist $T_0 > 0$ such that the BBM-KP I equation (3.2) has a unique solution $\xi \in C([0,T_0];H^s_{-1}(\mathbb{R}^2))$, $\partial_x^{-1}\xi_y \in C([0,T_0];H^{s-1}_{-1}(\mathbb{R}^2))$, with $\xi_t \in C([0,T_0];H^{s-2}(\mathbb{R}^2))$. Moreover, the map $g \mapsto \xi$ is continuous from $H^s_{-1}(\mathbb{R}^2)$ to $C([0,T_0] : H^s_{-1}(\mathbb{R}^2))$.*

4. Main result

In this section, we compare the solutions of the initial-value problems

$$\begin{aligned} \eta_t + \eta_x + \eta\eta_x + \eta_{xxx} - \partial_x^{-1}\eta_{yy} &= 0, \\ \xi_t + \xi_x + \xi\xi_x - \xi_{xxt} - \partial_x^{-1}\xi_{yy} &= 0 \end{aligned} \tag{4.1}$$

both with initial condition

$$\eta(x,y,0) = \xi(x,y,0) = \epsilon g(\epsilon^{1/2}x, \epsilon y). \tag{4.2}$$

Our main result in this paper is the following.

Theorem 4.1. *Let $g \in H^{k+5}_{-1}(\mathbb{R}^2)$, where $k \geq 0$ and let η and ξ be the unique solutions guaranteed by Theorems 3.1 and 3.2 for the initial-value problems (4.1). Then there exist positive constants C and T which depend only on k and g such that the solutions η and ξ of the initial-value problems*

$$\begin{aligned} \eta_t + \eta_x + \eta\eta_x + \eta_{xxx} - \partial_x^{-1}\eta_{yy} &= 0, \\ \xi_t + \xi_x + \xi\xi_x - \xi_{xxt} - \partial_x^{-1}\xi_{yy} &= 0, \\ \eta(x,y,0) = \xi(x,y,0) &= \epsilon g(\epsilon^{1/2}x, \epsilon y), \end{aligned} \tag{4.3}$$

satisfy the inequalities

$$\begin{aligned} \|\partial_x^j \eta(\cdot,\cdot,t) - \partial_x^j \xi(\cdot,\cdot,t)\|_0 &\le C\epsilon^{j/2+5/4}, \\ \|\partial_y^j \eta(\cdot,\cdot,t) - \partial_y^j \xi(\cdot,\cdot,t)\|_0 &\le C\epsilon^{j+5/4} \end{aligned} \tag{4.4}$$

for $0 < \epsilon \le 1$, *and* $0 \le t \le \epsilon^{-3/2}\min\{T, T_0\}$, *where* $0 \le j \le k$.

Before we prove Theorem 4.1, we introduce two new dependent variables

$$\begin{aligned} u(x,y,t) &= \epsilon^{-1}\eta(\epsilon^{-1/2}x + \epsilon^{-3/2}t, \epsilon^{-1}y, \epsilon^{-3/2}t), \\ v(x,y,t) &= \epsilon^{-1}\xi(\epsilon^{-1/2}x + \epsilon^{-3/2}t, \epsilon^{-1}y, \epsilon^{-3/2}t). \end{aligned} \tag{4.5}$$

A brief calculation shows that u and v satisfy, respectively, the initial-value problems

$$\begin{aligned} &u_t + uu_x + u_{xxx} - \partial_x^{-1}u_{yy} = 0, \\ &v_t + vv_x + v_{xxx} - \epsilon v_{xxt} - \partial_x^{-1}v_{yy} = 0, \\ &u(x,y,0) = v(x,y,0) = g(x,y). \end{aligned} \tag{4.6}$$

By virtue of Theorems 3.1 and 3.2, the existence and uniqueness of u and v are assured.

Theorem 4.2. *Let* $g \in H_{-1}^s(\mathbb{R}^2)$, *where* $s \ge 2$, *then the initial-value problem for both equations (4.6) have solutions in* $C([0,T_0]; H_{-1}^s(\mathbb{R}^2))$ *for some* $T_0 > 0$. *Moreover, if* $g \in H_{-1}^{k+5}(\mathbb{R}^2)$, *then there exist positive constants* C *and* T *depending only on* k *and* g *such that the difference* $u - v$ *satisfies*

$$\begin{aligned} \|\partial_x^j u - \partial_x^j v\|_0 &\le C\epsilon t, \\ \|\partial_y^j u - \partial_y^j v\|_0 &\le C\epsilon t \end{aligned} \tag{4.7}$$

for all ϵ *and* t *for which* $0 \le \epsilon \le t$ *and* $0 \le t \le \min[T; T_0]$ *where* $0 \le j \le k$.

We will make use of the following anisotropic Sobolev inequalities

$$\begin{aligned} |f|_\infty &\le 2\|f\|_0^{1/4}\|f_y\|_0^{1/2}\|f_{xx}\|_0^{1/4}, \\ |f_x|_\infty &\le \|f_{xx}\|_0^{1/4}\|f_y\|_0^{1/2}\|f_{xxx}\|_0^{1/4}, \end{aligned} \tag{4.8}$$

the proofs of which can be found in [10].

Proof of Theorem 4.2. Let $w = u - v$. Then, w is seen to satisfy

$$w_t + ww_x + w_{xxx} - \epsilon w_{xxt} - \partial_x^{-1} w_{yy} = -\epsilon u_{xxt} - (wu)_x, \tag{4.9}$$

$$w(x,y,0) = 0. \tag{4.10}$$

We now venture into the task of estimating $\|\partial_x^j w\|_0 + \|\partial_y^j w\|_0$, for $j = 0,1,2,3,4,\ldots$.

In light of Theorem 3.1, we note that

$$\|u\|_k \le c\|g\|_{H^k_{-1}(\mathbb{R}^2)} = A_k. \tag{4.11}$$

This fact justifies the following computations.

We first estimate $\|\partial_x^j w\|_0$. Apply the operator ∂_x^j to both sides of the differential equation (4.10), multiply the result by $\partial_x^j w$, and integrate the result over $\mathbb{R}^2$ and over $[0,t]$. After a few integrations by parts, and taking into account the fact that $w(x,y,0) \equiv 0$, it follows that

$$\int_{\mathbb{R}^2} \{(\partial_x^j w)^2 + \epsilon(\partial_x^{j+1} w)^2\} dx\,dy = 2\int_0^t \int_{\mathbb{R}^2} \partial_x^j w \left\{\epsilon \partial_x^{(j+2)} u_\tau - \partial_x^{(j+1)}\left(wu + \frac{1}{2}w^2\right)\right\} dx\,dy\,d\tau, \tag{4.12}$$

which holds for $j = 0,1,2,\ldots$. Similarly, apply the operator ∂_y^j on (4.10), multiply the result by $\partial_y^j w$, and integrate the result over $\mathbb{R}^2$ and over $[0,t]$. After a few integrations by parts, and taking into account the fact that $w(x,y,0) \equiv 0$, it follows also that

$$\int_{\mathbb{R}^2} \{(\partial_y^j w)^2 + \epsilon(\partial_y^j w_x)^2\} dx\,dy = 2\int_0^t \int_{\mathbb{R}^2} \partial_y^j w \left\{\epsilon \partial_y^j u_{xx\tau} - \partial_y^j\left(wu + \frac{1}{2}w^2\right)_x\right\} dx\,dy\,d\tau \tag{4.13}$$

for $j = 0,1,2,3,4,\ldots$. The relations (4.12) and (4.13) will be used repeatedly.

First, for $j = 0$, (4.12) or (4.13) may be used, since they will both given the same estimate. Making use of (4.12), there appears after two integrations by parts that

$$\int_{\mathbb{R}^2} (w^2 + \epsilon w_x^2) dx\,dy = \int_0^t \int_{\mathbb{R}^2} \{2\epsilon(wu_{xx\tau}) - (u_x w^2)\} dx\,dy\,d\tau, \tag{4.14}$$

from this, the following inequality is derived

$$\|w\|_0^2 \le \int_0^t (2\epsilon\|w\|_0 \|u_{xx\tau}\|_0 + |u_x|_\infty \|w\|_0^2) d\tau. \tag{4.15}$$

By a variant of Gronwall's lemma, it follows that

$$\|w\|_0 \le \epsilon\left(\frac{C_2}{C_1}\right)\left(e^{C_1 t}-1\right) \le \epsilon t C_2 e^{C_1} = M_0\epsilon t, \tag{4.16}$$

where C_1 and C_2 are bounds for $(1/2)\|u_x\|_\infty$ and $\|u_{xxt}\|_0$, respectively, and t is restricted to the range [0,1].

Using the equation satisfied by u, the following estimates can be derived:

$$\begin{aligned}\sup_{t\ge 0}\|u_{xxt}\|_0 &= \sup_{t\ge 0}\|u_{xyy}-3u_xu_{xx}-uu_{xxx}-u_{xxxxx}\|_0\\ &\le \sup_{t\ge 0}\Big[\|u_{xyy}\|_0+2\|u\|_0^{1/4}\|u_y\|_0^{1/2}\|u_{xx}\|_0^{1/4}\|u_{xxx}\|_0\\ &\qquad +6\|u_{xx}\|_0^{5/4}\|u_y\|_0^{1/2}\|u_{xxxx}\|_0^{1/4}+\|u_{xxxxx}\|_0\Big].\end{aligned} \tag{4.17}$$

Hence, C_2 may be defined by $\sup_{t\ge 0}\|u_{xxt}\|_0 \le C_2$. For C_1, we estimate it from the anisotropic Sobolev inequality

$$|u_x|_\infty \le 2\|u_{xx}\|_0^{1/4}\|u_y\|_0^{1/2}\|u_{xxxx}\|_0^{1/4} \le C_1. \tag{4.18}$$

We therefore infer by an application of Gronwall's lemma that

$$\|w\|_0 \le \epsilon\left(\frac{C_2}{C_1}\right)\left(e^{C_1 t}-1\right) \le \epsilon t C_2 e^{C_1} = M_0\epsilon t \le B \tag{4.19}$$

for $0 \le t \le 1$.

For $j=1$, integrate (4.12) by parts to get the following relation:

$$\int_{\mathbb{R}^2}[w_x^2+\epsilon w_{xx}^2]\,dx\,dy = \int_0^t\int_{\mathbb{R}^2}(2\epsilon w_x u_{xxx\tau}-w_x^3-3w_x^2u_x-ww_xu_{xx})\,dx\,dy\,d\tau. \tag{4.20}$$

Similarly, for $j=1$, integrate (4.13) by parts to get

$$\int_{\mathbb{R}^2}[w_y^2+\epsilon w_{xy}^2]\,dx\,dy = \int_0^t\int_{\mathbb{R}^2}(2\epsilon w_y u_{xxy\tau}-2w_xw_yu_y+w_y^2u_x+w_ywu_{xy}-w_y^2w_x)\,dx\,dy\,d\tau. \tag{4.21}$$

Adding these two equations above, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^2}[w_x^2+w_y^2+\epsilon w_{xx}^2+\epsilon w_{xy}^2]\,dx\,dy\\ &\quad= \int_0^t\int_{\mathbb{R}^2}\Big[2\epsilon w_xu_{xxx\tau}-w_x^3-3w_x^2u_x-w_xwu_{xx}+2\epsilon w_yu_{xxy\tau}\\ &\qquad -2w_xw_yu_y+w_y^2u_x+w_ywu_{xy}-w_y^2w_x\Big]\,dx\,dy\,d\tau.\end{aligned} \tag{4.22}$$

The integrand on the right-hand side of (4.22) may be bounded above by

$$\begin{aligned}
&2\epsilon\|w_x\|_0\|u_{xxx\tau}\|_0+2\epsilon\|w_y\|_0\|u_{xxy\tau}\|_0+[4|u_x|_\infty+|v_x|_\infty]\|w_x\|_0^2\\
&\quad+[2|u_x|_\infty+|v_x|_\infty]\|w_y\|_0^2+\|w_x\|_0\|w\|_0|u_{xx}|_\infty+\|w_y\|_0\|w\|_0|u_{xy}|_\infty\\
&\quad+2\|w_x\|_0\|w_y\|_0|u_y|_\infty.
\end{aligned}\tag{4.23}$$

Also from the anisotropic Sobolev inequalities, we infer that

$$\begin{aligned}
|u_y|_\infty &\le 2\|u_y\|_0^{1/4}\|u_{yy}\|_0^{1/2}\|u_{xxy}\|_0^{1/4}\le C,\\
|u_x|_\infty &\le 2\|u_{xx}\|_0^{1/4}\|u_y\|_0^{1/2}\|u_{xxx}\|_0^{1/4}\le C,\\
|u_{xx}|_\infty &\le 2\|u_{xxx}\|_0^{1/4}\|u_{xy}\|_0^{1/2}\|u_{xxxx}\|_0^{1/4}\le C,\\
|u_{xy}|_\infty &\le 2\|u_{xxy}\|_0^{1/4}\|u_{yy}\|_0^{1/2}\|u_{xxxy}\|_0^{1/4}\le C.
\end{aligned}\tag{4.24}$$

Now using the equation satisfied by u, we may derive the following estimates, valid for $0\le\tau\le 1$:

$$\begin{aligned}
&\sup_{t\ge0}\|u_{xxx\tau}\|_0\\
&\quad\le\sup_{t\ge0}\Big[\|u_{xxyy}\|_0+2\|u\|_0^{1/4}\|u_y\|_0^{1/2}\|u_{xx}\|_0^{1/4}\|u_{xxxx}\|_0\\
&\qquad+8\|u_{xx}\|_0^{1/4}\|u_y\|_0^{1/2}\|u_{xxxx}\|_0^{1/4}\|u_{xxx}\|_0+3\|u_{xx}\|_0^2+\|u_{xxxxxx}\|_0\Big]\le C,\\
&\sup_{t\ge0}\|u_{xxy\tau}\|_0\\
&\quad\le\sup_{t\ge0}\Big[\|u_{xxxy}\|_0+4\|u_{xy}\|_0^{3/2}\|u_{xxx}\|_0^{1/4}\|u_{xxxx}\|_0^{1/4}\\
&\qquad+6\|u_{xx}\|_0^{1/4}\|u_y\|_0^{1/2}\|u_{xxx}\|_0^{1/4}\|u_{xxy}\|_0+2\|u_y\|_0^{1/4}\|u_{yy}\|_0^{1/2}\|u_{yxx}\|_0^{1/4}\|u_{xxx}\|_0\\
&\qquad+2\|u_x\|_0^{1/4}\|u_y\|_0^{1/2}\|u_{xx}\|_0^{1/4}\|u_{xxxy}\|_0+\|u_{xxxxxy}\|_0+\|u_{yyy}\|_0\Big]\le C.
\end{aligned}\tag{4.25}$$

Putting all these estimates together, the right-hand side of (4.20) may be bounded above by

$$2\epsilon C_4(\|w_x\|_0+\|w_y\|_0)+C_3(\|w_x\|_0^2+\|w_y\|_0^2),\tag{4.26}$$

where C_3 and C_4 are order-one quantities. The following is then seen to hold from (4.19)

$$||w_x||_0^2 + ||w_y||_0^2 \leq \int_0^t [2\epsilon C_4(||w_x||_0 + ||w_y||_0) + C_3(||w_x||_0^2 + ||w_y||_0^2)]d\tau. \tag{4.27}$$

Let $A_1(t) = [\|w_x\|_0^2 + \|w_y\|_0^2]^{1/2}$. Observe that

$$(||w_x||_0 + ||w_y||_0)^2 = ||w_x||_0^2 + ||w_y||_0^2 + 2||w_x||_0||w_y||_0 \leq 2(||w_x||_0^2 + ||w_y||_0^2), \tag{4.28}$$

so that

$$||w_x||_0 + ||w_y||_0 \leq \sqrt{2}A_1(t). \tag{4.29}$$

Therefore,

$$\frac{d}{dt}[||w_x||_0^2 + ||w_y||_0^2] \leq 2\epsilon C_4(||w_x||_0 + ||w_y||_0) + C_3(||w_x||_0^2 + ||w_y||_0^2), \tag{4.30}$$

which is equivalent to

$$\frac{d}{dt}A_1^2(t) \leq 2\epsilon C_4\sqrt{2}A_1(t) + C_3A_1^2(t) \tag{4.31}$$

or

$$\frac{d}{dt}A_1(t) \leq \epsilon C_4\sqrt{2} + \frac{1}{2}C_3A_1(t), \tag{4.32}$$

and hence by a variant of Gronwall's inequality, we obtain

$$A_1(t) \leq 2\sqrt{2}\epsilon\left(\frac{C_4}{C_3}\right)(e^{(1/2)C_3t} - 1) \leq 2\sqrt{2}\epsilon tC_4e^{(1/2)C_3} = \epsilon tM_1. \tag{4.33}$$

We thus infer that

$$\begin{aligned} ||w_x||_0 &\leq \epsilon tM_1, \\ ||w_y||_0 &\leq \epsilon tM_1 \end{aligned} \tag{4.34}$$

for $0 \leq t \leq 1$. Since C_3 and C_4 are order-one quantities, M_1 is also an order-one quantity. For $j = 2$, integrate (4.12) and (4.13) by parts, combine the two operations to get the following:

$$\begin{aligned} &\int_{\mathbb{R}^2} [w_{xx}^2 + \epsilon w_{xxx}^2 + w_{yy}^2 + \epsilon w_{xyy}^2]dx\,dy \\ &\quad = \int_0^t \int_{\mathbb{R}^2} \Big[2\epsilon w_{xx}u_{xxxx\tau} + 2\epsilon w_{yy}u_{yyxx\tau} + 6w_xw_{xx}u_{xx} - 4w_yw_{yy}u_{xy} \\ &\qquad + 2ww_{xx}u_{xxx} - ww_{yy}u_{xyy} - 4w_xw_{xx}^2 - 2w_{yy}^2u_x\Big]dx\,dy\,d\tau. \end{aligned} \tag{4.35}$$

The right side of (4.35) may be bounded above by

$$\begin{aligned}
&2\epsilon\{\|w_{xx}\|_0\|u_{xxxx\tau}\|_0+\|w_{yy}\|_0\|u_{xxyy\tau}\|_0\}+6\|w_x\|_0\|w_{xx}\|_0|u_{xx}|_\infty\\
&\quad+4\|w_y\|_0\|w_{yy}\|_0|u_{xy}|_\infty+2\|w\|_0\|w_{xx}\|_0|u_{xxx}|_\infty+\|w\|_0\|w_{yy}\|_0|u_{xyy}|_\infty\\
&\quad+4\|w_{xx}\|_0^2\|w_x\|_\infty+2\|w_{yy}\|_0^2|u_x|_\infty.
\end{aligned}\tag{4.36}$$

From the anisotropic Sobolev inequalities, we get the following estimates:

$$\begin{aligned}
|u_{xyy}|_\infty &\le 2\|u_{xxyy}\|_0^{1/4}\|u_{yyy}\|_0^{1/2}\|u_{xxxyy}\|_0^{1/4}\le C,\\
|u_{xxx}|_\infty &\le 2\|u_{xxxx}\|_0^{1/4}\|u_{xxy}\|_0^{1/2}\|u_{xxxxx}\|_0^{1/4}\le C.
\end{aligned}\tag{4.37}$$

Using the differential equation satisfied by u, we also obtain the following estimates, valid for $0\le\tau\le 1$:

$$\begin{aligned}
&\sup_{\tau\ge0}\|u_{xxxx\tau}\|_0\\
&\quad\le\sup_{\tau\ge0}\Big\{\|u_{xxxxx}\|_0+\|u_{xxx}\|_0^{5/4}\|u_{xy}\|_0^{1/2}\|u_{xxxx}\|_0^{1/4}\\
&\qquad+4\|u_{xx}\|_0^{1/4}\|u_y\|_0^{1/2}\|u_{xxx}\|_0^{1/4}\|u_{xxxx}\|_0\\
&\qquad+2\|u_x\|_0^{1/4}\|u_y\|_0^{1/2}\|u_{xx}\|_0^{1/4}\|u_{xxxxx}\|_0\\
&\qquad+\|u_{xxxxxxx}\|_0+\|u_{xxyy}\|_0\le C\Big\}.
\end{aligned}\tag{4.38}$$

Similarly,

$$\begin{aligned}
&\sup_{\tau\ge0}\|u_{xxyy\tau}\|_0\\
&\quad\le\sup_{\tau\ge0}\Big\{\|u_{xxxyy}\|_0+8\|u_{xyy}\|_0\|u_{xxx}\|_0^{1/4}\|u_{xy}\|_0^{1/2}\|u_{xxxx}\|_0^{1/4}\\
&\qquad+14\|u_{xxy}\|_0^{5/4}\|u_{yy}\|_0^{1/2}\|u_{xxxy}\|_0^{1/4}+6\|u_{xx}\|_0^{1/4}\|u_y\|_0^{1/2}\|u_{xxx}\|_0^{1/4}\|u_{xxyy}\|_0\\
&\qquad+2\|u_{yy}\|_0^{1/4}\|u_{yyy}\|_0^{1/2}\|u_{xxyy}\|_0^{1/4}\|u_{xxx}\|_0+4\|u_y\|_0^{1/4}\|u_{yy}\|_0^{1/2}\|u_{xxy}\|_0^{1/4}\|u_{xxxy}\|_0\\
&\qquad+2\|u\|_0^{1/4}\|u_y\|_0^{1/2}\|u_{xx}\|_0^{1/4}\|u_{xxxyy}\|_0+\|u_{xxxxxyy}\|_0+\|u_{yyyy}\|_0\le C\Big\}.
\end{aligned}\tag{4.39}$$

Putting all these estimates together, the right of (4.35) is bounded by

$$2\epsilon C_6(\|w_{xx}\|_0+\|w_{yy}\|_0)+C_5(\|w_{xx}\|_0^2+\|w_{yy}\|_0^2),\tag{4.40}$$

where from previous estimates we already obtained $\|w\|_0 \le \epsilon t M_0 \le C_0$, $\|w_x\|_0 \le \epsilon t \widetilde{M}_1 \le C_1$, $\|w_y\|_0 \le \epsilon t M_1 \le C_1$ for $0 \le t \le 1$. Consequently, we have that

$$||w_{xx}||_0^2 + ||w_{yy}||_0^2 \le \int_0^t \left[2\epsilon C_6(||w_{xx}||_0 + ||w_{yy}||_0) + C_5(||w_{xx}||_0^2 + ||w_{yy}||_0^2)\right] d\tau, \tag{4.41}$$

where C_5, and C_6 are order-one constants. Let

$$A_2(t) = [||w_{xx}||_0^2 + ||w_{yy}||_0^2]^{1/2}. \tag{4.42}$$

Note that

$$(||w_{xx}||_0 + ||w_{yy}||_0)^2 = ||w_{xx}||_0^2 + ||w_{yy}||_0^2 + 2||w_{xx}||_0||w_{yy}||_0 \le 2(||w_{xx}||_0^2 + ||w_{yy}||_0^2), \tag{4.43}$$

so that

$$||w_{xx}||_0 + ||w_{yy}||_0 \le \sqrt{2}A_2(t), \tag{4.44}$$

and (4.44) is equivalent to

$$\frac{d}{dt}[||w_{xx}||_0^2 + ||w_{yy}||_0^2] \le 2\epsilon C_6(||w_{xx}||_0 + ||w_{yy}||_0) + C_5(||w_{xx}||_0^2 + ||w_{yy}||_0^2), \tag{4.45}$$

from which we have

$$\frac{d}{dt}A_2^2(t) \le \epsilon C_6\sqrt{2}A_2(t) + C_5A_2^2(t) \tag{4.46}$$

or

$$\frac{d}{dt}A_2(t) \le 2\epsilon C_6\sqrt{2} + \frac{1}{2}C_5A_2(t). \tag{4.47}$$

Apply Gronwall's inequality to obtain

$$A_2(t) \le 2\sqrt{2}\epsilon\left(\frac{C_6}{C_5}\right)(e^{(1/2)C_5t} - 1) \le 2\sqrt{2}\epsilon t C_6 e^{(1/2)C_5} = \epsilon t M_2. \tag{4.48}$$

From (4.44), we infer that

$$\begin{aligned} ||w_{xx}||_0 &\le \epsilon t M_2, \\ ||w_{yy}||_0 &\le \epsilon t M_2. \end{aligned} \tag{4.49}$$

For the general case j, the procedure for obtaining a bound on $\|\partial_x^j w\|_0 + \|\partial_y^j w\|_0$ is similar to that followed above in the cases $j = 0$, $j = 1$, and $j = 2$. Suppose inductively that for $j < k$, where $k > 1$, there have been established bounds of the form

$$\begin{aligned} &\|\partial_x^j w\|_0 + \|\partial_y^j w\|_0 \\ &\quad \le \epsilon \left(\frac{C_{2j}}{C_{2j-1}} \right) \left(e^{C_{2j-1} t} - 1 \right) \le \epsilon t C_{2j} e^{C_{2j-1}} = \epsilon t M_j \end{aligned} \tag{4.50}$$

for $t \in [0,1]$, where C_{2j-1} and C_{2j} are both order-one quantities. The goal now is to establish the proof for $j = k$. Before we proceed with the proof, we add (4.12) and (4.13) together to get the relation

$$\begin{aligned} &\int_{\mathbb{R}^2} \left[\left(\partial_x^k w\right)^2 + \epsilon \left(\partial_x^{(k+1)} w\right)^2 + \left(\partial_y^k w\right)^2 + \epsilon \left(\partial_y^k w_x\right)^2 \right] dx\, dy \\ &\quad = \int_0^t \int_{\mathbb{R}^2} \partial_x^k w \left[2\epsilon \partial_x^{(k+2)} u_\tau - 2\partial_x^{(k+1)} \left(wu + \frac{1}{2} w^2 \right) \right] \\ &\qquad + \int_0^t \int_{\mathbb{R}^2} \partial_y^k w \left[2\epsilon \partial_y^k u_{xx\tau} - 2\partial_y^k \left(wu + \frac{1}{2} w^2 \right)_x \right] dx\, dy\, d\tau. \end{aligned} \tag{4.51}$$

Using Leibniz rule, (4.51) may be written as

$$\begin{aligned} &\int_{\mathbb{R}^2} \left[\left(\partial_x^k w\right)^2 + \epsilon \left(\partial_x^{(k+1)} w\right)^2 + \left(\partial_y^k w\right)^2 + \epsilon \left(\partial_y^k w_x\right)^2 \right] dx\, dy \\ &\quad = 2\epsilon \int_0^t \int_{\mathbb{R}^2} \partial_x^k w \partial_x^{(k+2)} u_\tau dx\, dy\, d\tau \\ &\qquad - 2 \int_0^t \int_{\mathbb{R}^2} \sum_{j=0}^{k+1} \alpha_j \left\{ \frac{1}{2} \partial_x^{(k+1-j)} w \partial_x^j w + \partial_x^{(k+1-j)} w \partial_x^j u \right\} \left(\partial_x^k w\right) dx\, dy\, d\tau \\ &\qquad + 2\epsilon \int_0^t \int_{\mathbb{R}^2} \partial_y^k w \partial_y^k u_{xx\tau} dx\, dy\, d\tau \\ &\qquad - 2 \int_0^t \int_{\mathbb{R}^2} \sum_{j=0}^{k} \beta_j \left\{ \partial_y^{(k-j)} w \partial_y^j w_x + \partial_y^{(k-j)} w_x \partial_y^j u + \partial_y^{(k-j)} w \partial_y^j u_x \right\} \left(\partial_y^k w\right) dx\, dy\, d\tau. \end{aligned} \tag{4.52}$$

Here, the α_j and β_j are the constants that appear in Leibniz rule. Separating the top-order derivatives and directly the rest estimating, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left[\left(\partial_x^k w\right)^2+\epsilon\left(\partial_x^{(k+1)} w\right)^2+\left(\partial_y^k w\right)^2+\epsilon\left(\partial_y^k w_x\right)^2\right] dx\,dy \\
&\le 2\epsilon\int_0^t \|\partial_x^k w\|_0\|\partial_x^{(k+2)} u_\tau\|_0 d\tau-2\int_0^t\int_{\mathbb{R}^2}\left(w\partial_x^k w\partial_x^{(k+1)} w+u\partial_x^k w\partial_x^{(k+1)} w\right) dx\,dy\,d\tau \\
&\quad+\int_0^t\int_{\mathbb{R}^2}\sum_{j=1}^{k+1}\alpha_j\left|\partial_x^k w\partial_x^{(k+1-j)} w\partial_x^j w\right| dx\,dy\,d\tau+2\int_0^t\int_{\mathbb{R}^2}\sum_{j=1}^{k+1}\alpha_j\left|\partial_x^k w\partial_x^j u\partial_x^{(k+1-j)} w\right| dx\,dy\,d\tau \\
&\quad+2\epsilon\int_0^t\|\partial_y^k w\|_0\|\partial_y^k u_{xx\tau}\|_0 dx\,dy\,d\tau \\
&\quad-2\int_0^t\int_{\mathbb{R}^2}\left(u\partial_y^k w\partial_y^k w_x+u_x\partial_y^k w\partial_y^k w+w_x\partial_y^k w\partial_y^k w\right) dx\,dy\,d\tau \\
&\quad+2\int_0^t\int_{\mathbb{R}^2}\sum_{j=1}^{k}\beta_j\left|\partial_y^k w\partial_y^{(k-j)} w_x\partial_y^j u\right| dx\,dy\,d\tau+2\int_0^t\int_{\mathbb{R}^2}\sum_{j=1}^{k}\beta_j\left|\partial_y^k w\partial_y^j u_x\partial_y^{(k-j)} w\right| dx\,dy\,d\tau \\
&\quad+2\int_0^t\int_{\mathbb{R}^2}\sum_{j=1}^{k}\beta_j\left|\partial_y^k w\partial_y^{k-j} w\partial_y^j w_x\right| dx\,dy\,d\tau.
\end{aligned}\tag{4.53}$$

The induction hypothesis (4.50) assures us that on the time interval $[0,t]$, $\|\partial_x^j w\|_0$, $|\partial_x^i w|_\infty$, $\|\partial_y^j w\|_0$, $\|\partial_y^j w_x\|_0$, $\|\partial_y^j w_x\|_\infty$, and $|\partial_y^i w|_\infty$ are bounded by order-one constants if $0\le j\le k-1$, and $0\le i\le k-2$. Using the elementary inequalities, the following estimates may be obtained:

$$\begin{aligned}
\left|\partial_x^j u\right|_\infty &\le 2\|\partial_x^j u\|_0^{1/4}\|\partial_x^j u_y\|_0^{1/2}\|\partial_x^{j+2} u\|_0^{1/4}\le C, \\
\left|\partial_y^j u\right|_\infty &\le 2\|\partial_y^j u\|_0^{1/4}\|\partial_y^{j+1} u\|_0^{1/2}\|\partial_y^j u_{xx}\|_0^{1/4}\le C,
\end{aligned}\tag{4.54}$$

and from the differential equation satisfied by u, we get the following estimates:

$$\begin{aligned}
\sup_{\tau\ge 0}\|\partial_x^{k+2} u_\tau\|_0 \le \sup_{\tau\ge 0}\Bigg\{&\|\partial_x^{k+3} u\|_0+3\sum_{j=0}^{k}\alpha_j\|\partial_x^{k+1-j} u\|_0\|\partial_x^{j+2} u\|_0 \\
&+\sum_{j=0}^{k}\beta_j\|\partial_x^{k-j} u\|_0\|\partial_x^{j+3} u\|_0+\|\partial_x^{k+5} u\|_0+\|\partial_x^{k+1} u_{yy}\|_0\Bigg\}\le C.
\end{aligned}\tag{4.55}$$

Similarly,

$$\sup_{\tau\geq 0}\|\partial_y^k u_{xx\tau}\|_0 \leq \sup_{\tau\geq 0}\Bigg\{\|\partial_y^k u_{xxx}\|_0 + 3\sum_{j=0}^{k}\gamma_j\|\partial_y^{k-j}u_x\|_0\|\partial_y^j u_{xx}\|_0 + \sum_{j=0}^{k}\delta_j\|\partial_y^{k-j}u\|_0\|\partial_y^j u_{xxx}\|_0 + \|\partial_y^k u_{xxxxx}\|_0 + \|\partial_y^{k+2}u_x\|_0\Bigg\} \leq C \tag{4.56}$$

for $0 \leq j \leq k+1$, independent of $\tau \geq 0$, and the α_j, β_j, γ_j, and δ_j are the constants that appear in Leibniz rule. Because of Theorems 3.1 and 3.2, $|w_x|_\infty$, $|w_y|_\infty$, and $|w|_\infty$ are similarly bounded. The second and the sixth integral on the right-hand side of (4.53) are bounded, respectively, as follows:

$$\begin{aligned} -\int_0^t\int_{\mathbb{R}^2}(w_x+u_x)(\partial_x^k w)^2\,dx\,dy\,d\tau &\leq C\int_0^t\|\partial_x^k w\|_0^2\,d\tau,\\ -2\int_0^t\int_{\mathbb{R}^2}\left[\left(\frac{1}{2}u_x+w_x\right)(\partial_y^k w)^2\right]dx\,dy\,d\tau &\leq C\int_0^t\|\partial_y^k w\|_0^2\,d\tau, \end{aligned} \tag{4.57}$$

where, provided $0 \leq t \leq 1$, C may be inferred to be an order-one quantity. The other integrals on the right side of (4.53) may be estimated as follows:

$$\begin{aligned} &\int_0^t\int_{\mathbb{R}^2}\sum_{j=1}^{k+1}\alpha_j\left(|\partial_x^k w\partial_x^{(k+1-j)}w\partial_x^j w| + 2|\partial_x^k w\partial_x^j u\partial_x^{(k+1-j)}w|\right)dx\,dy\,d\tau\\ &\quad+2\int_0^t\int_{\mathbb{R}^2}\sum_{j=1}^{k}\beta_j\left(|\partial_y^k w\partial_y^{(k-j)}w_x\partial_y^j u| + |\partial_y^k w\partial_y^j u_x\partial_y^{(k-j)}w|\right)dx\,dy\,d\tau\\ &\quad+2\int_0^t\int_{\mathbb{R}^2}\sum_{j=1}^{k}\beta_j|\partial_y^k w\partial_y^{k-j}w\partial_y^j w_x|\,dx\,dy\,d\tau\\ &\leq C\int_0^t|w_x|_\infty\|\partial_x^k w\|_0^2\,d\tau + C\epsilon\int_0^t\left[\|\partial_x^k w\|_0\sum_{j=2}^{k+1}M_{k+1-j}M_j\tau^2\right]d\tau\\ &\quad+C\int_0^t|u_x|_\infty\|\partial_x^k w\|_0^2\,d\tau + C\epsilon\int_0^t\left[\|\partial_x^k w\|_0\sum_{j=2}^{k+1}N_{k+1-j}N_j\tau^2\right]d\tau\\ &\quad+C\epsilon\int_0^t\|\partial_y^k w\|_0\left[|u_x|_\infty\|\partial_y^{k-1}w\|_0\,d\tau + \sum_{j=2}^{k}P_{k-j}P_j\tau^2\right]d\tau\\ &\quad+C\epsilon\int_0^t\|\partial_y^k w\|_0\left[\|\partial_y^{k-1}w\|_0|u_{xy}|_\infty + \sum_{j=2}^{k}Q_{k-j}Q_j\tau^2\right]d\tau\\ &\quad+C\epsilon\int_0^t\|\partial_y^k w\|_0\left[\|\partial_y^{k-1}w\|_0|w_{xy}|_\infty + \sum_{j=2}^{k}R_{k-j}R_j\tau^2\right]d\tau, \end{aligned} \tag{4.58}$$

valid at least for $0 \le t \le 1$. The constants appearing in these inequalities are order one. Hence, for $0 \le t \le 1$,

$$||\partial_x^k w||_0^2 + ||\partial_y^k w||_0^2 \le \int_0^t \left\{ \epsilon C_{2k} [||\partial_x^k w||_0 + ||\partial_y^k w||_0] + C_{2k-1} [||\partial_x^k w||_0^2 + ||\partial_y^k w||_0^2] \right\} d\tau, \tag{4.59}$$

where C_{2k-1} and C_2k are order-one quantities. As previously seen, if

$$A_k(t) = [||\partial_x^k w||_0^2 + ||\partial_y^k w||_0^2]^{1/2}, \tag{4.60}$$

then an application of Gronwall's inequality will lead to

$$A_k(t) \le \sqrt{2}\epsilon \left(\frac{C_{2k}}{C_{2k-1}} \right) (e^{C_{2k-1}t} - 1) \le \sqrt{2}\epsilon t C_{2k} e^{C_{2k-1}} = \epsilon t M_k. \tag{4.61}$$

We therefore infer that

$$\begin{aligned} ||\partial_x^k w||_0 &\le \epsilon t M_k, \\ ||\partial_y^k w||_0 &\le \epsilon t M_k. \end{aligned} \tag{4.62}$$

The result (4.50) for $j = k$ now follows and the inductive step is completed.

It is worth noting that the constants C_{2k-1} and C_{2k} depend only on $\|g_{(j)}\|_0$ for $0 \le j \le k+5$. This concludes the proof of Theorem 4.2. □

Proof of Theorem 4.1. From (4.5), we have

$$\begin{aligned} u(x,y,t) &= \epsilon^{-1} \eta(\epsilon^{-1/2}x + \epsilon^{-3/2}t, \epsilon^{-1}y, \epsilon^{-3/2}t), \\ v(x,y,t) &= \epsilon^{-1} \xi(\epsilon^{-1/2}x + \epsilon^{-3/2}t, \epsilon^{-1}y, \epsilon^{-3/2}t). \end{aligned} \tag{4.63}$$

Inverting the above relations, we get

$$\begin{aligned} \eta(x,y,t) &= \epsilon u(\epsilon^{1/2}x - \epsilon^{3/2}t, \epsilon y, \epsilon^{3/2}t), \\ \xi(x,y,t) &= \epsilon v(\epsilon^{1/2}x - \epsilon^{3/2}t, \epsilon y, \epsilon^{3/2}t). \end{aligned} \tag{4.64}$$

It follows that

$$\begin{aligned} \partial_x^j \eta(x,y,t) &= \epsilon^{j/2+1} \partial_x^j u(\epsilon^{1/2}x - \epsilon^{3/2}t, \epsilon y, \epsilon^{3/2}t), \\ \partial_x^j \xi(x,y,t) &= \epsilon^{j/2+1} \partial_x^j v(\epsilon^{1/2}x - \epsilon^{3/2}t, \epsilon y, \epsilon^{3/2}t). \end{aligned} \tag{4.65}$$

Thus,

$$\begin{aligned} ||\partial_x^j(\eta(\cdot,\cdot,t) - \xi(\cdot,\cdot,t))||_0^2 = \epsilon^{j+2} \int_{\mathbb{R}^2} \Big[&\partial_x^j u(\epsilon^{1/2}x - \epsilon^{3/2}t, \epsilon y, \epsilon^{3/2}t) \\ &- \partial_x^j v(\epsilon^{1/2}x - \epsilon^{3/2}t, \epsilon y, \epsilon^{3/2}t) \Big]^2 dx\,dy. \end{aligned} \tag{4.66}$$

Now, change variables by letting $z_1 = \epsilon^2 x - \epsilon^{3/2} t$, $z_2 = \epsilon y$, and $dx\,dy = |J| dz_1 dz_2 = \epsilon^{-3/2} dz_1 dz_2$. The relation above then becomes

$$\begin{aligned}
\|\partial_x^j(\eta(\cdot,\cdot,t) - \xi(\cdot,\cdot,t))\|_0^2 &= \epsilon^{j+1/2} \int_{\mathbb{R}^2} \Big[\partial_x^j u(\epsilon^{1/2}x - \epsilon^{3/2}t, \epsilon y, \epsilon^{3/2}t) \\
&\qquad - \partial_x^j v(\epsilon^{1/2}x - \epsilon^{3/2}t, \epsilon y, \epsilon^{3/2}t)\Big]^2 dz_1 dz_2 \\
&= \epsilon^{j+1/2} \|\partial_x^j w(\cdot,\cdot,\epsilon^{3/2}t)\|_0^2 \\
&\le \epsilon^{j+1/2} (\epsilon \cdot M_j(\epsilon^{3/2}t))^2 = \epsilon^{j+5/2} M_j(\epsilon^{3/2}t)^2
\end{aligned} \tag{4.67}$$

for $0 \le \epsilon^{3/2} t \le 1$. Hence if $0 \le t \le \epsilon^{-3/2}$, then

$$\|\partial_x^j \eta(\cdot,\cdot,t) - \partial_x^j \xi(\cdot,\cdot,t)\|_0 \le \epsilon^{j/2+5/4} M_j(\epsilon^{3/2}t). \tag{4.68}$$

Similarly,

$$\begin{aligned}
\partial_y^j \eta(x,y,t) &= \epsilon^{j+1} \partial_y^j u(\epsilon^{1/2}x - \epsilon^{3/2}t, \epsilon y, \epsilon^{3/2}t), \\
\partial_y^j \xi(x,y,t) &= \epsilon^{j+1} \partial_y^j v(\epsilon^{1/2}x - \epsilon^{3/2}t, \epsilon y, \epsilon^{3/2}t),
\end{aligned} \tag{4.69}$$

so that

$$\begin{aligned}
\|\partial_y^j \eta(\cdot,\cdot,t) - \partial_y^j \xi(\cdot,\cdot,t)\|_0^2 &= \epsilon^{2j+2} \int_{\mathbb{R}^2} \Big[\partial_y^j u(\epsilon^{1/2}x - \epsilon^{3/2}t, \epsilon y, \epsilon^{3/2}t) \\
&\qquad - \partial_y^j v(\epsilon^{1/2}x - \epsilon^{3/2}t, \epsilon y, \epsilon^{3/2}t)\Big]^2 dx\,dy \\
&= \epsilon^{2j+1/2} \int_{\mathbb{R}^2} [\partial_y^j u(z_1, z_2, \epsilon^{3/2}t) - \partial_y^j v(z_1, z_2, \epsilon^{3/2}t)]^2 dz_1 dz_2 \\
&= \epsilon^{2j+1/2} \|\partial_y^j w(\cdot,\cdot,\epsilon^{3/2}t)\|_0^2 \\
&\le \epsilon^{2j+1/2} (\epsilon \cdot M_j(\epsilon^{3/2}t))^2 = \epsilon^{2j+5/2} M_j(\epsilon^{3/2}t)^2
\end{aligned} \tag{4.70}$$

for $0 \le \epsilon^{3/2} t \le 1$. Hence if $0 \le t \le \epsilon^{-3/2}$, then

$$\|\partial_y^j \eta(\cdot,\cdot,t) - \partial_y^j \xi(\cdot,\cdot,t)\|_0 \le \epsilon^{j+5/4} M_j(\epsilon^{3/2}t), \tag{4.71}$$

where M_j is of order one. □

Corollary 4.3. *Assume the hypothesis in Theorem 4.1. Then, there are constants K_j such that*

$$|\partial_x^j \eta(x,y,t) - \partial_x^j \xi(x,y,t)|_\infty \le \epsilon^{2+j/2} K_j(\epsilon^{3/2}t) \tag{4.72}$$

for $0 \le j \le k-2$ and $0 \le t \le \epsilon^{-3/2} \min\{T, T_0\}$, and $j \le k/2$.

Proof. Note that

$$|\partial_x^j \eta(x,y,t) - \partial_x^j(x,y,t)|_\infty \le 2\|\partial_x^j(\eta-\xi)\|_0^{1/4}\|\partial_x^j(\eta_y-\xi_y)\|_0^{1/2}\|\partial_x^{j+2}(\eta-\xi)\|_0^{1/4}. \tag{4.73}$$

Now observe that

$$\|\partial_x^j\partial_y(\eta-\xi)\|_0 \le \|\partial_x^{2j}(\eta-\xi)\|_0^{1/2}\|\partial_y^2(\eta-\xi)\|_0^{1/2}, \tag{4.74}$$

so that

$$\begin{aligned}
|\partial_x^j(\eta-\xi)|_\infty &\le 2\|\partial_x^j(\eta-\xi)\|_0^{1/4}\|\partial_x^{2j}(\eta-\xi)\|_0^{1/4}\|\|_0^{1/4}\|\partial_y^2(\eta-\xi)\|_0^{1/4}\|\partial_x^{j+2}(\eta-\xi)\|_0^{1/4} \\
&\le \left[\epsilon^{j/2+5/4}m_j\right]^{1/4}\left[\epsilon^{2j/2+5/4}m_{2j}\right]^{1/4}\left[\epsilon^{13/4}m_1\right]^{1/4}\left[\epsilon^{(j+2)/2+5/4}m_j\right]^{1/4} \\
&= \epsilon^{j/8+5/16+2j/8+5/16+13/16+(j+2)/8+5/16}\widetilde{m} \\
&= \epsilon^{j/8+5/16+2j/8+5/16+13/16+(j+2)/8+5/16}\widetilde{m} = \epsilon^{2+j/2}\widetilde{m}(\epsilon^{3/2}t).
\end{aligned} \tag{4.75}$$

From which, with $j=0$, we get the following interesting case:

$$|\eta(x,y,t)-\xi(x,y,t)|_\infty \le K_0\epsilon^{7/2}t \tag{4.76}$$

holding for $0 \le t \le \epsilon^{-3/2}$. Inequality (4.76) is exactly what was obtained when comparing solutions of the two linearized model evolution equations. Note in particular that at $t = t_1 = \epsilon^{-3/2}$

$$|\eta(x,y,t_1)-\xi(x,y,t_1)|_\infty \le K_0\epsilon^2, \tag{4.77}$$

showing explicitly that the two solutions are the same in the formal order of accuracy ϵ^2 achieved by either model. Similarly, we state without proof the following corollary. □

Corollary 4.4. *Assume the hypothesis in Theorem 4.1. Then, there are constants K_j such that*

$$|\partial_y^j\eta(x,y,t) - \partial_y^j\xi(x,y,t)|_\infty \le \epsilon^{2+j}K_j(\epsilon^{3/2}t) \tag{4.78}$$

for $0 \le j \le k-2$ and $0 \le t \le \epsilon^{-3/2}\min\{T,T_0\}$, and $j \le k/2-2$.

5. Remark

The same result is obtained by analyzing the solutions of the initial-value problems for the KP II and the BBM KP II equations. Indeed, that analysis was carried out by Mammeri in [11]. This was made known to us after the submission of this paper.

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