Research Article

Contra- ω -Continuous and Almost Contra- ω -Continuous

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Received 29 May 2007; Accepted 31 July 2007

Recommended by Sehie Park

The notion of contra continuous functions was introduced and investigated by Dontchev. In this paper, we apply the notion of ω -open sets in topological space to present and study a new class of functions called almost contra ω -continuous functions as a new generalization of contra continuity.

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1. Introduction

Dontchev [1] introduced the notions of contra continuity and strong S-closedness in topological spaces. He defined a function $f : X \rightarrow Y$ is contra continuous if the preimage of every open set of Y is closed in X . A new weaker form of this class of functions called contra semicontinuous function is introduced and investigated by Dontchev and Noiri [2]. Caldas and Jafari [3] have introduced and studied contra β -continuous function. Jafari and Noiri [4, 5] introduced and investigated the notions of contra super continuous, contra precontinuous, and contra α -continuous functions. Almost contra precontinuous functions were introduced by Ekici [6] and recently have been investigated further by Noiri and Popa [7]. Nasef [8] has introduced and studied contra γ -continuous function. In This direction, we will introduce the concept of almost contra ω -continuous functions via the notion of ω -open set and study some properties of contra ω -continuous and almost contra ω -continuous.

All through this paper, (X, τ) and (Y, σ) stand for topological spaces with no separation axioms assumed, unless otherwise stated. Let $A \subseteq X$, the closure of A and the interior of A will be denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. A is regular open if $A = \text{Int}(\text{Cl}(A))$ and A is regular closed if its complement is regular open; equivalently

A is regular closed if $A = \text{Cl}(\text{Int}(A))$, see [9]. Let (X, τ) be a space and let A be a subset of X . A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is called ω -closed [10] if it contains all its condensation points. The complement of an ω -closed set is called ω -open. It is well known that a subset W of a space (X, τ) is ω -open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω -open subsets of a space (X, τ) , denoted by τ_ω , forms a topology on X finer than τ . We set $\omega O(X, x) = \{U : x \in U \text{ and } U \in \tau_\omega\}$. The ω -closure and ω -interior, that can be defined in a manner to $\text{Cl}(A)$ and $\text{Int}(A)$, respectively, will be denoted by $\text{Cl}_\omega(A)$ and $\text{Int}_\omega(A)$, respectively. Several characterizations and properties of ω -closed subsets were provided in [10–12].

2. Contra ω -continuous

Definition 2.1. A function $f : X \rightarrow Y$ is called ω -continuous [12] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq V$.

Definition 2.2. A function $f : X \rightarrow Y$ is called contra- ω -continuous (resp., contra-continuous [1]) if $f^{-1}(V)$ is ω -closed (resp., closed) in X for each open set of Y .

Definition 2.3. A function $f : X \rightarrow Y$ is said to be almost continuous [13] if $f^{-1}(V)$ is open in X for each regular open set V of Y .

LEMMA 2.4 [4]. *The following properties hold for subsets A, B of a space X :*

- (1) $x \in \text{Ker}(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in C(X, x)$;
- (2) $A \subseteq \text{Ker}(A)$ and $A = \text{Ker}(A)$ if A is open in X ;
- (3) if $A \subseteq B$, then $\text{Ker}(A) \subseteq \text{Ker}(B)$.

THEOREM 2.5. *The following are equivalent for a function $f : X \rightarrow Y$:*

- (1) f is contra- ω -continuous;
- (2) for every closed subset F of Y , $f^{-1}(F) \in \omega O(X)$;
- (3) for each $x \in X$ and each $F \in C(Y, f(x))$, there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq F$;
- (4) $f(\text{Cl}_\omega(A)) \subseteq \text{Ker}(f(A))$ for every subset A of X ;
- (5) $\text{Cl}_\omega(f^{-1}(B)) \subseteq f^{-1}(\text{Ker}(B))$ for every subset B of Y .

Proof. The implications (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (2) Let F be any closed set of Y and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in \omega O(X, x)$ such that $f(U_x) \subseteq F$. Therefore, we obtain $f^{-1}(F) = \cup \{U_x \mid x \in f^{-1}(F)\}$ and $f^{-1}(F)$ is ω -open, since τ_ω is a topological space.

(2) $\Rightarrow$ (4) Let A be any subset of X . Suppose that $y \notin \text{Ker}(f(A))$. Then by Lemma 2.4 there exists $F \in C(Y, f(x))$ such that $f(A) \cap F = \emptyset$. Thus, we have $A \cap f^{-1}(F) = \emptyset$ and since $f^{-1}(F)$ is ω -open then we have $\text{Cl}_\omega(A) \cap f^{-1}(F) = \emptyset$. Therefore, we obtain $f(\text{Cl}_\omega(A)) \cap F = \emptyset$ and $y \notin f(\text{Cl}_\omega(A))$. This implies that $f(\text{Cl}_\omega(A)) \subseteq \text{Ker}(f(A))$.

(4) $\Rightarrow$ (5) Let B be any subset of Y . By (4) and Lemma 2.4, we have $f(\text{Cl}_\omega(f^{-1}(B))) \subseteq \text{Ker}(f(f^{-1}(B))) \subseteq \text{Ker}(B)$ thus $\text{Cl}_\omega(f^{-1}(B)) \subseteq f^{-1}(\text{Ker}(B))$.

(5) $\Rightarrow$ (1) Let V be any open set of Y . Then, by Lemma 2.4 we have $\text{Cl}_\omega(f^{-1}(V)) \subseteq f^{-1}(\text{Ker}(V)) = f^{-1}(V)$ and $\text{Cl}_\omega(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is ω -closed in X . $\square$

The following examples show that contra- ω -continuous and contra-precontinuous functions [4] (resp., contra-semicontinuous [2], contra- α -continuous [5], contra- γ -continuous [8]) are independent notions.

Example 2.6. Let $X = \{a, b\}$ with $\tau = \{X, \phi, \{a\}\}$ and the real number $\mathbb{R}$ with the standard topology, consider the map $f : \mathbb{R} \rightarrow X$ defined by $f(x) = b$ if $x \in \mathbb{Q}$ where $\mathbb{Q}$ is the set of all rational numbers and $f(x) = a$ if $x \notin \mathbb{Q}$. Then f is contra-precontinuous but not f contra- ω -continuous since $\{b\}$ is a closed set of (X, τ) and $f^{-1}(\{b\}) = \mathbb{Q}$ is not ω -open. but $\mathbb{Q}$ is preopen set in $\mathbb{R}$.

Example 2.7. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, and $Y = \{1, 2\}$ be the Sierpinski space with the topology $\sigma = \{\phi, \{1\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be defined by $f(a) = 1$ and $f(b) = 2 = f(c)$. Then f is contra ω -continuous but not contra-precontinuous, since $\{2\}$ is a closed set of (Y, σ) and $f^{-1}(\{2\}) = \{c, b\}$ is not preopen (X, τ) .

Example 2.8. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, X\}$, and $\sigma = \{\phi, \{c\}, \{b\}, \{c, b\}, X\}$. Then the identity function $f : (X, \tau) \rightarrow (X, \sigma)$ is contra- ω -continuous but not contra-continuous.

Example 2.9. $X = \{a, b\}$ with $\tau = \{X, \phi, \{a\}\}$ and the real number $\mathbb{R}$ with the standard topology, consider the map $f : \mathbb{R} \rightarrow X$ defined by $f(x) = b$ if $x \in [0, 1)$ and $f(x) = a$ if $x \notin [0, 1)$. Then f is contra-semicontinuous but not f contra- ω -continuous since $\{b\}$ is a closed set of (X, τ) and $f^{-1}(\{b\}) = [0, 1)$ is not ω -open. but $[0, 1)$ is semi-open set in $\mathbb{R}$.

Example 2.10. Let $X = \{a, b\}$ with the indiscrete topology τ and $\sigma = \{\phi, \{a\}, X\}$. Then the identity function $f : (X, \tau) \rightarrow (X, \sigma)$ is contra ω -continuous but not contra semicontinuous, since $A = \{a\} \in \sigma$ but A is not semiclosed in (X, τ) .

Example 2.11. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, b, c\}, \{b, c, d\}, X\}$. Define a function $f : (X, \tau) \rightarrow (X, \tau)$ as follows: $f(a) = b, f(b) = a, f(c) = d$, and $f(d) = c$. Then f is contra ω -continuous but not contra α -continuous, since $\{c, d\}$ is a closed set of (X, τ) and $f^{-1}(\{c, d\}) = \{c, d\}$ is not α -open.

$$\begin{array}{ccccc}
 \text{contra-}\omega\text{-continuity} & & & & \\
 \uparrow & & & & \\
 \text{contra-continuity} & \Rightarrow & \text{contra-}\alpha\text{-continuity} & \Rightarrow & \text{contra-percontinuity} \\
 & & \downarrow & & \downarrow \\
 & & \text{contra-semicontinuity} & \Rightarrow & \text{contra-}\gamma\text{-continuity} \\
 & & & & \downarrow \\
 & & & & \text{contra-}\beta\text{-continuity}
 \end{array} \tag{2.1}$$

THEOREM 2.12. *If a function $f : X \rightarrow Y$ is contra- ω -continuous and Y is regular, then f is ω -continuous.*

Proof. Let x be an arbitrary point of X and let V be an open set of Y containing $f(x)$; since Y is regular, there exists an open set W in Y containing $f(x)$ such that $\text{Cl}(W) \subseteq V$.

Since f is contra- ω -continuous, so by Theorem 2.5(3) there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq \text{Cl}(W)$. Then $f(U) \subseteq \text{Cl}(W) \subseteq V$. Hence, f is ω -continuous. $\square$

Definition 2.13. A space (X, τ) is said to be ω -space (resp., locally ω -indiscrete) if every ω -open set is open (resp., closed) in X .

For any space (X, τ) , we have $\tau \subseteq \tau_\omega$. So the following results follows immediately.

THEOREM 2.14. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is contra- ω -continuous if and only if $f : (X, \tau_\omega) \rightarrow (Y, \sigma)$ is contra-continuous.

THEOREM 2.15. If a function $f : X \rightarrow Y$ is contra- ω -continuous and X is ω -space, then f is contra-continuous.

THEOREM 2.16. Let X be locally ω -indiscrete. If a function $f : X \rightarrow Y$ is contra- ω -continuous, then f is continuous.

Definition 2.17. A function $f : X \rightarrow Y$ is called almost- ω -continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq \text{Int}_\omega(\text{Cl}(V))$.

Definition 2.18. A function $f : X \rightarrow Y$ is said to be pre- ω -open if the image of each ω -open set is ω -open.

THEOREM 2.19. If a function $f : X \rightarrow Y$ is a pre- ω -open contra- ω -continuous function, then f is almost ω -continuous.

Proof. Let x be any arbitrary point of X and V be an open set containing $f(x)$. Since f is contra- ω -continuous, then by Theorem 2.5(3) there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq \text{Cl}(V)$. Since f is pre- ω -open, $f(U)$ is ω -open in Y . Therefore, $f(U) = \text{Int}_\omega f(U) \subseteq \text{Int}_\omega(\text{Cl}(f(U))) \subseteq \text{Int}_\omega(\text{Cl}(V))$. This shows that f is almost ω -continuous. $\square$

Definition 2.20. A function $f : X \rightarrow Y$ is said to be almost weakly ω -continuous if for each $x \in X$ and each open V of $f(x)$ there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq \text{Cl}(V)$.

THEOREM 2.21. If a function $f : X \rightarrow Y$ is contra- ω -continuous, then f is almost weakly ω -continuous.

Proof. Let V be any open set of Y . Since $\text{Cl}(V)$ is closed in Y , by Theorem 2.5(3) $f^{-1}(\text{Cl}(V))$ is ω -open in X and set $U = f^{-1}(\text{Cl}(V))$, then we have $f(U) \subseteq \text{Cl}(V)$. This shows that f is almost weakly ω -continuous.

Since the family of all ω -open subsets of a space (X, τ) , denoted by τ_ω , forms a topology on X finer than τ , then the ω -frontier of A , where $A \subseteq X$, is defined by $Fr_\omega(A) = \text{Cl}_\omega(A) \cap \text{Cl}_\omega(X - A)$. $\square$

THEOREM 2.22. The set of all points of x of X at which $f : X \rightarrow Y$ is not contra- ω -continuous is identical with the union of the ω -frontier of the inverse images of closed sets of Y containing $f(x)$.

Proof. Suppose f is not contra- ω -continuous at $x \in X$. There exists $F \in C(Y, f(x))$ such that $f(U) \cap (Y - F) \neq \emptyset$ for every $U \in \omega O(X, x)$ by Theorem 2.5. This implies that $U \cap f^{-1}(Y - F) \neq \emptyset$. Therefore, we have $x \in \text{Cl}_\omega(f^{-1}(Y - F)) = \text{Cl}_\omega(X - f^{-1}(F))$. However,

since $x \in f^{-1}(F) \subseteq \text{Cl}_\omega(f^{-1}(F))$, thus $x \in \text{Cl}_\omega(f^{-1}(F)) \cap \text{Cl}_\omega(f^{-1}(Y - F))$. Therefore, we obtain $x \in \text{Fr}_\omega(f^{-1}(F))$. Suppose that $x \in \text{Fr}_\omega f(f^{-1}(F))$ for some $F \in C(Y, f(x))$, and f is contra- ω -continuous at x , then there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq F$. Therefore, we have $x \in U \subseteq f^{-1}(F)$ and hence $x \in \text{Int}_\omega(f^{-1}(F)) \subseteq X - \text{Fr}_\omega(f^{-1}(F))$. This is a contradiction. This mean that f is not contra- ω -continuous. $\square$

THEOREM 2.23. *Let $f : X \rightarrow Y$ be a function and let $g : X \rightarrow X \times Y$ be the graph function of f defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is contra ω -continuous, then f is contra ω -continuous.*

Proof. Let U be an open set in Y , then $X \times U$ is an open set in $X \times Y$. Since g is contra ω -continuous. It follows that $f^{-1}(U) = g^{-1}(X \times U)$ is an ω -closed in X . Thus, f is contra ω -continuous. $\square$

THEOREM 2.24. *If $f : X \rightarrow Y$ and $g : X \rightarrow Y$ are contra ω -continuous and Y is Urysohn, then $E = \{x \in X : f(x) = g(x)\}$ is ω -closed in X .*

Proof. Let $x \in X - E$. Then $f(x) \neq g(x)$. Since Y is Urysohn, there exist open sets V and W such that $f(x) \in V, g(x) \in W$, and $\text{Cl}(V) \cap \text{Cl}(W) = \emptyset$. Since f and g is contra ω -continuous, then $f^{-1}(\text{Cl}(V))$ and $g^{-1}(\text{Cl}(W))$ are ω -open sets in X . Let $U = f^{-1}(\text{Cl}(V))$ and $G = g^{-1}(\text{Cl}(W))$. Then U and V are ω -open sets containing x . Set $A = U \cap G$, thus A is ω -open in X . Hence, $f(A) \cap g(A) = f(U \cap G) \cap g(U \cap G) \subseteq f(U) \cap g(G) = \text{Cl}(V) \cap \text{Cl}(W) = \emptyset$; therefore, $A \cap E = \emptyset$ and $x \notin \text{Cl}_\omega(E)$. Hence, E is ω -closed in X . $\square$

A subset A of a topological space X is said to be ω -dense in X if $\text{Cl}_\omega(A) = X$.

THEOREM 2.25. *Let $f : X \rightarrow Y$ and $g : X \rightarrow Y$ be functions. If Y is Urysohn, f and g are contra ω -continuous and $f = g$ on ω -dense set $A \subseteq X$, then $f = g$ on X .*

Proof. Since f and g are contra ω -continuous and Y is Urysohn, by the previous theorem, $E = \{x \in X : f(x) = g(x)\}$ is ω -closed in X . By assumption, we have $f = g$ on ω -dense set $A \subseteq X$. Since $A \subseteq E$ and A is ω -dense set in X , then $X = \text{Cl}_\omega(A) \subseteq \text{Cl}_\omega(E) = E$. Hence, $f = g$ on X . $\square$

Definition 2.26. A space X is called ω -connected provided that X is not the union of two disjoint nonempty ω -open sets.

THEOREM 2.27. *If $f : X \rightarrow Y$ is a contra ω -continuous function from an ω -connected space X onto any space Y , then Y is not a discrete space.*

Proof. Suppose that Y is discrete. Let A be a proper nonempty open and closed subset of Y . Then $f^{-1}(A)$ is a proper nonempty ω -clopen subset of X , which is a contradiction to the fact that X is ω -connected. $\square$

THEOREM 2.28. *If $f : X \rightarrow Y$ is contra- ω -continuous surjection and X is ω -connected, then Y is connected.*

Proof. Suppose that Y is not connected space. Then there exist nonempty disjoint open sets V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen in Y . Since f is contra- ω -continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are ω -open in X . Moreover, $f^{-1}(V_1)$ and

$f^{-1}(V_2)$ are nonempty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that X is not ω -connected. This is a contradiction. This means that Y is connected. $\square$

THEOREM 2.29. *A space X is ω -connected, if every contra- ω -continuous from a space X into any T_0 -space Y is constant.*

Proof. Suppose that X is not ω -connected and every contra- ω -continuous function from X into Y is constant. Since X is not ω -connected, there exists a proper nonempty ω -clopen subset A of X . Let $Y = \{a, b\}$ and $\tau = \{Y, \phi, \{a\}, \{b\}\}$ be a topology for Y . Let $f : X \rightarrow Y$ be a function such that $f(A) = \{a\}$ and $f(X - A) = \{b\}$. Then f is nonconstant and contra- ω -continuous such that Y is T_0 which is a contradiction. Hence, X must be ω -connected. $\square$

Definition 2.30. A space X is said to be ω - T_2 if for each pair of distinct points x and y in X , there exist $U \in \omega O(X, x)$ and $V \in \omega O(X, y)$ such that $U \cap V = \phi$.

THEOREM 2.31. *Let X and Y be topological spaces. If*

- (1) *for each pair of distinct points x and y in X there exists a function f of X into Y such that $f(x) \neq f(y)$,*
- (2) *Y is an Urysohn space,*
- (3) *f is contra- ω -continuous at x and y , then X is ω - T_2 .*

Proof. let x and y be any distinct points in X . Then, there exists a Urysohn space Y and a function $f : X \rightarrow Y$ such that $f(x) \neq f(y)$ and f is contra- ω -continuous at x and y . Let $a = f(x)$ and $b = f(y)$. Then $a \neq b$. Since Y is Urysohn space, there exist open sets V and W containing a and b , respectively, such that $\text{Cl}(V) \cap \text{Cl}(W) = \phi$. Since f is contra- ω -continuous at x and y , then there exist ω -open sets A and B containing a and b , respectively, such that $f(A) \subseteq \text{Cl}(V)$ and $f(B) \subseteq \text{Cl}(W)$. Then $f(A) \cap f(B) = \phi$, so $A \cap B = \phi$. Hence, X is ω - T_2 . $\square$

COROLLARY 2.32. *Let $f : X \rightarrow Y$ be contra- ω -continuous injection. If Y is an Urysohn space, then X is ω - T_2 .*

3. Almost contra ω -continuous

In this section, we introduce a new type of continuity called almost contra ω -continuous which is weaker than contra ω -continuous.

Definition 3.1. A function $f : X \rightarrow Y$ is said to be almost contra- ω -continuous (resp., almost contra-precontinuous [6]) $f^{-1}(V) \in \omega C(X)$ (resp., $f^{-1}(V) \in PC(X)$) for every $V \in RO(Y)$.

THEOREM 3.2. *The following are equivalents for a function $f : X \rightarrow Y$:*

- (1) *f is almost contra- ω -continuous;*
- (2) *$f^{-1}(F) \in \omega O(X, x)$ for every $F \in RC(Y)$;*
- (3) *for each $x \in X$ and each regular closed set F in Y containing $f(x)$, there exists an ω -open set U in X containing x such that $f(U) \subseteq F$;*
- (4) *for each $x \in X$ and each regular open set V in Y noncontaining $f(x)$, there exists an ω -closed set K in X noncontaining x such that $f^{-1}(V) \subseteq K$.*

Proof. (1) $\Leftrightarrow$ (2). Let F be any regular closed set of Y . Then $Y - F$ is regular open. By (1), $f^{-1}(Y - F) = X - f^{-1}(F) \in \omega C(X)$. We have $f^{-1}(F) \in \omega O(X)$. The converse is obvious.

(2) $\Rightarrow$ (3). Let F be any regular closed set in Y containing $f(x)$. Then by (2) $f^{-1}(F) \in \omega O(X)$ and $x \in f^{-1}(F)$. Take $U = f^{-1}(F)$. Then $f(U) \subseteq F$.

(3) $\Rightarrow$ (2). Let F be any regular closed set in Y and $x \in f^{-1}(F)$. From (3) there exists an ω -open U_x in X containing x such that $f(U_x) \subseteq F$, thus $U_x \subseteq f^{-1}(F)$. We have $f^{-1}(F) \subseteq \bigcup_{x \in f^{-1}(F)} U_x$. This implies that $f^{-1}(F)$ is ω -open.

(3) $\Leftrightarrow$ (4). Let V be any regular open set in Y noncontaining $f(x)$. Then $Y - V$ is a regular closed set containing $f(x)$. By (3), there exists an ω -open set U in X containing x such that $f(U) \subseteq Y - V$. Hence, $U \subseteq f^{-1}(Y - V) \subseteq X - f^{-1}(V)$ and then $f^{-1}(V) \subseteq X - U$. Take $H = X - U$. We obtain that H is an ω -closed set in X noncontaining x . The converse is obvious. $\square$

The following examples show that almost contra- ω -continuous and almost contra-precontinuous functions are independent notions.

Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then $RC(X, \tau) = \{X, \phi, \{b, c\}, \{a, c\}\}$ and $\omega O(X, \tau) = \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X , $PO(X, \tau) = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \rightarrow (X, \tau)$ be the identity map. Then f is almost contra- ω -continuous function which is not almost contra-precontinuous, since $\{a, c\}$ is a regular closed set of (X, τ) and $f^{-1}(\{a, c\}) = \{a, c\} \notin PO(X, \tau)$.

Example 3.4. Let $\mathbb{R}$ be the real number with usual topology and $X = \{a, b, c\}$ with $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$, then $RO(X) = \{\phi, X, \{a\}, \{b\}\}$. Let $f : \mathbb{R} \rightarrow X$ be defined as $f(x) = a$ if $x \in \mathbb{Q}$ and $f(x) = c$ if $x \notin \mathbb{Q}$. Then f is almost contra-precontinuous function which is not almost contra ω -continuous, since $\{a\}$ is a regular closed set in (X, τ) and $f^{-1}(\{a\}) = \mathbb{Q}$ which is not ω -open but preopen in $\mathbb{R}$.

$$\begin{array}{ccccc}
 \text{contra-}\omega\text{-continuity} & \Rightarrow & \text{almost contra-}\omega\text{-continuity} & \Rightarrow & \text{almost week-}\omega\text{-continuity} \\
 \updownarrow & & & & \updownarrow \\
 \text{contra-continuity} & \Longrightarrow & (\theta, s)\text{-continuity} & \Longrightarrow & \text{week-continuity} \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{contra-percontinuity} & \Rightarrow & \text{almost contra-precontinuity} & \Rightarrow & \text{almost week-continuity} \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{contra-}\gamma\text{-continuity} & \Rightarrow & \text{almost contra-}\gamma\text{-continuity} & \Rightarrow & \text{almost week-}\gamma\text{-continuity}
 \end{array} \tag{3.1}$$

A space (X, τ) is anti-locally countable [11] if all nonempty open subsets are uncountable. Note that $\mathbb{R}$ with usual topology is anti-locally countable space.

LEMMA 3.5 [11]. If (X, τ) is an anti-locally countable space, then $\text{Cl}_\omega(A) = \text{Cl}(A)$ for every ω -open subset of X and $\text{Int}(A) = \text{Int}_\omega(A)$ for every ω -closed subset of X .

Definition 3.6 [11]. A space (X, τ) is called locally countable, if each point $x \in X$ has a countable open neighborhood.

LEMMA 3.7 [11]. If (X, τ) is a locally countable space, then τ_ω is the discrete topology on X .

Definition 3.8. A function $f : X \rightarrow Y$ is said to be regular set-connected if $f^{-1}(V)$ is clopen in X for each regular open set V of Y .

THEOREM 3.9. Let (X, τ) be an anti-locally countable space, if a function $f : X \rightarrow Y$ is almost contra- ω -continuous and almost continuous, then f is regular set-connected.

Proof. Let V be any regular open set in Y . Since f is almost contra- ω -continuous and contra continuous $f^{-1}(V)$ is ω -closed and open. Thus $\text{Cl}_\omega(f^{-1}(V)) = (f^{-1}(V))$, since (X, τ) be an anti-locally countable space then by Lemma 3.5, we have $\text{Cl}_\omega(f^{-1}(V)) = \text{Cl}(f^{-1}(V))$. Hence $f^{-1}(V)$ is clopen. We obtain that f is regular set-connected. $\square$

Definition 3.10 [14]. A space X is said to be weakly Hausdorff if each element of X is an intersection of regular closed sets.

Definition 3.11. A space X is said to be ω - T_1 if for each pair of distinct points x and y of X , there exists ω -open sets U and V containing x and y , respectively, such that $y \notin U$ and $x \notin V$.

THEOREM 3.12. If $f : X \rightarrow Y$ is an almost contra- ω -continuous injection and Y is weakly Hausdorff, then X is ω - T_1 .

Proof. Suppose that Y is weakly Hausdorff. For any distinct points x and y in X , there exists V, W which are regular closed in Y such that $f(x) \in V, f(y) \notin V, f(x) \notin W$, and $f(y) \in W$. Since f is almost contra- ω -continuous, then $f^{-1}(V)$ and $f^{-1}(W)$ are ω -open subsets of X such that $x \in f^{-1}(V), y \notin f^{-1}(V), x \notin f^{-1}(W)$, and $y \in f^{-1}(W)$. This show that X is ω - T_1 . $\square$

COROLLARY 3.13. If $f : X \rightarrow Y$ is an contra- ω -continuous injection and Y is weakly Hausdorff, then X is ω - T_1 .

THEOREM 3.14. If $f : X \rightarrow Y$ is almost contra- ω -continuous surjection and X is ω -connected, then Y is connected.

Proof. Suppose that Y is not connected space. There exist nonempty disjoint open sets V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen sets. Thus they are regular open in Y . Since f is almost contra- ω -continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are ω -open in X . Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that X is not ω -connected. This is a contradiction. This means that Y is connected. $\square$

Definition 3.15. A space X is said to be

- (1) ω -compact if every ω -open cover of X has a finite subcover;
- (2) countably ω - compact if every countable cover of X by ω -open sets has a finite subcover;
- (3) ω -Lindelof if every ω -open cover of X has a countable subcover;
- (4) S-Lindelof [6] if every cover of X by regular closed sets has a countable subcover;

- (5) countably S-closed [15] if every countable cover of X by regular closed sets has a finite subcover;
- (6) S-closed [16] if every regular closed cover of X has a finite subcover.

THEOREM 3.16. *Let $f : X \rightarrow Y$ be an almost contra- ω -continuous surjection. The following statements hold:*

- (1) *if X is ω -compact, then Y is S-closed;*
- (2) *if X is ω -Lindelof, then Y is S-Lindelof;*
- (3) *if X is countably ω -compact, then Y is countably S-closed.*

Proof. We prove only (1). let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of Y . Since f is almost contra- ω -continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is an ω -open cover of X and hence there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$ therefore we have $Y = \cup\{V_\alpha : \alpha \in I_0\}$ and Y is S-closed. $\square$

Definition 3.17. A space X is said to be

- (1) ω -closed compact if every ω -closed cover of X has a finite subcover;
- (2) countably ω -closed compact if every countable cover of X by ω -closed sets has a finite subcover;
- (3) ω -closed-Lindelof if every cover of X by ω -closed sets has a countable subcover;
- (4) nearly compact [17] if every regular open cover of X has a finite subcover;
- (5) nearly countably compact [17] if every countable cover of X by regular open sets has a finite subcover;
- (6) nearly Lindelof [17] if every cover of X by regular open sets has a countably subcover.

THEOREM 3.18. *Let $f : X \rightarrow Y$ be an almost contra- ω -continuous surjection. The following statements hold:*

- (1) *if X is ω -closed compact, then Y is nearly compact;*
- (2) *if X is ω -closed-Lindelof, then Y nearly Lindelof;*
- (3) *if X is countably ω -closed compact, then Y is nearly countably compact.*

Proof. We prove only (1). Let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of Y . Since f is almost contra- ω -continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is an ω -closed cover of X . Since X is ω -closed compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Thus, we have $Y = \cup\{V_\alpha : \alpha \in I_0\}$ and Y is nearly compact. $\square$

Definition 3.19 [14]. A space X is said to be mildly compact (mildly countably compact, mildly Lindelof) if every clopen cover (resp., clopen countably cover, clopen cover) of X has a finite (resp., a finite, a countable) subcover.

THEOREM 3.20. *Let (X, τ) be an anti-locally countable space, if $f : X \rightarrow Y$ be an almost contra- ω -continuous and almost continuous surjection and X is mildly compact (resp., mildly countably compact, mildly Lindelof), then Y is nearly compact (resp., nearly countably compact, nearly Lindelof) and S-closed (resp., countably S-closed, S-Lindelof).*

Proof. Let V be any regular closed set on Y . Then since f is almost contra- ω -continuous and almost continuous, then $f^{-1}(V)$ is ω -open and closed in X . By Lemma 3.5, we have $\text{Int}(f^{-1}(V)) = \text{Int}_\omega(f^{-1}(V)) = f^{-1}(V)$. Hence, $f^{-1}(V)$ is clopen. Let $\{V_\alpha : \alpha \in I\}$ be

any regular closed (resp., regular open) cover of Y . Then $\{F^{-1}(V_\alpha : \alpha \in I)\}$ is a clopen cover of X and since X is mildly compact, there exists a finite subset I_0 of I such that $X = \cup \{f^{-1}(V_\alpha) : \alpha \in I_0\}$. since f is surjection, we obtain $Y = \cup \{V_\alpha : \alpha \in I_0\}$. This shows that Y is S -closed (resp., nearly compact). The other proofs are similar. $\square$

THEOREM 3.21. *If $f : X \rightarrow Y$ is contra- ω -continuous and A is ω -compact relative to X , then $f(A)$ is strongly S -closed in Y .*

Proof. Let $\{V_i : i \in I\}$ be any cover of $f(A)$, by closed sets of the subspace $f(A)$. For $i \in I$, there exists a closed set A_i of Y such that $V_i = A_i \cap f(A)$. For each $x \in A$, there exists $i(x) \in I$ such that $f(x) \in A_{i(x)}$ and by Theorem 2.5, there exists $U_x \in \omega O(X, x)$ such that $f(U_x) \subseteq A_{i(x)}$. Since the family $\{U_x : x \in A\}$ is a cover of A by ω -open sets of X , there exists a finite subset A_0 of A such that $A \subseteq \cup \{U_x : x \in A_0\}$. Therefore, we obtain $f(A) \subseteq \cup \{f(U_x) : x \in A_0\}$. which is a subset of $\cup \{A_{i(x)} : x \in A_0\}$. Thus $f(A) = \cup \{V_{i(x)} : x \in A_0\}$ and hence $f(A)$ is strongly S -closed. $\square$

COROLLARY 3.22. *If $f : X \rightarrow Y$ is contra- ω -continuous surjection and X is ω -compact, then Y is strongly S -closed.*

4. Contra-closed graphs

Recall that for a function $f : X \rightarrow Y$, the subset $\{(x, f(x)) : x \in X\} \subseteq X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 4.1. The graph $G(f)$ of a function $f : X \rightarrow Y$ is said to be contra- ω -closed if for each $(x, y) \in (X, Y) - G(f)$, there exist $U \in \omega O(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap G(f) = \phi$.

The following results can be easily verified.

LEMMA 4.2 [6]. *Let $G(f)$ be the graph of f , for any subset $A \subseteq X$ and $B \subseteq Y$, we have $f(A) \cap B = \phi$ if and only if $(A \times B) \cap G(f) = \phi$.*

LEMMA 4.3. *The graph $G(f)$ of $f : X \rightarrow Y$ is contra- ω -closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in \omega O(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \phi$.*

THEOREM 4.4. *If $f : X \rightarrow Y$ is contra- ω -continuous and Y is Urysohn, then $G(f)$ is contra- ω -closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and there exists open sets V, W such that $f(x) \in V, y \in W$, and $\text{Cl}(V) \cap \text{Cl}(W) = \phi$. Since f is contra- ω -continuous, there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq \text{Cl}(V)$. Therefore, we obtain $f(U) \cap \text{Cl}(W) = \phi$. This shows that $G(f)$ is contra- ω -closed. $\square$

THEOREM 4.5. *If $f : X \rightarrow Y$ is ω -continuous and Y is T_1 , then $G(f)$ is contra- ω -closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and there exists open set V of Y , such that $f(x) \in V, y \notin V$. Since f is ω -continuous, there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq V$. Therefore, $f(U) \cap (Y - V) = \emptyset$ and $Y - V \in C(Y, y)$. This shows that $G(f)$ is contra- ω -closed in $X \times Y$. $\square$

THEOREM 4.6. *If $f : X \rightarrow Y$ has a contra ω -closed graph, then the inverse image of a strongly S -closed set A of Y is ω -closed in X .*

Proof. Assume that A is a strongly S -closed set of Y and $x \notin f^{-1}(A)$. For each $a \in A, (x, a) \notin G(f)$. By Lemma 4.3 there exist $U_a \in \omega O(X, x)$ and $V_a \in C(Y, a)$ such that $f(U_a) \cap V_a = \emptyset$. Since $\{A \cap V_a \mid a \in A\}$ is a closed cover of the subspace A , there exists a finite subset $A_0 \subseteq A$ such that $A \subseteq \bigcup \{V_a \mid a \in A_0\}$. Set $U = \bigcap \{U_a \mid a \in A_0\}$, and U is ω -open since τ_ω is topology and $f(U) \cap A = \emptyset$. Therefore, $U \cap f^{-1}(A) = \emptyset$; and hence, $x \notin \text{Cl}_\omega(f^{-1}(A))$. This shows that $f^{-1}(A)$ is ω -closed. $\square$

THEOREM 4.7. *Let Y be a strongly S -closed space. If a function $f : X \rightarrow Y$ has a contra- ω -closed graph, then f is contra ω -continuous.*

Proof. Suppose that Y is strongly S -closed space and $G(f)$ is contra ω -closed. First we show that an open set of Y is strongly S -closed. Let U be an open set of Y and $\{V_i \mid i \in I\}$ be a cover of U by closed sets V_i of U . For each $i \in I$, there exists a closed set K_i of X such that $V_i = K_i \cap U$. Then the family $\{K_i \mid i \in I\} \cup (Y - U)$ is a closed cover of Y . Since Y is strongly S -closed, there exists a finite subset $I_0 \subseteq I$ such that $Y = \bigcup \{K_i \mid i \in I_0\} \cup (Y - U)$. Therefore, we obtain $U = \bigcup \{V_i \mid i \in I_0\}$. This shows that U is strongly S -closed. Now for any open set U by Theorem 4.6 $f^{-1}(U)$ is ω -closed in X ; therefore, f is contra ω -continuous. $\square$

Definition 4.8. The graph $G(f)$ of a function $f : X \rightarrow Y$ is said to be strongly contra- ω -closed if for each $(x, y) \in (X, Y) - G(f)$, there exist $U \in \omega O(X, x)$ and $V \in RC(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.

LEMMA 4.9. *The graph $G(f)$ of $f : X \rightarrow Y$ is strongly contra- ω -closed graph in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in \omega O(X, x)$ and $V \in RC(Y, y)$ such that $f(U) \cap V = \emptyset$.*

THEOREM 4.10. *If $f : X \rightarrow Y$ is almost weakly- ω -continuous and Y is Urysohn, then $G(f)$ is strongly contra- ω -closed in $X \times Y$.*

Proof. Suppose that $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is Urysohn, there exist open sets V and W in Y containing y and $f(x)$, respectively, such that $\text{Cl}(V) \cap \text{Cl}(W) = \emptyset$. Since f is almost weakly- ω -continuous, by Definition 2.20 there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq \text{Cl}(W)$. This shows that $f(U) \cap \text{Cl}(V) = f(U) \cap \text{Cl}(\text{Int}(V)) = \emptyset$, where $\text{Cl}(\text{Int}(V)) \in RC(Y)$ and hence by Lemma 4.9 we have $G(f)$ is strongly contra- ω -closed. $\square$

THEOREM 4.11. *If $f : X \rightarrow Y$ is almost contra- ω -continuous, then f is almost weakly- ω -continuous.*

Proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. Then $\text{Cl}(V)$ is a regular closed set of Y containing $f(x)$. Since f is almost contra- ω -continuous, by Theorem 3.2 there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq \text{Cl}(V)$. By Definition 2.20 f is almost weakly- ω -continuous. $\square$

COROLLARY 4.12. *If $f : X \rightarrow Y$ is almost contra- ω -continuous and Y is Urysohn, then $G(f)$ is strongly contra- ω -closed.*

The following result can be easily verified.

LEMMA 4.13. *a function $f : X \rightarrow Y$ is almost ω -continuous, if and only if for each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq V$.*

THEOREM 4.14. *If $f : X \rightarrow Y$ is almost ω -continuous, and Y is Hausdorff, then $G(f)$ is strongly contra- ω -closed.*

Proof. Suppose that $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is Hausdorff, there exist open sets V and W in Y containing y and $f(x)$, respectively, such that $V \cap W = \emptyset$; hence, $\text{Cl}(V) \cap \text{Int}(\text{Cl}(W)) = \emptyset$. Since f is almost ω -continuous, and W is regular open by Lemma 4.13 there exists $U \in \omega O(X, x)$ such that $f(U) = W \subseteq \text{Int}(\text{Cl}(W))$. This shows that $f(U) \cap \text{Cl}(V) = \emptyset$ and hence by Lemma 4.9 we have $G(f)$ is strongly contra- ω -closed. $\square$

We recall that a topological space (X, τ) is said to be extremely disconnected (E.D) if the closure of every open set of X is open in X .

THEOREM 4.15. *Let Y be E.D. Then a function $f : X \rightarrow Y$ is almost contra- ω -continuous if and only if it is almost ω -continuous.*

Proof. Let $x \in X$ and V be any regular open set of Y containing $f(x)$. Since Y is E.D then V is clopen and hence V is regular closed. By Theorem 3.2, there exists $U \in \omega O(X, x)$ such that $f(U) \subseteq V$. Then Lemma 4.13 implies that f is almost ω -continuous. Conversely, let F be any regular closed set of Y . Since Y is E.D, F is also regular open and $f^{-1}(F)$ is ω -open in X . This shows that f is almost contra- ω -continuous. $\square$

THEOREM 4.16. *If $f : X \rightarrow Y$ is an injective almost contra- ω -continuous function with the strongly contra- ω -closed graph, then (X, τ) is ω - T_2 .*

Proof. Let x and y be distinct points of X . Then, since f is injective, we have $f(x) \neq f(y)$. Then we have $(x, f(y)) \in (X \times Y) - G(f)$. Since $G(f)$ is strongly contra- ω -closed, by Lemma 4.9 there exists $U \in \omega O(X, x)$ and a regular closed set V containing $f(y)$ such that $f(U) \cap V = \emptyset$. Since f is almost contra- ω -continuous, by Theorem 3.2 there exists $G \in \omega O(X, y)$ such that $f(G) \subseteq V$. Therefore, we have $f(U) \cap f(G) = \emptyset$; hence, $U \cap G = \emptyset$. This shows that (X, τ) is ω - T_2 . $\square$

Acknowledgment

This work is financially supported by the Malaysian Ministry of Science, Technology and Environment, Science Fund Grant no. 04-01-02-SF0177.

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