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Research Article

General Symmetric Starter of Orthogonal Double Covers of Complete Bipartite Graph

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An orthogonal double cover (ODC) of the complete graph is a collection of graphs such that every two of them share exactly one edge and every edge of the complete graph belongs to exactly two of the graphs. In this paper, we consider the case where the graph to be covered twice is the complete bipartite graph $K_{mn,mn}$ (for any values of m, n) and all graphs in the collection are isomorphic to certain spanning subgraphs. Furthermore, the ODCs of $K_{n,n}$ by certain disjoint stars are constructed.

1. Introduction

Let $\mathcal{G}$ be a collection of n spanning subgraphs (called *pages*) of the complete graph on n vertices. $\mathcal{G}$ is called an ODC if

(1) every edge of K_n is an edge in exactly two of the pages,

(2) any two pages share exactly one edge.

If all pages in $\mathcal{G}$ are isomorphic to a graph G, then $\mathcal{G}$ is said to be an ODC by G. Clearly, G must have exactly $n-1$ edges. The existence of ODCs was considered by many authors [1–9]. Although progress on the existence problem has been made, it is far from being solved completely [4].

A generalization of the notion of an ODC to arbitrary underlying graphs is as follows. Let H be an arbitrary graph with n vertices and let $\mathcal{G} = \{G_0, \ldots, G_{n-1}\}$ be a collection of n spanning subgraphs of H. $\mathcal{G}$ is called an *ODC* of H if there exists a bijective mapping $\varphi : V(H) \to \mathcal{G}$ such that

(i) every edge of H is contained in exactly two of the graphs $G_0, \ldots, G_{n-1}$;

(ii) for every choice of different vertices a, b of H,

$$|E(\varphi(a)) \cap E(\varphi(b))| = \begin{cases} 1 & \text{if } a,b \in E(H), \text{ or} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

We consider the complete bipartite graph $K_{n,n}$ whose vertices are the elements of the set $\mathbb{Z}_n \times \{0,1\}$ and whose edges are the pairs of $E(K_{n,n}) = \{\{(u,0),(v,1)\} : u,v \in \mathbb{Z}_n\}$. If there is no danger of ambiguity, the vertex and the element $(v,i) \in \mathbb{Z}_n \times \{0,1\}$ will be represented by v_i and the edge $\{(u,0),(v,1)\} \in K_{n,n}$ will be represented by (u,v). If $\mathcal{G}$ is an ODC of $K_{n,n}$ by graph G, then the double cover property (i) implies that the graph G has exactly n edges, and the orthogonality property (ii) implies that the ODC is $\mathcal{G} = \{\varphi(a,i) : (a,i) \in \mathbb{Z}_n \times \{0,1\}\}$, where each subgraph $\varphi(a,i)$ is isomorphic to G, the subgraphs $\varphi(a,0)$, $a \in \mathbb{Z}_n$, are an edge decomposition of $K_{n,n}$ as the subgraphs $\varphi(a,1)$ are, $a \in \mathbb{Z}_n$, and for all $a,b \in \mathbb{Z}_n$, each subgraph $\varphi(a,0)$ has precisely one edge in common with each subgraph $\varphi(b,1)$.

In this paper, we construct a general symmetric starter of an ODC of $K_{mn,mn}$ which is discussed in Section 3. In Section 4, the ODC of $K_{mn,mn}$ by $K_{m,n}$ is constructed. Finally, in Section 5, we have constructed the ODCs of $K_{2n,2n}$ by $G = 2S_n \cup E_{2n-2}$ (vertex disjoint union of two stars), and of $K_{2n+1,2n+1}$ by $G = S_4 \cup S_2 \cup S_1 \cup S_{2n-6} \cup E_{2n-3}$ (vertex disjoint union of stars).

2. ODC of $K_{n,n}$ by symmetric starters

Let G be a spanning subgraph of $K_{n,n}$ and any $a \in \mathbb{Z}_n$. Then the graph $G+a$ with $E(G+a) = \{\{(u+a,0),(v+a,1)\} : \{(u,0),(v,1)\} \in E(G)\}$ is called the *a-translate* of G. The length of an edge $e = (u,v) = \{(u,0),(v,1)\} \in E(K_{n,n})$ is defined to be $d(e) = v - u \in \mathbb{Z}_n$.

G is called a *half starter* with respect to $\mathbb{Z}_n$ if $|E(G)| = n$ and the lengths of all edges in G are mutually different, that is, $\{d(e) : e \in E(G)\} = \mathbb{Z}_n$.

Theorem 2.1 (see [6]). *If G is a half starter, then the set of all translates of G forms an edge decomposition of $K_{n,n}$. That is, $\bigcup_{a\in\Gamma} E(G+a) = E(K_{n,n})$.*

If n is a positive integer, then an ordered n-tuple is a sequence of n integer numbers $(v_0, v_1, \ldots, v_{n-1})$. The set of all ordered n-tuples is denoted by $\mathbb{Z}_n^n = \underbrace{\mathbb{Z}_n \times \mathbb{Z}_n \times \ldots \times \mathbb{Z}_n}_{n \text{ times}}$.

Here, for any vector $v(G) = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{Z}_n^n = \underbrace{\mathbb{Z}_n \times \mathbb{Z}_n \times \ldots \times \mathbb{Z}_n}_{n \text{ times}}$, the half starter $v(G)$ is the spanning subgraph of $K_{n,n}$ having edge set $E(G) = \{(v_i,0),(v_i+i,1)\}$ for all $i \in \mathbb{Z}_n$.

Two half starter vectors $v(G_0)$ and $v(G_1)$ are said to be orthogonal if $\{v_\gamma(G_0) - v_\gamma(G_1) : \gamma \in \Gamma\} = \Gamma$, where $\Gamma = \{\gamma_0, \ldots, \gamma_{n-1}\}$ is an additive group of order n.

Theorem 2.2 (see [6]). *If two half starters $v(G_0)$ and $v(G_1)$ are orthogonal, then $\mathcal{G} = \{G_{a,i} : (a,i) \in \Gamma \times \mathbb{Z}_2\}$ with $G_{a,i} = G_i + a$ is an ODC of $K_{n,n}$.*

The subgraph G_s of $K_{n,n}$ with $E(G_s) = \{\{u_0,v_1\} : \{v_0,u_1\} \in E(G)\}$ is called the symmetric graph of G. Note that if G is a half starter, then G_s is also a half starter.

A half starter G is called a symmetric starter with respect to Γ if $v(G)$ and $v(G_s)$ are orthogonal.

Theorem 2.3 (see [6]). *Let n be a positive integer and let G be a half starter represented by $v(G) = (v_0, v_1, \ldots, v_{n-1})$. Then G is symmetric starter if and only if $\{v_\gamma - v_{-\gamma} + \gamma : \gamma \in \Gamma\} = \Gamma$.*

3. General symmetric starter of the ODC of $K_{mn,mn}$

Let $v(G) = (v_0, v_1, \ldots, v_{n-1})$ be a symmetric starter of an ODC of $K_{n,n}$ by G with respect to $\mathbb{Z}_n$. If we define the graph G' to be a spanning subgraph of $K_{mn,mn}$ such that $E(G') = \{\{(v_i, 0), (v_i + i + \alpha n, 1)\} : i \in \mathbb{Z}_n,\ \alpha \in \mathbb{Z}_m,\ v_i + i + \alpha n \in \mathbb{Z}_{mn}\}$, then we can deduce the following theorem.

Theorem 3.1. *If $v(G) = (v_0, v_1, \ldots, v_{n-1}) \in \mathbb{Z}_n^n$ is a symmetric starter for an ODC of $K_{n,n}$ by G, then*

$$v(G') = \left(\overbrace{v_0, v_1, \ldots, v_{n-1}}^{1}, \overbrace{v_0, v_1, \ldots, v_{n-1}}^{2}, \ldots, \overbrace{v_0, v_1, \ldots, v_{n-1}}^{m}\right) \in \mathbb{Z}_{mn}^{mn} \tag{3.1}$$

is a symmetric starter of an ODC of $K_{mn,mn}$ by G', where $E(G') = \{\{(v_i, 0), (v_i + i + \alpha n, 1)\} : i \in \mathbb{Z}_n,\ \alpha \in \mathbb{Z}_m,\ v_i + i + \alpha n \in \mathbb{Z}_{mn}\}$.

Proof. Since $v(G) = (v_0, v_1, \ldots, v_{n-1})$ is a symmetric starter for an ODC of $K_{n,n}$ by G with respect to the group $\mathbb{Z}_n$, we have that

$$\{v_s - v_{-s} + s : s \in \mathbb{Z}_n\} = \mathbb{Z}_n. \tag{3.2}$$

Every element $s \in \mathbb{Z}_{mn}$ has a unique representation of the form $s = \alpha n + i$, where $0 \le \alpha \le m$ and $0 \le i \le n$. By definition,

$$\begin{aligned} v_{\alpha n+i}(G') &= v_i \quad \forall \alpha n + i \in \mathbb{Z}_{mn}, \\ v_{-(\alpha n+i)} &= v_{n-i} \quad \forall \alpha n + i \in \mathbb{Z}_{mn}. \end{aligned} \tag{3.3}$$

(Note that there is no ambiguity here because $v_i(G') = v_i(G)$ for all i, $0 \le i < n$.) Then for any $s = \alpha n + i \in \mathbb{Z}_{mn}$,

$$v_{\alpha n+i} - v_{-(\alpha n+i)} + (\alpha n + i) = v_i - v_{-i} + (\alpha n + i). \tag{3.4}$$

Let u be any integer of $\mathbb{Z}_{mn}$. It has a unique representation of the form $u = \beta n + k$, where $0 \le \beta < m$ and $0 \le k < n$. Since

$$\{v_j - v_{-j} + j : j \in \mathbb{Z}_n\} = \mathbb{Z}_n, \tag{3.5}$$

there exists an integer $\ell \in \mathbb{Z}_n$ such that

$$v_\ell - v_{-\ell} + \ell = k \in \mathbb{Z}_n. \tag{3.6}$$

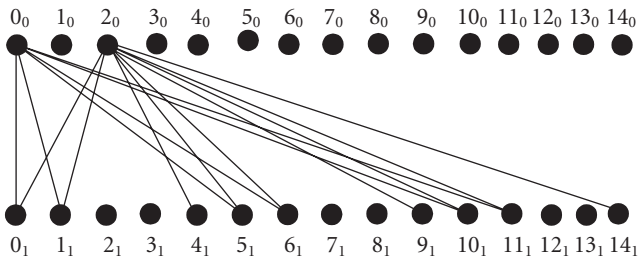

Figure 3.1. Symmetric starter of an ODC of $K_{15,15}$ by $G' = (K_{2,9} - E(K_{1,3})) \cup E_{19}$.

In the group $\mathbb{Z}_{mn}$, the value of the expression $v_\ell - v_{-\ell} + \ell$ is $\gamma n + k$ for some integer γ, $0 \le \gamma < m$. Let $s = (\beta - \gamma)n + \ell \in \mathbb{Z}_{mn}$. Then

$$\begin{aligned} v_s - v_{-s} + s &= v_{(\beta-\gamma)n+\ell} - v_{-(\beta-\gamma)n-\ell} + (\beta-\gamma)n + \ell \\ &= v_\ell - v_{-\ell} + (\beta-\gamma)n + \ell \\ &= (\beta-\gamma)n + \gamma n + k = u. \end{aligned} \tag{3.7}$$

Therefore, every integer $u \in \mathbb{Z}_{mn}$ is in the set $\{v_s - v_{-s} + s : s \in \mathbb{Z}_{mn}\}$. Hence, $v(G')$ is a symmetric starter. □

From Theorem 3.1, we can get the following corollary.

Corollary 3.2. *For $m \ge 1$, $n \ge 4$, the vector*

$$v(G') = \left(\overbrace{0,0,2,2,\ldots,2}^{1},\overbrace{0,0,2,2,\ldots,2}^{2},\ldots,\overbrace{0,0,2,2,\ldots,2}^{m}\right) \in \mathbb{Z}_{mn}^{mn} \tag{3.8}$$

is a symmetric starter of an ODC of $K_{mn,mn}$ by G'.

Proof. As we have proved in [8], the vector $v(G) = (0,0,2,2,\ldots,2)$ is a symmetric starter of an ODC of $K_{n,n}$ by G, where G is the vertex disjoint union of a graph with n independent vertices E_n (the empty graph on n isolated vertices) and $K_{2,n-2} - K_{1,n-4}$, where this represents the graph obtained by removing the edges of a subgraph $K_{1,n-4}$ from $K_{2,n-2}$. Then applying Theorem 3.1 proves the claim. The graph G' can be described as the vertex disjoint union of $mn + 2m - 2$ independent vertices and $K_{2,m(n-2)} - K_{1,m(n-4)}$, that is, $G' = E_{mn+2m-2} \cup (K_{2,m(n-2)} - K_{1,m(n-4)})$. □

Example 3.3. Since the vector $v(G) = (0,0,2,2,2)$ is a symmetric starter of an ODC of $K_{5,5}$ by $G = (K_{2,3} - E(K_{1,1})) \cup E_5$ (E_n are n isolated vertices), then the vector $v(G') = (0,0,2,2,2,0,0,2,2,2,0,0,2,2,2)$ is also a symmetric starter of an ODC of $K_{15,15}$ by $G' = (K_{2,9} - E(K_{1,3})) \cup E_{19}$, see Figure 3.1.

4. Orthogonal double cover of $K_{mn,mn}$ by $K_{n,m}$

In this section, we construct a new class of ODCs of the complete bipartite graph $K_{mn,mn}$, for any values of m, n, and all graphs in the collection are isomorphic to the spanning

subgraph $K_{n,m}$. Considering ODCs by complete bipartite graphs seems natural, since the relational databases can be represented by a complete bipartite graph.

As an application of Theorem 3.1, we can get the following theorem.

Theorem 4.1. *For any positive integers m and n, the vector*

$$v(K_{n,m}) = \left(\overbrace{0,n-1,n-2,\ldots,2,1}^{1},\overbrace{0,n-1,n-2,\ldots,2,1}^{2},\ldots,\overbrace{0,n-1,n-2,\ldots,2,1}^{m}\right) \tag{4.1}$$

is a symmetric starter of an ODC of $K_{mn,mn}$ by $K_{n,m}$ over the group $\mathbb{Z}_{mn}$.

Proof. For any integer, n we have $(n-i)+i=n=0$ in $\mathbb{Z}_n$ for all $i\in\mathbb{Z}_n$. This confirms that $v(G)=(0,n-1,n-2,\ldots,2,1)$ is a half starter with respect to $\mathbb{Z}_n$ for graph $G=K_{n,1}$. Now, for any $i\in\mathbb{Z}_n$, $v_i-v_{-i}+i=(n-i)-i+i=n-i\in\mathbb{Z}_n$, where as usual $(n-0)-0+0=n=0\in\mathbb{Z}_n$. Therefore, $\{v_i-v_{-i}+i : i\in\mathbb{Z}_n\}=\mathbb{Z}_n$. Therefore, the vector $v(K_{n,1})$ is a symmetric half starter for an ODC of $K_{n,n}$ by $K_{n,1}$ with respect to group $\mathbb{Z}_n$. The result now follows by Theorem 3.1. □

Theorem 4.1 can be generalized as follows.

Theorem 4.2. *Let $H=\{0,n,2n,\ldots,(m-1)n\}\subset\mathbb{Z}_{mn}$ and let $H+i=\{h+i : h\in H\}\subset\mathbb{Z}_{mn}$ for any $i\in\mathbb{Z}_{mn}$. For $i=0,1,\ldots,n-1$, let $z_i\in(H+i)$. Then $v(K_{n,m})=(\overbrace{z_0,z_{n-1},z_{n-2},\ldots,z_2,z_1}^{1},\overbrace{z_0,z_{n-1},z_{n-2},\ldots,z_2,z_1}^{2},\ldots,\overbrace{z_0,z_{n-1},z_{n-2},\ldots,z_2,z_1}^{m})$ is a symmetric starter of an ODC of $K_{mn,mn}$ by $K_{n,m}$.*

Example 4.3. For any positive integer n, the vector $v(K_{2,n})=(0,2n-1,0,2n-1,\ldots,0,2n-1)$ is a symmetric starter of an ODC of $K_{2n,2n}$ by $K_{2,n}$.

Example 4.4. For any positive integer n, the vector $v(K_{3,n})=(0,3n-1,3n-2,0,3n-1,3n-2,\ldots,0,3n-1,3n-2)$ is a symmetric starter of an ODC of $K_{3n,3n}$ by $K_{3,n}$.

Example 4.5. For any positive integer n, the vector $v(K_{4,n})=(0,4n-1,4n-2,4n-3,0,4n-1,4n-2,4n-3,\ldots,0,4n-1,4n-2,4n-3)$ is a symmetric starter of an ODC of $K_{4n,4n}$ by $K_{4,n}$.

5. ODC of $K_{n,n}$ by certain stars

In this section, the ODCs of $K_{2n,2n}$ by $2S_n\cup E_{2n-2}$ and of $K_{2n+1,2n+1}$ by $G=S_4\cup S_2\cup S_1\cup S_{2n-6}\cup E_{2n-3}$ are constructed.

For all positive integers $n\geq 2$, let us define the graph $G=2S_n\cup E_{2n-2}$ (vertex disjoint union of two stars with $2n-2$ isolated vertices) to be a spanning subgraph of the complete bipartite graph $K_{2n,2n}$, where there are two cases.

Firstly, when n is even, as shown in Figure 5.1, we have $E(2S_n\cup E_{2n-2})=\{((n-2)_0,j_1) : j\text{is odd}\}\cup\{(n_0,j_1) : j \text{ is even}\}$.

Secondly, when n is odd, as shown in Figure 5.2, we have $E(2S_n\cup E_{2n-2})=\{((n-2)_0,j_1) : j \text{ is even}\}\cup\{(n_0,j_1) : j \text{ is odd}\}$. Then we can prove the following theorem.

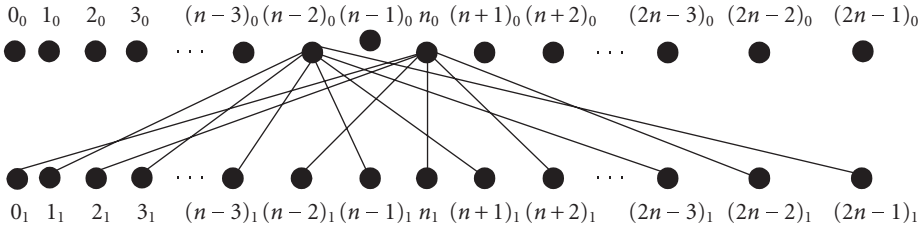

Figure 5.1. Symmetric starter of an ODC of $K_{2n,2n}$ by $G = 2S_n \cup E_{2n-2}$ with respect to $\mathbb{Z}_{2n}$, n is even.

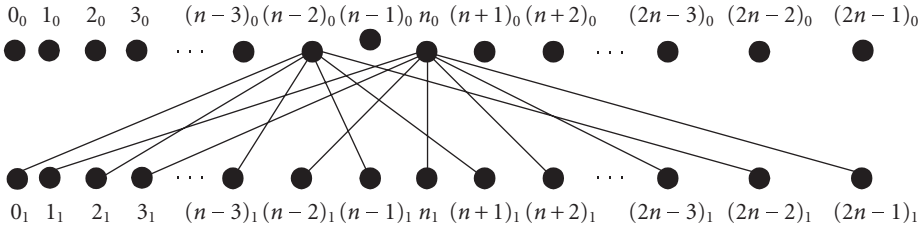

Figure 5.2. Symmetric starter of an ODC of $K_{2n,2n}$ by $G = 2S_n \cup E_{2n-2}$ with respect to $\mathbb{Z}_{2n}$, n is odd.

Theorem 5.1. *For all positive integers $n \geq 2$, the vector $v(G) = (n, n-2, n, n-2, n, n-2, \ldots, n, n-2) \in \mathbb{Z}_{2n}$ is a symmetric starter of an ODC of $K_{2n,2n}$ by $G = 2S_n \cup E_{2n-2}$.*

Proof. For any $i \in \mathbb{Z}_{2n}$ and from the vector $v(G) = (n, n-2, n, n-2, n, n-2, \ldots, n, n-2) \in \mathbb{Z}_{2n}$, we can define

$$v_i(G) = v_{-i}(G) = \begin{cases} n & \text{if } i \text{ is even, or} \\ n-2 & \text{if } i \text{ is odd.} \end{cases} \tag{5.1}$$

Then we have $\{v_i - v_{-i} + i : i \in \mathbb{Z}_{2n}\} = \mathbb{Z}_{2n}$. Applying Theorem 2.3 proves the claim. □

An immediate consequence of the above theorem is the following general result.

Theorem 5.2. *For any positive integer n, let Y be the subgroup of order n in $\mathbb{Z}_{2n}$, that is, let $Y = \{0, 2, 4, \ldots, 2n-2\} \subset \mathbb{Z}_{2n}$. For any two elements $a, b \in Y$, the vector $v(G) = (a, b, a, b, \ldots, a, b) \in \mathbb{Z}_{2n}^{2n}$, is a symmetric starter of an ODC of $K_{2n,2n}$ by $G = 2S_n \cup E_{2n-2}$. The same result is true if $a, b \in (Y+1) = \{1, 3, 5, \ldots, 2n-1\}$.*

Proof. Since Y is a subgroup of $\mathbb{Z}_{2n}$ and $a \in Y$, $\{a + y : y \in Y\} = Y$. Since $b \in Y$, $\{b + z : z \in (Y+1)\} = (Y+1)$. These observations allow us to determine the nature of the graph G.

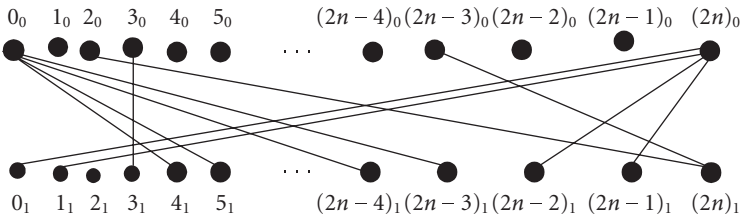

Figure 5.3. Symmetric starter of an ODC of $K_{2n+1,2n+1}$ by $G = S_4 \cup S_2 \cup S_1 \cup S_{2n-6} \cup E_{2n-3}$ with respect to $\mathbb{Z}_{2n+1}$.

By definition, the graph G is the spanning subgraph of $K_{2n,2n}$ whose edge set is

$$\begin{aligned} E(G) &= \{\{(a,0),(a+y,1)\} : y \in Y\} \cup \{\{(b,0),(b+z,1)\} : z \in (Y+1)\} \\ &= \{\{(a,0),(y,1)\} : y \in Y\} \cup \{\{(b,0),(z,1)\} : z \in (Y+1)\}. \end{aligned} \tag{5.2}$$

Clearly, $G = 2S_n \cup E_{2n-2}$. For any element $v_s \in v(G)$, observe that $v_s = v_{-s}$. Hence,

$$\begin{aligned} v_s - v_{-s} + s &= a - a + s = s \quad \forall s \in Y, \\ v_s - v_{-s} + s &= b - b + s = s \quad \forall s \in (Y+1). \end{aligned} \tag{5.3}$$

Therefore, $v(G)$ is a symmetric starter for an ODC of $K_{2n,2n}$ by $G = 2S_n \cup E_{2n-2}$. The same result holds if $a, b \in (Y+1)$. □

For all positive integers $n \geq 4$, let us define the graph $G = S_4 \cup S_2 \cup S_1 \cup S_{2n-6} \cup E_{2n-3}$ (vertex disjoint union of stars with $2n-3$ isolated vertices) to be a spanning subgraph of $K_{2n+1,2n+1}$ as shown in Figure 5.3. Then we can prove the following theorem.

Theorem 5.3. *For all positive integers $n \geq 4$, the vector $v(G) = (3, 2n, 2n, 2n-3, 0, 0, 0, \ldots, 0, 0, 0, 2, 2n, 2n) \in \mathbb{Z}_{2n+1}$ is a symmetric starter of an ODC of $K_{2n+1,2n+1}$ by $G = S_4 \cup S_2 \cup S_1 \cup S_{2n-6} \cup E_{2n-3}$.*

Proof. For all positive integers $n \geq 4$, and from the vector $v(G) = (3, 2n, 2n, 2n-3, 0, 0, 0, \ldots, 0, 0, 0, 2, 2n, 2n) \in \mathbb{Z}_{2n+1}$, we can define

$$v_i(G) = v_{-i}(G) = \begin{cases} 3 & \text{if } i = 0, \text{ or} \\ 2n & \text{if } i = 1, 2, 2n-1, 2n, \text{ or} \\ 2n-3 & \text{if } i = 3, \text{ or} \\ 2 & \text{if } i = 2n-2, \text{ or} \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

Then for any $i \in \mathbb{Z}_{2n+1}$, we have

$$v_i(G) - v_{-i}(G) + i = \begin{cases} -i & \text{if } i = 3, 2n-2, \text{ or} \\ i & \text{otherwise.} \end{cases} \tag{5.5}$$

Applying Theorem 2.3 proves the claim. Note that for any $i \in \mathbb{Z}_{2n+1}$, the ith graph isomorphic to $G = S_4 \cup S_2 \cup S_1 \cup S_{2n-6} \cup E_{2n-3}$ has the edges

$$\begin{aligned}\{\{(i+2n)_0,(i+j)_1\} : j = 0,1,2n-2,2n-1\} &\cup \{\{(i+j)_0,(i+2n)_1\} : j = 2,2n-3\} \\ &\cup \{(i+3)_0,(i+3)_1\} \\ &\cup \{\{i_0,(i+j)_1\} : 4 \le j \le 2n-3\}.\end{aligned} \tag{5.6}$$

□

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