Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 43013, 19 pages
doi:10.1155/2007/43013

Research Article

Bitraces on Partial O^*-Algebras

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Received 12 March 2006; Revised 12 December 2006; Accepted 27 February 2007

Recommended by Lokenath Debnath

Unbounded bitraces on partial O^*-algebras are considered, a class of ideals defined by them is exhibited, and several relationships between certain commutants, bicommutants, and tricommutants associated with the $*$-representations and $*$-antirepresentations determined by the bitraces are established. Moreover, a notion of a partial W^*-algebra of unbounded densely defined linear maps on a Hilbert space, as a generalization of a W^*-algebra, is introduced and a set of criteria for classifying such algebras by means of the type of bitraces that are defined on them is proposed.

1. Introduction

Unbounded traces, which form a special class of *weights*, on W^*-algebras [1–4] of continuous linear operators on a Hilbert space give rise to $*$-*representations* and $*$-*antirepresentations* of these algebras. The $*$-maps play a fundamental role in the classification of W^*-algebras.

In this paper, we consider certain partial O^*-algebras, equipped with *unbounded bitraces*. For details about partial $*$-algebras and some of their applications, see, for example, [5–11]. Unbounded bitraces are a special class of *biweights*. A discussion of some features of biweights on partial $*$-algebras is given in [5] and [12, Chapter 9], and a class of biweights possessing trace representations associated with positive Hilbert-Schmidt operators is described in [12, Example 9.1.13]. The *unbounded bitraces* introduced in the sequel are biweights with trace representations and some additional properties indicated in Definition 3.2. They have a role to play in the classification of partial O^*-algebras, as we briefly indicate towards the end of the paper. Moreover, as in the case of W^*-algebras, they also give rise to $*$-*representations* and $*$-*antirepresentations* of the partial

O^*-algebras on which they are defined. In applying these constructs, the considerations in this paper involve the use of *commutants*. In the partial O^*-algebraic context in which we work, several versions of *commutants* and *higher commutants* are possible. This is in contrast to the situation in W^*-algebras.

Our main result, which generalizes a well-known result [1, Theorem 5.3.3] for W^*-algebras, gives several relationships between certain commutants, bicommutants, and tricommutants associated with the $*$-representations and $*$-antirepresentations determined by unbounded bitraces. This type of result, which is of independent interest, is needed in an alternative approach to the formulation of the Tomita-Takesaki theory for partial O^*-algebras that is different from the one described in [5].

The rest of this paper is structured as follows. In Section 2, we establish our notation and recall a number of notions that are needed in the sequel. In particular, we describe the various commutants and bicommutants that will be encountered later. Unbounded bitraces are introduced in Section 3. To illustrate the discussion in this section, we demonstrate how a certain class of bitraces may arise. Section 4 contains our main result (Theorem 4.6), mentioned above. Rounding off the discussion in this section, we furnish a notion of partial W^*-algebras, which are generalizations of W^*-algebras, as well as a set of criteria for classifying such algebras, based on the type of bitraces that are defined on them.

2. Preliminaries

The basic structure employed in the sequel is a quadruplet $(\mathscr{A},\Gamma,*,\cdot)$, called a *partial $*$-algebra* [5]. This is an involutive complex linear space $\mathscr{A}$ with involution $*$, a relation $\Gamma \subseteq \mathscr{A}\times\mathscr{A}$ on $\mathscr{A}$, and a partial multiplication "$\cdot$" on $\mathscr{A}$, such that

(1) $(x,y)\in\Gamma \Leftrightarrow x\cdot y\in\mathscr{A}$;

(2) $(x,y)\in\Gamma \Leftrightarrow (y^*,x^*)\in\Gamma$, and then $(x\cdot y)^* = y^*\cdot x^*$;

(3) $(x,y)\in\Gamma$ and $(x,z)\in\Gamma \Rightarrow (x,\alpha y+\beta z)\in\Gamma$ and then $x\cdot(\alpha y+\beta z) = \alpha(x\cdot y)+\beta(x\cdot z)$, for all $\alpha,\beta\in\mathbb{C}$.

Remark 2.1. In view of (1), a partial $*$-algebra is, in general, *nonassociative*, thereby making its study largely dependent on its several classes of multipliers introduced as follows.

For a partial $*$-algebra $(\mathscr{A},\Gamma,*,\cdot)$, a subset $\mathscr{C}\subseteq\mathscr{A}$, and a point $x\in\mathscr{A}$, let

$$\begin{aligned} L(x) &= \{y\in\mathscr{A} : (y,x)\in\Gamma\},\\ R(x) &= \{y\in\mathscr{A} : (x,y)\in\Gamma\},\\ L(\mathscr{C}) &= \bigcap_{x\in\mathscr{C}} L(x),\\ R(\mathscr{C}) &= \bigcap_{x\in\mathscr{C}} R(x),\\ M(\mathscr{C}) &= L(\mathscr{C})\cap R(\mathscr{C}). \end{aligned} \tag{2.1}$$

Then $L(x)$ (resp., $R(x)$) is the set of *left* (resp., *right*) *multipliers* of x; $L(\mathscr{C})$ (resp., $R(\mathscr{C})$) is the set of *left* (resp., *right*) *multipliers* of $\mathscr{C}$, and $M(\mathscr{C})$ is the set of *universal multipliers* (or simply *multipliers*) of $\mathscr{C}$.

Example 2.2. (a) If $\Gamma = \mathcal{A} \times \mathcal{A}$, then each of the sets $L(x)$, $R(x)$, $x \in \mathcal{A}$, and $L(\mathcal{C})$, $R(\mathcal{C})$, $M(\mathcal{C})$, $\mathcal{C} \subseteq \mathcal{A}$, reduces to $\mathcal{A}$, showing that $\mathcal{A}$ is an $*$-algebra. Hence, all $*$-algebras are partial $*$-algebras.

(b) If $(\mathcal{A}_0, \mathbf{t})$ is a topological $*$-algebra, with topology $\mathbf{t}$ and completion $\mathcal{A}$, then $\mathcal{A}$ is called a *quasi-$*$-algebra*, and the relation

$$\Gamma = \{(x,y) \in \mathcal{A} \times \mathcal{A} : \text{either } x \in \mathcal{A},\ y \in \mathcal{A}_0 \text{ or } x \in \mathcal{A}_0,\ y \in \mathcal{A}\} \tag{2.2}$$

induces a partial multiplication "$\cdot$" on $\mathcal{A}$, converting the quadruplet $(\mathcal{A}, \Gamma, *, \cdot)$ into a partial $*$-algebra. In this way, it is seen that every quasi-$*$-algebra is a partial $*$-algebra.

(c) A concrete partial $*$-algebra arises as follows. Let $\mathcal{D}$ be a complex pre-Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ (assumed linear on the right), norm $\|\cdot\|$, and completion $\mathcal{H}$. Denote by $L^+(\mathcal{D}, \mathcal{H})$ the set of all linear maps A, each with range in $\mathcal{H}$, such that domain $(A) = \mathcal{D}$ and domain $(A^*) \supset \mathcal{D}$. Equipped with the involution $A \mapsto A^+ = A^* \lceil \mathcal{D}$ and the usual notions of addition and scalar multiplication, $L^+(\mathcal{D}, \mathcal{H})$ is a complex involutive linear space. Let

$$\Gamma = \{(A,B) \in L^+(\mathcal{D}, \mathcal{H}) \times L^+(\mathcal{D}, \mathcal{H}) : B\mathcal{D} \subset \text{domain } (A^{+*}),\ A^*\mathcal{D} \subset \text{domain } (B^*)\}. \tag{2.3}$$

Then, the relation Γ induces, and is induced by, a partial multiplication "$\cdot$" on $L^+(\mathcal{D}, \mathcal{H})$ given by

$$A \cdot B = A^{+*} B \text{ for } (A,B) \in \Gamma. \tag{2.4}$$

The quadruplet $(L^+(\mathcal{D}, \mathcal{H}), \Gamma, +, \cdot)$ is therefore a partial $*$-algebra, which will be denoted henceforth by $L_w^+(\mathcal{D}, \mathcal{H})$.

Contained in $L_w^+(\mathcal{D}, \mathcal{H})$ is the set

$$L^+(\mathcal{D}) = \{A \in L^+(\mathcal{D}, \mathcal{H}) : \text{range } (A) \subseteq \mathcal{D},\ A^*\mathcal{D} \subseteq \mathcal{D}\}, \tag{2.5}$$

which is, indeed, an $*$-algebra with the involution $A \mapsto A^+ = A^* \lceil \mathcal{D}$ and the composition of maps as multiplication.

Definition 2.3. Let $(\mathcal{A}_1, \Gamma_1, *, \cdot)$ and $(\mathcal{A}_2, \Gamma_2, \#, \circ)$ be partial $*$-algebras. Then, a linear map $\sigma : \mathcal{A}_1 \to \mathcal{A}_2$ is called

(a) a *homomorphism* (resp., an *antihomomorphism*) if $(\sigma(x), \sigma(y)) \in \Gamma_2$ (resp., $(\sigma(y), \sigma(x)) \in \Gamma_2$) whenever $(x,y) \in \Gamma_1$ and
   (i) $\sigma(x \cdot y) = \sigma(x) \circ \sigma(y)$ (resp., $\sigma(x \cdot y) = \sigma(y) \circ \sigma(x)$);
   (ii) $\sigma(x^*) = \sigma(x)^\#$,
   (iii) $\sigma(e_1) = e_2$, if $\mathcal{A}_j$ is unital with unit $e_j \in \mathcal{A}_j$, $j = 1,2$;

(b) a *representation* (resp., *antirepresentation*) of $(\mathcal{A}_1, \Gamma_1, *, \cdot)$ if σ is a homomorphism (resp., antihomomorphism) and $(\mathcal{A}_2, \Gamma_2, \#, \circ)$ is $L_w^+(\mathcal{D}, \mathcal{H})$ for some pre-Hilbert space $\mathcal{D}$ having $\mathcal{H}$ as its completion.

One speaks of a *faithful* homomorphism (resp., antihomomorphism) σ (resp., representation π) if $x \in \mathcal{A}_1$ and $\sigma(x) = 0 \Rightarrow x = 0$ (resp., $\pi(x) = 0 \Rightarrow x = 0$). A faithful homomorphism (resp., antihomomorphism) σ from $\mathcal{A}_1 \to \mathcal{A}_2$ whose inverse σ^{-1} is a homomorphism (antihomomorphism) from $\mathcal{A}_2 \to \mathcal{A}_1$ is called an *isomorphism* (resp., *anti-isomorphism*).

Remark 2.4. In view of (1), we will speak henceforth of a partial $*$-algebra $(\mathcal{A}, *, \cdot)$, or simply $\mathcal{A}$, when the operations have previously been specified, instead of $(\mathcal{A}, \Gamma, *, \cdot)$.

Definition 2.5. Let $(\mathcal{A}, *, \cdot)$ be a partial $*$-algebra. A subspace of $\mathcal{A}$ will be called a *partial $*$-subalgebra*, or simply a *subalgebra*, if it is also a partial $*$-algebra when endowed with the same involution $*$ and partial multiplication "$\cdot$" as already on $\mathcal{A}$.

Remark 2.6. In the sequel, we will be concerned with certain subalgebras of $L^+(\mathcal{D})$ and $L_w^+(\mathcal{D}, \mathcal{H})$.

Definition 2.7. A subalgebra

(a) of $L^+(\mathcal{D})$ is called an *O^*-algebra* on $\mathcal{D}$;

(b) of $L_w^+(\mathcal{D}, \mathcal{H})$ is called a *partial O^*-algebra* on $\mathcal{D}$.

Remark 2.8. We consider mainly *partial O^*-algebras* on $\mathcal{D}$; different classes of these arise depending, for example, on the topologies on them or $\mathcal{D}$. We first describe three topologies on partial O^*-algebras that are employed in the sequel.

Notation 2.9. Let $\mathcal{M} \subset L_w^+(\mathcal{D}, \mathcal{H})$ be a partial O^*-algebra on $\mathcal{D}$ and

$$\begin{aligned}
\mathcal{D}^\infty(\mathcal{M}) &= \left\{ (\xi_n) \subset \mathcal{D} : \sum_{n=1}^{\infty} \left(\|\xi_n\|^2 + \|x\xi_n\|^2 \right) < \infty \ \forall x \in \mathcal{M} \right\}, \\
p_{\xi,\eta}(x) &= |\langle \xi, x\eta \rangle|, \quad \xi, \eta \in \mathcal{D},\ x \in \mathcal{M}, \\
p_\xi^*(x) &= \|x\xi\| + \|x^+\xi\|, \quad \xi \in \mathcal{D},\ x \in \mathcal{M}, \\
p_{(\xi_n),(\eta_n)}(x) &= \left| \sum_{n=1}^{\infty} \langle \xi_n, x\eta_n \rangle \right|, \quad (\xi_n), (\eta_n) \in \mathcal{D}^\infty(\mathcal{M}),\ x \in \mathcal{M}.
\end{aligned} \tag{2.6}$$

The functionals $p_{\xi,\eta}$, p_ξ^*, and $p_{(\xi_n),(\eta_n)}$, with $\xi, \eta \in \mathcal{D}$ and $(\xi_n), (\eta_n) \in \mathcal{D}^\infty(\mathcal{M})$ are seminorms on $\mathcal{M}$.

Definition 2.10. Let $\mathcal{M}$ be a partial O^*-algebra on $\mathcal{D}$. The locally convex topology on $\mathcal{M}$ determined by $p_{\xi,\eta}$, $\xi, \eta \in \mathcal{D}$ (resp., p_ξ^*, $\xi \in \mathcal{D}$; resp., $p_{(\xi_n),(\eta_n)}$, $(\xi_n), (\eta_n) \in \mathcal{D}^\infty(\mathcal{M})$) is called the *weak topology* t_w (resp., *strong $*$ topology* t_s^*; resp., *σ-weak topology* $t_{\sigma w}$).

Notation 2.11. Let $\mathcal{M}$ be a partial O^*-algebra on $\mathcal{D}$ and

$$\|\xi\|_x = \|x\xi\|, \quad x \in \mathcal{M},\ \xi \in \mathcal{D}. \tag{2.7}$$

Let $t_{\mathcal{M}}$ be the locally convex topology on $\mathcal{D}$ generated by the seminorms $\{\|\cdot\|_x : x \in \mathcal{M}\}$.

Definition 2.12. A partial O^*-algebra $\mathcal{M}$ on $\mathcal{D}$ is called

(a) *closed* if the locally convex space $(\mathcal{D}, t_{\mathcal{M}})$ is complete;

(b) *standard* if $\mathcal{M}$ is closed and $\overline{x^+} = x^*$, for each $x \in \mathcal{M}$.

Remark 2.13. We consider standard partial O^*-algebras in the sequel.

Definition 2.14. A member e (resp., p) of a partial O^*-algebra $\mathcal{M}$ on $\mathcal{D}$ will be called a *unit* (resp., *projection*) if $e \in M(\mathcal{M})$, $e^+ = e$, and $e \cdot x = x = x \cdot e$, for all $x \in \mathcal{M}$ (resp., $p \in L^+(\mathcal{D})$ and $p = p^+ = p \cdot p$).

A partial O^*-algebra $\mathcal{M}$ on $\mathcal{D}$ is *unital* if it has a unit.

Notation 2.15. (a) If $\mathcal{N}$ is a subset of $L_w^+(\mathcal{D},\mathcal{H})$, the set of all projections in $\mathcal{N}$ will be denoted by $\mathrm{Proj}(\mathcal{N})$.

(b) Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two subsets of $L_w^+(\mathcal{D},\mathcal{H})$. Then $\mathcal{N}_1 \cdot \mathcal{N}_2$ will denote the linear span of the set $\{a \cdot b : a \in \mathcal{N}_1,\ b \in \mathcal{N}_2 \text{ with } a \in L(b)\}$. When $\mathcal{N}_2$ is a singleton $\{x\}$, the set $\mathcal{N}_1 \cdot \{x\}$ will be written simply as $\mathcal{N}_1 \cdot x$.

Definition 2.16. Let $\mathcal{M}$ be a partial O^*-algebra on $\mathcal{D}$ and $\mathcal{B}$ a subspace of $\mathcal{M}$. Then $\mathcal{B}$ is a *left ideal* (resp., a *right ideal*; resp., an *ideal*) of $\mathcal{M}$ if $L(\mathcal{M}) \cdot \mathcal{B} \subseteq \mathcal{B}$ (resp., $\mathcal{B} \cdot R(\mathcal{M}) \subseteq \mathcal{B}$; resp., $\mathcal{B}$ is both a *left* ideal and a *right* ideal).

Remark 2.17. (a) Let $\mathcal{M}$ be a unital $t_{\sigma w}$-closed partial O^*-algebra on $\mathcal{D}$ and $p \in \mathrm{Proj}\,(\mathcal{M})$. Then the following statements are easily verified:

(i) $\mathcal{M} \cdot p$ is $t_{\sigma w}$-closed;

(ii) if $a, b \in \mathcal{M}$ with $a \in L(b \cdot p)$, then $a \in L(b)$ and $(a \cdot b) \cdot p = a \cdot (b \cdot p)$;

(iii) from (i) and (ii), $\mathcal{M} \cdot p$ is a $t_{\sigma w}$-closed left ideal of $\mathcal{M}$ for any $p \in \mathrm{Proj}\,(\mathcal{M})$.

(b) The converse of (a)(iii) can also be established under some conditions.

Commutants and bicommutants. Unlike in the case of W^*-algebras, several possibilities arise for the *commutants* and *bicommutants* of partial O^*-algebras. We limit ourselves in this paper to the following notions.

Commutants. Let $\mathcal{C}$ be a $+$-invariant subset of $L_w^+(\mathcal{D},\mathcal{H})$. Then the commutants $\mathcal{C}'_\sigma$ and $\mathcal{C}'_c$ are defined as follows:

$$\begin{aligned}
\mathcal{C}'_\sigma &= \{X \in L_w^+(\mathcal{D},\mathcal{H}) : \langle X\xi, A^+\eta\rangle = \langle A\xi, X^+\eta\rangle,\ \forall A \in \mathcal{C},\ \xi,\eta \in \mathcal{D}\},\\
\mathcal{C}'_c &= \{X \in L^+(\mathcal{D}) \cap \mathrm{B}(\mathcal{H}) : \langle A \cdot X\xi, \eta\rangle = \langle \xi, A^+ \cdot X^+\eta\rangle,\ \forall A \in \mathcal{C},\ \xi,\eta \in \mathcal{D}\}.
\end{aligned} \tag{2.8}$$

These commutants are related, since $\mathcal{C}'_c = \mathcal{C}'_\sigma \cap L^+(\mathcal{D}) \cap \mathrm{B}(\mathcal{H})$.

Bicommutants. Associated with the above commutants are the four bicommutants $\mathcal{C}''_{\sigma\sigma}$, $\mathcal{C}''_{cc}$, $\mathcal{C}''_{c\sigma}$, and $\mathcal{C}''_{\sigma c}$, which are defined as follows:

$$\begin{aligned}
\mathcal{C}''_{\sigma\sigma} &= \{X \in L_w^+(\mathcal{D},\mathcal{H}) : \langle X\xi, A^+\eta\rangle = \langle A\xi, X^+\eta\rangle,\ \forall A \in \mathcal{C}'_\sigma,\ \xi,\eta \in \mathcal{D}\},\\
\mathcal{C}''_{cc} &= \{X \in L^+(\mathcal{D}) \cap \mathrm{B}(\mathcal{H}) : \langle A \cdot X\xi, \eta\rangle = \langle \xi, A^+ \cdot X^+\eta\rangle,\ \forall A \in \mathcal{C}'_c,\ \xi,\eta \in \mathcal{D}\},\\
\mathcal{C}''_{c\sigma} &= \{X \in L_w^+(\mathcal{D},\mathcal{H}) : \langle X\xi, A^+\eta\rangle = \langle A\xi, X^+\eta\rangle,\ \forall A \in \mathcal{C}'_c,\ \xi,\eta \in \mathcal{D}\},\\
\mathcal{C}''_{\sigma c} &= \{X \in L^+(\mathcal{D}) \cap \mathrm{B}(\mathcal{H}) : \langle A \cdot X\xi, \eta\rangle = \langle \xi, A^+ \cdot X^+\eta\rangle,\ \forall A \in \mathcal{C}'_\sigma,\ \xi,\eta \in \mathcal{D}\}.
\end{aligned} \tag{2.9}$$

There are the following relations among these bicommutants:

(i) $\mathscr{C}''_{\sigma c} = \mathscr{C}''_{\sigma\sigma} \cap L^+(\mathscr{D}) \cap \mathrm{B}(\mathscr{H})$;

(ii) $\mathscr{C}''_{\sigma\sigma} \subseteq \mathscr{C}''_{c\sigma} \subset L^+_w(\mathscr{D},\mathscr{H})$ and $\mathscr{C}''_{\sigma c} \subseteq \mathscr{C}''_{cc} \subseteq L^+(\mathscr{D}) \cap \mathrm{B}(\mathscr{H})$;

(iii) $\mathscr{C}''_{\sigma c} \subseteq \mathscr{C}''_{\sigma\sigma}$ and $\mathscr{C}''_{cc} \subseteq \mathscr{C}''_{c\sigma}$.

The sets $\mathscr{C}'_{\sigma}$, $\mathscr{C}''_{\sigma\sigma}$, and $\mathscr{C}''_{c\sigma}$ are t_s^*-closed in $L^+_w(\mathscr{D},\mathscr{H})$. Similarly, the sets $\mathscr{C}'_{c}$, $\mathscr{C}''_{\sigma c}$, and $\mathscr{C}''_{cc}$ are t_s^*-closed in $L^+(\mathscr{D}) \cap \mathrm{B}(\mathscr{H})$. Moreover, each of the sets $\mathscr{C}''_{\sigma\sigma}$ and $\mathscr{C}''_{c\sigma}$ contains $\mathscr{C}$.

Remark 2.18. Additional information about commutants and bicommutants is available in [13].

3. Ideals determined by bitraces

Let $\mathscr{M}$ be a unital partial O^*-algebra on $\mathscr{D}$, with unit e, $\mathscr{M}_+ = \{x \in \mathscr{M} : \langle \xi, x\xi \rangle \geq 0$, for all $\xi \in \mathscr{D}\}$ and $\mathbb{C}^*$ denotes the extended complex plane.

Notation 3.1. The symbol $\mathrm{wgt}(\mathscr{M})$ will denote the set of all maps $\varphi : \mathscr{M} \times \mathscr{M} \to \mathbb{C}^*$ satisfying

(i) $\varphi(x, \alpha y) = \alpha\varphi(x,y)$, $\alpha \in \mathbb{C}$, $x, y \in \mathscr{M}$, with $0 \cdot (\pm\infty) = 0$;

(ii) $\varphi(x,y) = \overline{\varphi(y,x)}$, $x, y \in \mathscr{M}$;

(iii) $\varphi(x \cdot y, z) = \varphi(y, x^+ \cdot z)$, $x, y, z \in \mathscr{M}$, with $x \in L(y)$, $x^+ \in L(z)$;

(iv) $\varphi(x,x) \in \mathbb{R}_+ \cup \{+\infty\}$, $x \in \mathscr{M}$;

(v) $\varphi(e,x) \in \mathbb{R}_+ \cup \{+\infty\}$, $x \in \mathscr{M}_+$;

(vi) $\varphi(e, x+y) = \varphi(e,x) + \varphi(e,y)$, $x, y \in \mathscr{M}_+$.

Definition 3.2. (a) A member of $\mathrm{wgt}(\mathscr{M})$ will be called a *weight* on $\mathscr{M}$.

(b) A pair $(\tau, \mathscr{N}_\tau)$ will be called a *bitrace* on $\mathscr{M}$ provided that

(i) $\tau \in \mathrm{wgt}\,(\mathscr{M})$;

(ii) $\tau(x,y) = \tau(y^+, x^+)$, $x, y \in \mathscr{M}$;

(iii) $\mathscr{N}_\tau$ is an ideal of $\mathscr{M}$;

(iv) the restriction of τ to $\mathscr{N}_\tau \times \mathscr{N}_\tau$ (denoted in the sequel again by τ) is a positive sesquilinear form on $\mathscr{N}_\tau$.

Notation 3.3. (i) The set of all bitraces on $\mathscr{M}$ will be denoted by $\mathrm{btr}(\mathscr{M})$.

(ii) If $(\tau, \mathscr{N}_\tau)$ is a bitrace on $\mathscr{M}$, then $\mathscr{N}_\tau$ will be called the *definition ideal* of the bitrace.

Remark 3.4. (i) We will sometimes refer to τ as a bitrace, instead of the pair $(\tau, \mathscr{N}_\tau)$, and write $\tau \in \mathrm{btr}\,(\mathscr{M})$.

(ii) If $\mathscr{A}$ is a W^*-algebra of operators on a Hilbert space and ω is a trace on $\mathscr{A}$, then the set $\{x \in \mathscr{A} : \omega(x^*x) < \infty\}$ is an ideal of $\mathscr{A}$.

Antithetically, when $\mathscr{M}$ is a partial O^*-algebra on a pre-Hilbert space $\mathscr{D}$, equipped with a weight τ satisfying Definition 3.2(b)(ii), then the set $\mathscr{M}^\tau = \{x \in \mathscr{M} : \tau(x,x) < \infty\}$ is not even a left ideal of $\mathscr{M}$.

(iii) To illustrate the foregoing discussion, the following result demonstrates how a certain class of bitraces on some partial O^*-algebras may arise.

Theorem 3.5. *Let $\mathscr{M}$ be a standard, unital, partial O^*-algebra and $\tau \in \mathrm{wgt}(\mathscr{M})$ be such that Definition 3.2(b)(ii) holds and $\tau((z \cdot x) \cdot b, (z \cdot x) \cdot b) < \infty$, whenever $e \neq z \in L(\mathscr{M})$,*

$e \neq b \in R(\mathcal{M})$, $x \in \mathcal{M}$. Then, the pair $(\tau, \mathcal{N}_\tau)$, where $\mathcal{N}_\tau = \{x \in \mathcal{M} : \tau(a \cdot x, a \cdot x) < \infty$ and $\tau(x \cdot b, x \cdot b) < \infty$, for all $e \neq a \in L(\mathcal{M})$ and $e \neq b \in R(\mathcal{M})\}$, is a bitrace on $\mathcal{M}$.

Proof. We only need to show that $\mathcal{N}_\tau$ is an ideal of $\mathcal{M}$. First note that if $x \in \mathcal{N}_\tau$, then $x^+ \in \mathcal{N}_\tau$, showing that $\mathcal{N}_\tau$ is a $+$-invariant set. This is because if $x \in \mathcal{N}_\tau$, $e \neq a \in L(\mathcal{M})$ and $e \neq b \in R(\mathcal{M})$, whence $a^+ \in R(\mathcal{M})$ and $b^+ \in L(\mathcal{M})$, then, in view of Definition 3.2(b)(ii), it follows that

$$\begin{aligned} \tau(a \cdot x^+, a \cdot x^+) &= \tau(x \cdot a^+, x \cdot a^+) < \infty, \\ \tau(x^+ \cdot b, x^+ \cdot b) &= \tau(b^+ \cdot x, b^+ \cdot x) < \infty. \end{aligned} \tag{3.1}$$

Next, we show that if $x \in \mathcal{N}_\tau$ and $z \in L(\mathcal{M})$, then $z \cdot x \in \mathcal{N}_\tau$. To this end, let $x \in \mathcal{N}_\tau$. Then $z \in L(x)$ and $a \in L(z \cdot x)$. From $z \in L(x)$, we get the inclusions

$$x\mathcal{D} \subset \text{domain } (z^{+*}), \tag{3.2}$$

$$z^*\mathcal{D} \subset \text{domain } (x^*), \tag{3.3}$$

and since $a \in L(z \cdot x)$, there are the inclusions

$$(z \cdot x)\mathcal{D} \subset \text{domain } (a^{+*}), \tag{3.4}$$

$$a^+\mathcal{D} \subset \text{domain } ((z \cdot x)^*). \tag{3.5}$$

As $\mathcal{M}$ is standard, that is, as $\overline{m^+} = m^*$ for every $m \in \mathcal{M}$, (3.2) is equivalent to $z^{**}x\mathcal{D} \subset \mathcal{H}$ and (3.4) is equivalent to $a^{+*}(z \cdot x)\mathcal{D} = a^{+*}z^{+*}x\mathcal{D} = a^{+*}z^{**}x\mathcal{D} \subset \mathcal{H}$, that is, $((a \cdot z) \cdot x)\mathcal{D} \subset \mathcal{H}$, showing that $x\mathcal{D} \subset \text{domain } ((a \cdot z)^{+*})$. Analogously, (3.5) is equivalent to $(z \cdot x)^* a^+\mathcal{D} = (z^{+*}x)^* a^+\mathcal{D} = x^* z^{+**} a^+\mathcal{D} = x^* z^* a^+\mathcal{D} \subset \mathcal{H}$, that is, $(a \cdot z)^+\mathcal{D} \subset \text{domain } (x^*)$. The two relations

$$x\mathcal{D} \subset \text{domain } ((a \cdot z)^{+*}), \qquad (a \cdot z)^+\mathcal{D} \subset \text{domain } (x^*) \tag{3.6}$$

show that $a \cdot z \in \mathcal{M}$ and $a \cdot z \in L(x)$. As

$$a \cdot (z \cdot x) = a^{+*}z^{+*}x = a^{+*}z^{**}x = (a \cdot z) \cdot x \text{ on } \mathcal{D}, \tag{3.7}$$

it follows that for $x \in \mathcal{N}_\tau$, $z \in L(\mathcal{M})$, and $a \in L(\mathcal{M})$,

$$\tau(a \cdot (z \cdot x), a \cdot (z \cdot x)) = \tau((a \cdot z) \cdot x, (a \cdot z) \cdot x) < \infty. \tag{3.8}$$

Moreover, for $x \in \mathcal{N}_\tau$, $z \in L(\mathcal{M})$, and $b \in R(\mathcal{M})$, we have, by hypothesis, that

$$\tau((z \cdot x) \cdot b, (z \cdot x) \cdot b) < \infty. \tag{3.9}$$

The last two inequalities show that $z \cdot x \in \mathcal{N}_\tau$, whenever $x \in \mathcal{N}_\tau$ and $z \in L(\mathcal{M})$. As $\mathcal{N}_\tau$ is $+$-invariant, it follows that $x \cdot z \in \mathcal{N}_\tau$ whenever $x \in \mathcal{N}_\tau$ and $z \in R(\mathcal{M})$.

Finally, let $x, y \in \mathcal{N}_\tau$, $e \neq a \in L(\mathcal{M})$, and $e \neq b \in R(\mathcal{M})$. As

$$\begin{aligned} \tau(a\cdot(x+y), a\cdot(x+y)) &\leq 2(\tau(a\cdot x, a\cdot x)+\tau(a\cdot y, a\cdot y)), \\ \tau((x+y)\cdot b, (x+y)\cdot b) &\leq 2(\tau(x\cdot b, x\cdot b)+\tau(y\cdot b, y\cdot b)), \end{aligned} \tag{3.10}$$

it follows that $x+y \in \mathcal{N}_\tau$. This concludes the proof that $\mathcal{N}_\tau$ is an ideal of $\mathcal{M}$. □

Remark 3.6. (a) The finiteness condition on $\tau \in \mathrm{btr}(\mathcal{M})$ in Theorem 3.5 will be met if, for example, τ is such that the maps $x \mapsto a\cdot x$ and $y \mapsto y\cdot b$ of $\mathcal{M}$ to $\mathcal{M}$ are continuous in the $\|\cdot\|_\tau$-topology, for arbitrary $a \in L(\mathcal{M})$ and $b \in R(\mathcal{M})$, where $\|\cdot\|_\tau$ is the norm introduced in Section 4.

(b) Let $\mathcal{M}$ and $\tau \in \mathrm{btr}\,(\mathcal{M})$ be as in Theorem 3.5. Suppose that e is the unit of $\mathcal{M}$.

(i) If $\tau(e,e) < \infty$, then it does not follow that $\tau(x,x) < \infty$, for all $x \in \mathcal{M}$, as would be the case were $\mathcal{M}$ is a W^*-algebra of operators on a Hilbert space.

(ii) If $e \in \mathcal{N}_\tau$, then $\tau(x,x) < \infty$, for all $x \in \mathcal{M}$.

(c) Theorem 3.5 furnishes a class of bitraces on standard, unital, partial O^*-algebras, which are in general noncommutative. In Theorem 3.11, we also exhibit a one-parameter family of bitraces on some selfadjoint, noncommutative partial O^*-algebras. These examples of bitraces are of interest in the quantum statistics of thermodynamic systems [14].

Notation 3.7. Let $\mathcal{H}$ be a separable Hilbert space, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and H is a selfadjoint linear operator with domain in $\mathcal{H}$ such that $\exp(-\beta H)$ is nuclear for every $\beta > 0$. Write (f_n) and (λ_n) for the normalized eigenvectors and corresponding eigenvalues of H. By the nuclearity of $\exp(-\beta H)$ for each $\beta > 0$, (f_n) is an orthonormal basis for $\mathcal{H}$ and

$$\sum_{n=1}^{\infty} \exp(-\beta\lambda_n) < \infty, \quad \beta > 0. \tag{3.11}$$

Let $\mathcal{D} = \bigcap_{\beta>0} D(e^{\beta H})$, where $D(A)$ denotes the domain of A. Since $\mathcal{D}$ contains the linear span of (f_n), the set $\mathcal{D}$ is dense in $\mathcal{H}$.

Introduce the weak partial O^*-algebra $L_w^+(\mathcal{D},\mathcal{H})$ and let

$$L(\mathcal{D}) = \{a \in L(\mathcal{D},\mathcal{H}) : a\mathcal{D} \subset \mathcal{D}\}. \tag{3.12}$$

Also recall that

$$L^+(\mathcal{D}) = \{a \in L(\mathcal{D}) : a^+\mathcal{D} \subset \mathcal{D}\}. \tag{3.13}$$

Let $\mathcal{M} \subset L_w^+(\mathcal{D},\mathcal{H})$ be a partial O^*-algebra on $\mathcal{D}$. For simplicity, we assume that $\mathcal{M}$ is selfadjoint, that is, $\bigcap_{a\in\mathcal{M}} D(a^*) = \mathcal{D}$. Then, with $\mathcal{M}_b = \{a \in \mathcal{M} : \overline{a} \in B(\mathcal{H})\}$, we have

$$\begin{aligned} R(\mathcal{M}) &= \mathcal{M}_b \cap L(\mathcal{D}), \\ L(\mathcal{M}) &= \{a \in \mathcal{M}_b : a^+\mathcal{D} \subset \mathcal{D}\}, \\ M(\mathcal{M}) &= L(\mathcal{M}) \cap R(\mathcal{M}) = \mathcal{M}_b \cap L^+(\mathcal{D}). \end{aligned} \tag{3.14}$$

In what follows, we assume that

$$e^{-\beta H}x = xe^{-\beta H}, \quad x \in \mathcal{M}_b. \tag{3.15}$$

Define a one-parameter family $\{\tau^\beta : \beta > 0\}$ of members of wgt $(\mathcal{M})$ by [15]

$$\tau^\beta(x,y) = \sum_{n=1}^{\infty} \langle xf_n, yf_n\rangle e^{-\beta\lambda_n}, \quad x,y \in \mathcal{M},\ \beta > 0, \tag{3.16}$$

and let

$$\mathcal{M}^{\tau^\beta} = \{x \in \mathcal{M} : \tau^\beta(x,x) < \infty\}. \tag{3.17}$$

Define $\mathcal{N}_{\tau^\beta}$ by

$$\mathcal{N}_{\tau^\beta} = \{x \in \mathcal{M}^{\tau^\beta} : \tau^\beta(a\cdot x, a\cdot x) < \infty,\ \tau^\beta(x\cdot b, x\cdot b) < \infty,\ \forall a \in L(\mathcal{M}),\ b \in R(\mathcal{M})\}. \tag{3.18}$$

Introduce the Hilbert space $(\mathcal{H}_{\tau^\beta}, \langle\cdot,\cdot\rangle_{\tau^\beta})$ as in Section 4. We will first establish some results.

Lemma 3.8. *The set $R(\mathcal{M}) \cap \mathcal{N}_{\tau^\beta}$ is dense in $(\mathcal{H}_{\tau^\beta}, \langle\cdot,\cdot\rangle_{\tau^\beta})$.*

Proof. Let $x \in \mathcal{H}_{\tau^\beta}$ be arbitrary. For $f \in \mathcal{D}$, xf lies in $\mathcal{H}$, and hence has a representation of the form

$$xf = \sum_{n=1}^{\infty} \langle x^+ f_n, f\rangle f_n, \tag{3.19}$$

whence

$$\|xf\|^2 = \sum_{n=1}^{\infty} |\langle x^+ f_n, f\rangle|^2 < \infty. \tag{3.20}$$

Define x_m, $1 \le m < \infty$, in $B(\mathcal{H})$ as follows:

$$x_m g = \sum_{n=1}^{m} \langle x^+ f_n, g\rangle f_n, \tag{3.21}$$

for arbitrary $g \in \mathcal{H}$. Then $x_m \in \mathcal{M}_b \cap L(\mathcal{D})$, whence $x_m \in R(\mathcal{M})$, $1 \le m < \infty$, and $L(x_m) = \mathcal{M}$. Let $a \in R(\mathcal{M})$. As $x_m \cdot af_j = \sum_{n=1}^{m} \langle x^+ f_n, af_j \rangle f_n$, we get

$$\begin{aligned}
\tau^\beta(x_m \cdot a, x_m \cdot a) &= \sum_{j=1}^{\infty} \langle x_m \cdot af_j, x_m \cdot af_j \rangle \mathrm{e}^{-\beta\lambda_j} \\
&= \sum_{j=1}^{\infty} \sum_{n=1}^{m} \sum_{k=1}^{m} \langle af_j, x^+ f_n \rangle \langle x^+ f_k, af_j \rangle \langle f_n, f_k \rangle \mathrm{e}^{-\beta\lambda_j} \\
&= \sum_{j=1}^{\infty} \sum_{n=1}^{m} |\langle af_j, x^+ f_n \rangle|^2 \mathrm{e}^{-\beta\lambda_j} \le \sum_{j=1}^{\infty} \sum_{n=1}^{m} \|af_j\|^2 \|x^+ f_n\|^2 \mathrm{e}^{-\beta\lambda_j} \\
&\le \left(\sum_{n=1}^{m} \|x^+ f_n\|^2 \right) \left(\sum_{n=1}^{\infty} \mathrm{e}^{-\beta\lambda_n} \right) \|a\|^2 < \infty.
\end{aligned} \tag{3.22}$$

Also, as $a \cdot x_m g = \sum_{n=1}^{m} \langle x^+ f_n, g \rangle af_n$, we get

$$\begin{aligned}
\tau^\beta(a \cdot x_m, a \cdot x_m) &= \sum_{j=1}^{\infty} \langle a \cdot x_m f_j, a \cdot x_m f_j \rangle \mathrm{e}^{-\beta\lambda_j} \\
&= \sum_{n=1}^{m} \sum_{k=1}^{m} \left(\sum_{j=1}^{\infty} \langle xf_j, f_n \rangle \langle f_k, xf_j \rangle \mathrm{e}^{-\beta\lambda_j} \right) \langle af_n, af_k \rangle \\
&\le \sum_{n=1}^{m} \sum_{k=1}^{m} \left(\sum_{j=1}^{\infty} \|xf_j\|^2 \mathrm{e}^{-\beta\lambda_j} \right) |\langle af_n, af_k \rangle| \\
&= \left(\sum_{n=1}^{m} \sum_{k=1}^{m} |\langle af_n, af_k \rangle| \right) \tau^\beta(x,x) < \infty.
\end{aligned} \tag{3.23}$$

We conclude that $x_m \in \mathcal{N}_{\tau^\beta}$, whence $x_m \in R(\mathcal{M}) \cap \mathcal{N}_{\tau^\beta}$. Finally,

$$\begin{aligned}
\|x - x_m\|_{\tau^\beta}^2 &= \sum_{j=1}^{\infty} \langle xf_j - x_m f_j, xf_j - x_m f_j \rangle \mathrm{e}^{-\beta\lambda_j} \\
&= \sum_{j=1}^{\infty} \left\| \sum_{n=m+1}^{\infty} \langle f_n, xf_j \rangle f_n \right\|^2 \mathrm{e}^{-\beta\lambda_j} \\
&= \sum_{j=1}^{\infty} \sum_{n=m+1}^{\infty} |\langle f_n, xf_j \rangle|^2 \mathrm{e}^{-\beta\lambda_j} \\
&= \sum_{n=m+1}^{\infty} \sum_{j=1}^{\infty} |\langle f_j, x^+ f_n \rangle|^2 \mathrm{e}^{-\beta\lambda_j}.
\end{aligned} \tag{3.24}$$

Hence

$$\lim_{m \to \infty} \|x - x_m\|_{\tau^\beta}^2 = 0, \tag{3.25}$$

and as $x \in \mathcal{H}_{\tau^\beta}$ was arbitrary, it follows that $R(\mathcal{M}) \cap \mathcal{N}_{\tau^\beta}$ is dense in $\mathcal{H}_{\tau^\beta}$. This ends the proof. □

Lemma 3.9. *For $x, y \in \mathcal{N}_{\tau^\beta}$, $\tau^\beta(x,y) = \tau^\beta(y^+, x^+)$.*

Proof. Let $x, y \in \mathcal{N}_{\tau^\beta}$ and (x_p), (y_q) be sequences in $R(\mathcal{M}) \cap \mathcal{N}_{\tau^\beta}$ such that

$$\lim_{p\to\infty} ||x - x_p||_{\tau^\beta} = 0 = \lim_{q\to\infty} ||x - x_q||_{\tau^\beta}. \tag{3.26}$$

We may assume that (x_p) and (y_q) converge to x and y, respectively, in the $|||\cdot|||_{\tau^\beta}$-topology on $\mathcal{N}_{\tau^\beta}$, where

$$|||z|||_{\tau^\beta}^2 = \sum_{n=1}^{\infty} (||zf_n||^2 + ||z^+ f_n||^2) e^{-\beta\lambda_n}, \quad z \in \mathcal{N}_{\tau^\beta}. \tag{3.27}$$

Then

$$\begin{aligned} \left| \sum_{n=1}^{\infty} (\langle (y - y_q) f_n, (x - x_p) f_n \rangle) e^{-\beta\lambda_n} \right| &\le |||y - y_q|||_{\tau^\beta} |||x - x_p|||_{\tau^\beta}, \\ \left| \sum_{n=1}^{\infty} (\langle (y - y_q)^+ f_n, (x - x_p)^+ f_n \rangle) e^{-\beta\lambda_n} \right| &\le |||y - y_q|||_{\tau^\beta} |||x - x_p|||_{\tau^\beta}. \end{aligned} \tag{3.28}$$

From these, and with $\operatorname{tr}(z) = \sum_{n=1}^{\infty} \langle f_n, z f_n \rangle$, we get

$$\begin{aligned} \tau^\beta(x,y) &= \lim_{p,q\to\infty} \tau^\beta(x_p, y_q) = \lim_{p,q\to\infty} \sum_{n=1}^{\infty} \langle x_p e^{-\beta H/2} f_n, y_q e^{-\beta H/2} f_n \rangle \\ &= \lim_{p,q\to\infty} \operatorname{tr}\left((x_p e^{-\beta H/2})^+ \cdot (y_q e^{-\beta H/2}) \right) \\ &= \lim_{p,q\to\infty} \operatorname{tr}(e^{-\beta H} y_q x_p^+) = \lim_{p,q\to\infty} \sum_{n=1}^{\infty} \langle f_n, e^{-\beta H} y_q x_p^+ \rangle \\ &= \lim_{p,q\to\infty} \sum_{n=1}^{\infty} \langle y_q^+ f_n, x_p^+ f_n \rangle e^{-\beta\lambda_n} = \lim_{p,q\to\infty} \tau^\beta(y_q^+, x_p^+) \\ &= \tau^\beta(y^+, x^+), \quad \text{by (3.28).} \end{aligned} \tag{3.29}$$

This concludes the proof. □

Remark 3.10. The following result concludes our demonstration of the existence of a concrete bitrace on a noncommutative partial O^*-algebra.

Theorem 3.11. *The set of pairs $\{(\mathcal{N}_{\tau^\beta}, \tau^\beta) : \beta > 0\}$ is a one-parameter family of bitraces on $\mathcal{M}$.*

Proof. Let $x \in \mathcal{N}_{\tau^\beta}$ and $a, b \in L(\mathcal{M})$. Then

$$\begin{aligned}\tau^\beta(b\cdot(a\cdot x), b\cdot(a\cdot x)) &= \sum_{n=1}^{\infty} \langle b\cdot(a\cdot x)f_n, b\cdot(a\cdot x)f_n\rangle \mathrm{e}^{-\beta\lambda_n} \\ &= \sum_{n=1}^{\infty} \|b^{+*}a^{+*}xf_n\|^2 \mathrm{e}^{-\beta\lambda_n} \\ &\le \|b^{+*}a^{+*}\|^2 \sum_{n=1}^{\infty} \|xf_n\|^2 \mathrm{e}^{-\beta\lambda_n} \\ &= \|b^{+*}a^{+*}\|^2 \tau^\beta(x,x) < \infty.\end{aligned} \tag{3.30}$$

Similarly, for $x \in \mathcal{N}_{\tau^\beta}$, $a \in L(\mathcal{M})$, and $b \in R(\mathcal{M})$, one checks that

$$\tau^\beta((a\cdot x)\cdot b, (a\cdot x)\cdot b) \le \|a^{+*+*}\|^2 \sum_{n=1}^{\infty} \|x^{+*}bf_n\|^2 \mathrm{e}^{-\beta\lambda_n}. \tag{3.31}$$

Then since $b \in R(\mathcal{M})$, whence

$$bf_n = \sum_{\alpha\in\Lambda} \langle f_\alpha, bf_n\rangle f_\alpha, \tag{3.32}$$

for some finite subset Λ of the natural numbers, we get

$$\tau^\beta((a\cdot x)\cdot b, (a\cdot x)\cdot b) \le 2\sum_{\alpha\in\Lambda} \|x^{+*}f_\alpha\|^2 \left(\sum_{n=1}^{\infty} \mathrm{e}^{-\beta\lambda_n}\right) \|a^{+*+*}\|^2 \|b\|^2 < \infty. \tag{3.33}$$

Hence $a\cdot x \in \mathcal{N}_{\tau^\beta}$ whenever $x \in \mathcal{N}_{\tau^\beta}$ and $a \in L(\mathcal{M})$. From this, we conclude that for each $\beta > 0$, $\mathcal{N}_{\tau^\beta}$ is an ideal since, by Lemma 3.9, $z^+ \in \mathcal{N}_{\tau^\beta}$ whenever $z \in \mathcal{N}_{\tau^\beta}$. Hence $\{(\mathcal{N}_{\tau^\beta}, \tau^\beta) : \beta > 0\}$ is indeed a one-parameter family of bitraces on $\mathcal{M}$. This ends the proof. □

4. Representations determined by bitraces

Let $\mathcal{M}$ be a partial O^*-algebra and $(\tau, \mathcal{N}_\tau)$ a bitrace on $\mathcal{M}$. Introduce $\mathcal{I}_\tau$ as $\mathcal{I}_\tau = \{x \in \mathcal{N}_\tau : \tau(x,x) = 0\}$ and define $\lambda_\tau : \mathcal{N}_\tau \to \mathcal{N}_\tau/\mathcal{I}_\tau$ by $\lambda_\tau(x) = x + \mathcal{I}_\tau$. Let $[\lambda_\tau(\mathcal{N}_\tau)]$ be the linear span of $\lambda_\tau(\mathcal{N}_\tau)$. A sesquilinear form (linear on the right) is defined on $[\lambda_\tau(\mathcal{N}_\tau)]$ through its action on $\lambda_\tau(\mathcal{N}_\tau)$ as follows:

$$\langle \lambda_\tau(x), \lambda_\tau(y)\rangle_\tau = \tau(x,y), \quad x, y \in \mathcal{N}_\tau. \tag{4.1}$$

Then, $\mathcal{H}_\tau$ denotes the completion of $[\lambda_\tau(\mathcal{N}_\tau)]$ in the norm topology furnished by the norm $\|\cdot\|_\tau$ induced by $\langle\cdot,\cdot\rangle_\tau$.

Remark 4.1. (1) If $\tau \in \mathrm{btr}\,(\mathcal{M})$ is faithful, that is, if $\tau(x,x) = 0 \Rightarrow x = 0$, then $\mathcal{I}_\tau = \{0\}$, whence $\lambda_\tau(x) = x$ on $\mathcal{N}_\tau$.

(2) For the GNS construction in respect of an arbitrary biweight, see [12, Section 9.1]. In the case of bitraces, which are our focus in this paper, we will show that to each *regular* bitrace $(\tau, \mathcal{N}_\tau)$ on $\mathcal{M}$ there correspond two representations π_τ and ρ_τ of $\mathcal{M}$. These enable

us to introduce, for example, the conjugation operator J which is of much importance in the following discussion.

Definition 4.2. (1) A bitrace $(\tau,\mathcal{N}_\tau)$ on $\mathcal{M}$ will be called *regular* if

(i) $\mathcal{N}_\tau \neq \{0\}$;

(ii) there is a subspace $\mathcal{G}_\tau \subset M(\mathcal{M}) \cap \mathcal{N}_\tau$ such that

(a) the linear span $[\lambda_\tau(\mathcal{G}_\tau)]$ of $\lambda_\tau(\mathcal{G}_\tau)$ is dense in $\mathcal{H}_\tau$;

(b) (α) $\tau(x_1 \cdot b_1, x_2 \cdot b_2) = \tau(b_1, (x_1^+ \cdot x_2) \cdot b_2)$;

(β) $\tau(b_1 \cdot z_1, b_2 \cdot z_2) = \tau(b_1, b_2 \cdot (z_2 \cdot z_1^+))$ for all $b_1, b_2 \in \mathcal{G}_\tau$, $x_1, x_2 \in \mathcal{M}$ with $x_1^+ \in L(x_2)$, and $z_1, z_2 \in \mathcal{M}$ with $z_2 \in L(z_1^+)$.

(2) If $(\tau,\mathcal{N}_\tau) \in \mathrm{btr}\,(\mathcal{M})$ is *regular*, then the subspace $\mathcal{G}_\tau$ will be called a *core*.

Notation 4.3. (a) The set of all regular members of $\mathrm{btr}(\mathcal{M})$ will be denoted by $\mathrm{Btr}\,(\mathcal{M})$. We will sometimes abbreviate $(\tau,\mathcal{N}_\tau) \in \mathrm{Btr}\,(\mathcal{M})$ by τ.

(b) If $\tau \in \mathrm{Btr}\,(\mathcal{M})$ and $\mathcal{G}_\tau$ is a core for τ, then the symbol $\mathcal{D}_\tau$ will denote the linear space $[\lambda_\tau(\mathcal{G}_\tau)]$.

(c) Let $\tau \in \mathrm{Btr}\,(\mathcal{M})$ and $\mathcal{G}_\tau$ be a core for τ. On the dense subspace $\mathcal{D}_\tau \subset \mathcal{H}_\tau$, define the linear maps $\pi_\tau(x)$ and $\rho_\tau(x)$, $x \in \mathcal{M}$, by

$$\begin{aligned} \pi_\tau(x)\lambda_\tau(y) &= \lambda_\tau(x \cdot y), \quad x \in \mathcal{M},\ y \in \mathcal{G}_\tau \\ \rho_\tau(x)\lambda_\tau(y) &= \lambda_\tau(y \cdot x), \quad x \in \mathcal{M},\ y \in \mathcal{G}_\tau, \end{aligned} \tag{4.2}$$

and denote $\overline{\pi_\tau(x)}$ (resp., $\overline{\rho_\tau(x)}$) again by $\pi_\tau(x)$ (resp., $\rho_\tau(x)$), $x \in \mathcal{M}$. Then, π_τ (resp., ρ_τ) is a representation (resp., antirepresentation) of $\mathcal{M}$ in $L^+(\mathcal{D}_\tau, \mathcal{H}_\tau)$.

(d) Since $\tau(x^+, x^+) = \tau(x,x)$, $x \in \mathcal{N}_\tau$, it follows that the involution in $\mathcal{N}_\tau$ is $\|\cdot\|_\tau$-isometric, and hence extends to an antilinear isometry $J : \mathcal{H}_\tau \to \mathcal{H}_\tau$, $x \mapsto x^+$, satisfying $J^2 = I$, the identity map on $\mathcal{H}_\tau$.

Remark 4.4. Employing the foregoing notation, we introduce the following notions.

Definition 4.5. Let $\mathcal{M}$ be a unital partial O^*-algebra on $\mathcal{D}$, with unit e. A bitrace $\tau \in \mathrm{btr}\,(\mathcal{M})$ will be called

(i) *finite* if $e \in \mathcal{N}_\tau$;

(ii) *semifinite* if there is a net $\{t_\alpha\} \subset \mathcal{G}_\tau \cap \mathcal{M}_+$, satisfying $\{\pi_\tau(t_\alpha)\} \subset L^+(\mathcal{D}_\tau) \cap \mathrm{B}(\mathcal{H})$ and $\|\pi_\tau(t_\alpha)\| \leq 1$ for each α, such that $\{\pi_\tau(t_\alpha)\}$ converges strongly to the identity element of $\mathrm{B}(\mathcal{H}_\tau)$;

(iii) *normal* if for each t_s^*-convergent increasing net $\{t_\alpha\} \subset \mathcal{G}_\tau \cap \mathcal{M}_+$, satisfying $\{\pi_\tau(t_\alpha)\} \subset L^+(\mathcal{D}_\tau) \cap \mathrm{B}(\mathcal{H})$ and $\|\pi_\tau(t_\alpha)\| \leq 1$ for each α, with limit $t \in \mathcal{G}_\tau \cap \mathcal{M}_+$, the net $\{\tau(t_\alpha, x)\}$ converges to $\tau(t,x)$ for every $x \in \mathcal{G}_\tau$.

Theorem 4.6. *Let $\mathcal{M}$ be a standard, unital, partial O^*-algebra and $(\tau,\mathcal{N}_\tau) \in \mathrm{Btr}(\mathcal{M})$ a semifinite normal bitrace on $\mathcal{M}$. Then, π_τ (resp., ρ_τ) is a normal representation (resp., antirepresentation) of $\mathcal{M}$ into $L^+(\mathcal{D}_\tau, \mathcal{H}_\tau)$ satisfying the following properties:*

(i) *$J\pi_\tau(x)J = \rho_\tau(x^+)$, and $J\rho_\tau(x)J = \pi_\tau(x^+)$, $x \in \mathcal{M}$;*

(ii) *$\pi_\tau(\mathcal{M})''_{\sigma c} \subset \rho_\tau(\mathcal{M})'_c \subset \pi_\tau(\mathcal{M})''_{c\sigma}$;*

(iii) $\pi_\tau(\mathcal{M})''''_{cc\sigma} \subset \rho_\tau(\mathcal{M})''_{c\sigma} \subset \pi_\tau(\mathcal{M})''''_{\sigma c\sigma}$;
(iv) $\rho_\tau(\mathcal{M})''_{\sigma c} \subset \pi_\tau(\mathcal{M})'_c \subset \rho_\tau(\mathcal{M})''_{c\sigma}$;
(v) $\rho_\tau(\mathcal{M})''''_{cc\sigma} \subset \pi_\tau(\mathcal{M})''_{c\sigma} \subset \rho_\tau(\mathcal{M})''''_{\sigma c\sigma}$.

Remark 4.7. (i) If $\mathcal{A}$ is a W^*-algebra of operators, and π and ρ are the $*$-representation and $*$-antirepresentation, respectively, determined by some unbounded trace on $\mathcal{A}$, then we simply have the following relationships: $\pi(\mathcal{A})' = \rho(\mathcal{A})$ and $\rho(\mathcal{A})' = \pi(\mathcal{A})$ [2]. Parts (ii) to (v) of Theorem 4.6 reduce to these results if $\mathcal{A}$ is a W^*-algebra.

(ii) Theorem 4.6 will be established by means of several lemmas.

Lemma 4.8. *Under the hypotheses of Theorem 4.6, $\pi_\tau(\mathcal{M}) \subset \rho_\tau(\mathcal{M})'_\sigma$ and $\rho_\tau(\mathcal{M}) \subset \pi_\tau(\mathcal{M})'_\sigma$.*

Proof. For $x_1, x_2 \in \mathcal{M}$ and $y, z \in \mathcal{G}_\tau$, we have

$$\begin{aligned}
\langle \rho_\tau(x_1)\lambda_\tau(z), \pi_\tau(x_2)\lambda_\tau(y)\rangle_\tau &= \langle \lambda_\tau(z\cdot x_1), \lambda_\tau(x_2\cdot y)\rangle \\
&= \tau(z\cdot x_1, x_2\cdot y) = \tau(y^+\cdot x_2^+, x_1^+\cdot z^+) \\
&= \tau(x_2^+, y\cdot(x_1^+\cdot z^+)) = \tau(x_2^+, (y\cdot x_1^+)\cdot z^+) \\
&= \tau(x_2^+\cdot z, y\cdot x_1^+) = \langle \lambda_\tau(x_2^+\cdot z), \lambda_\tau(y\cdot x_1^+)\rangle_\tau \\
&= \langle \pi_\tau(x_2)^+\lambda_\tau(z), \rho_\tau(x_1)^+\lambda_\tau(y)\rangle_\tau
\end{aligned} \tag{4.3}$$

showing that $\pi_\tau(\mathcal{M}) \subset \rho_\tau(\mathcal{M})'_\sigma$ and $\rho_\tau(\mathcal{M}) \subset \pi_\tau(\mathcal{M})'_\sigma$. □

Lemma 4.9. *Let $t \in \mathcal{G}_\tau \cap \mathcal{M}_+$ and $X \in \rho_\tau(\mathcal{M})'_c$, with $X^+ = X$. Then, under the hypotheses of Theorem 4.6, $J\pi_\tau(t)X\lambda_\tau(t) = \pi_\tau(t)X\lambda_\tau(t)$.*

Proof. This is seen as follows. First note that $\pi_\tau(t)X\lambda_\tau(t)$ is well defined as $X \in L^+(\mathcal{D}_\tau) \cap \mathrm{B}(\mathcal{H}_\tau)$ and $\pi_\tau(t)$ has $\mathcal{D}_\tau$ as its domain. For $y \in \mathcal{G}_\tau$, we have

$$\begin{aligned}
\langle J\pi_\tau(t)X\lambda_\tau(t), \lambda_\tau(y)\rangle_\tau &= \langle J\lambda_\tau(y), \pi_\tau(t)X\lambda_\tau(t)\rangle_\tau \\
&= \langle \lambda_\tau(y^+), \pi_\tau(t)X\lambda_\tau(t)\rangle_\tau = \langle \pi_\tau(t)\lambda_\tau(y^+), X\lambda_\tau(t)\rangle_\tau \\
&= \langle \rho_\tau(y^+)\lambda_\tau(t), X\lambda_\tau(t)\rangle_\tau = \langle \lambda_\tau(t), \rho_\tau(y)X\lambda_\tau(t)\rangle_\tau \\
&= \langle X\lambda_\tau(t), \rho_\tau(y)\lambda_\tau(t)\rangle_\tau, \quad \text{as } X \in \rho_\tau(\mathcal{M})'_c, \\
&= \langle X\lambda_\tau(t), \pi_\tau(t)\lambda_\tau(y)\rangle_\tau \\
&= \langle \pi_\tau(t)X\lambda_\tau(t), \lambda_\tau(y)\rangle_\tau, \quad \text{for arbitrary } y \in \mathcal{G}_\tau,
\end{aligned} \tag{4.4}$$

showing that

$$J\pi_\tau(t)X\lambda_\tau(t) = \pi_\tau(t)X\lambda_\tau(t). \tag{4.5}$$

□

Lemma 4.10. *Under the hypotheses of Lemma 4.9, $\pi_\tau(t)X\pi_\tau(t)$ lies in $\pi_\tau(\mathcal{M})''_{\sigma\sigma}$.*

Proof. As $\mathcal{M}$ is semifinite, there is a net $\{x_\alpha\}$ in $\mathcal{G}_\tau$ such that $\pi_\tau(t)X\lambda_\tau(t) = \lim_\alpha \lambda_\tau(x_\alpha)$. By Lemma 4.9,

$$\pi_\tau(t)X\lambda_\tau(t) = \lim_\alpha \frac{1}{2}(\lambda_\tau(x_\alpha) + J\lambda_\tau(x_\alpha)) = \lim_\alpha \frac{1}{2}\lambda_\tau(x_\alpha + x_\alpha^+). \tag{4.6}$$

Thus we may assume, as we do henceforth, that the net $\{x_\alpha\}$ satisfies $x_\alpha^+ = x_\alpha$.

We note that the net $\{\rho_\tau(y)\lambda_\tau(x_\alpha)\}$ is Cauchy. This is because for any two indices α,β,

$$\begin{aligned} ||\rho_\tau(y)\lambda_\tau(x_\alpha) - \rho_\tau(y)\lambda_\tau(x_\beta)|| &= ||\pi_\tau(x_\alpha)\lambda_\tau(y) - \pi_\tau(x_\beta)\lambda_\tau(y)|| \\ &= ||(\pi_\tau(x_\alpha) - \pi_\tau(x_\beta))\lambda_\tau(y)||, \end{aligned} \tag{4.7}$$

and $\{\pi_\tau(x_\alpha)\} \subset L^+(\mathcal{D}_\tau) \cap \mathrm{B}(\mathcal{H}_\tau)$ converges strongly to the identity of $\mathrm{B}(\mathcal{H}_\tau)$. Hence, since $\lim_\alpha \rho_\tau(y)\lambda_\tau(x_\alpha)$ exists, as it is just seen, $\lim_\alpha \lambda_\tau(x_\alpha) = \pi_\tau(t)X\lambda_\tau(t)$, and $\pi_\tau(t)X\lambda_\tau(t)$ is in the domain of $\rho_\tau(y)$, it follows that

$$\lim_\alpha \rho_\tau(y)\lambda_\tau(x_\alpha) = \rho_\tau(y)\lim_\alpha \lambda_\tau(x_\alpha) = \rho_\tau(y)\pi_\tau(t)X\lambda_\tau(t) \tag{4.8}$$

because $\rho_\tau(y)$ is closed.

Let $y,z \in \mathcal{G}_\tau$ and $A \in \pi_\tau(\mathcal{M})'_\sigma$. Then,

$$\begin{aligned} &\langle \pi_\tau(t)X\pi_\tau(t)\lambda_\tau(y), A^+\lambda_\tau(z)\rangle_\tau \\ &\quad = \langle \pi_\tau(t)X\rho_\tau(y)\lambda_\tau(t), A^+\lambda_\tau(z)\rangle_\tau \\ &\quad = \langle \pi_\tau(t)\rho_\tau(y)X\lambda_\tau(t), A^+\lambda_\tau(z)\rangle_\tau, \quad \text{since } t \in \mathcal{G}_\tau \cap \mathcal{M}_+,\ X \in \rho_\tau(\mathcal{M})'_c, \\ &\quad = \langle \rho_\tau(y)\pi_\tau(t)X\lambda_\tau(t), A^+\lambda_\tau(z)\rangle_\tau \\ &\quad = \lim_\alpha \langle \rho_\tau(y)\lambda_\tau(x_\alpha), A^+\lambda_\tau(z)\rangle_\tau \\ &\quad = \lim_\alpha \langle \pi_\tau(x_\alpha)\lambda_\tau(y), A^+\lambda_\tau(z)\rangle_\tau \\ &\quad = \lim_\alpha \langle A\lambda_\tau(y), \pi_\tau(x_\alpha)^+\lambda_\tau(z)\rangle_\tau, \quad \text{since } A \in \pi_\tau(\mathcal{M})'_\sigma, \\ &\quad = \lim_\alpha \langle A\lambda_\tau(y), \rho_\tau(z)\lambda_\tau(x_\alpha)\rangle_\tau, \quad \text{since } x_\alpha^+ = x_\alpha, \\ &\quad = \langle A\lambda_\tau(y), \rho_\tau(z)\pi_\tau(t)X\lambda_\tau(t)\rangle_\tau \\ &\quad = \langle A\lambda_\tau(y), \pi_\tau(t)X\rho_\tau(z)\lambda_\tau(t)\rangle_\tau \\ &\quad = \langle A\lambda_\tau(y), \pi_\tau(t)X\pi_\tau(t)\lambda_\tau(z)\rangle_\tau \end{aligned} \tag{4.9}$$

showing that $\pi_\tau(t)X\pi_\tau(t)$ lies in $\pi_\tau(\mathcal{M})''_{\sigma\sigma}$. □

Proof of Theorem 4.6. As τ is semifinite, there is a net $\{t_\alpha\} \subset \mathcal{G}_\tau \cap \mathcal{M}_+$, satisfying $\{\pi_\tau(t_\alpha)\} \subset L^+(\mathcal{D}_\tau) \cap \mathrm{B}(\mathcal{H})$ and $\|\pi_\tau(t_\alpha)\| \le 1$ for each α, such that $\{\pi_\tau(t_\alpha)\}$ converges strongly to

the identity element of $\mathrm{B}(\mathcal{H}_\tau)$. Hence,

$$\text{weak-}\lim_\alpha \pi_\tau(t_\alpha)X\pi_\tau(t_\alpha) = X, \tag{4.10}$$

since, for arbitrary $\eta \in \mathcal{D}_\tau$,

$$\begin{aligned}
&\lim_\alpha \langle \eta, \pi_\tau(t_\alpha)X\pi_\tau(t_\alpha)\lambda_\tau(y)\rangle_\tau \\
&\quad = \lim_\alpha \langle \pi_\tau(t_\alpha)^+\eta, X\pi_\tau(t_\alpha)\lambda_\tau(y)\rangle_\tau = \lim_\alpha \langle \pi_\tau(t_\alpha)\eta, X\pi_\tau(t_\alpha)\lambda_\tau(y)\rangle_\tau \\
&\quad = \left\langle \lim_\alpha \pi_\tau(t_\alpha)\eta, \lim_\alpha X\pi_\tau(t_\alpha)\lambda_\tau(y)\right\rangle_\tau = \left\langle \eta, \lim_\alpha X\pi_\tau(t_\alpha)\lambda_\tau(y)\right\rangle_\tau \\
&\quad = \lim_\alpha \langle \eta, X\pi_\tau(t_\alpha)\lambda_\tau(y)\rangle_\tau = \lim_\alpha \langle X\eta, \pi_\tau(t_\alpha)\lambda_\tau(y)\rangle_\tau \\
&\quad = \left\langle X\eta, \lim_\alpha \pi_\tau(t_\alpha)\lambda_\tau(y)\right\rangle_\tau = \langle X\eta, \lambda_\tau(y)\rangle_\tau = \langle \eta, X\lambda_\tau(y)\rangle_\tau,
\end{aligned} \tag{4.11}$$

where use has been made of the facts that the net $\{\pi_\tau(t_\alpha)\}$ strongly converges to the identity map and the net $\{X\pi_\tau(t_\alpha)\lambda_\tau(y)\}$ is Cauchy in $\mathcal{H}_\tau$ for each $y \in \mathcal{G}_\tau$, since X is bounded.

Hence,

$$\lim_\alpha \langle A^+\lambda_\tau(z), \pi_\tau(t_\alpha)X\pi_\tau(t_\alpha)\lambda_\tau(y)\rangle_\tau = \langle A^+\lambda(z), X\lambda_\tau(y)\rangle_\tau \tag{4.12}$$

for arbitrary $A \in \pi_\tau(\mathcal{M})'_\sigma$ and $y, z \in \mathcal{G}_\tau$, whence, by Lemma 3.9,

$$\lim_\alpha \langle \pi_\tau(t_\alpha)X\pi_\tau(t_\alpha)\lambda_\tau(z), A\lambda_\tau(y)\rangle_\tau = \langle A^+\lambda_\tau(z), X\lambda_\tau(y)\rangle_\tau, \tag{4.13}$$

showing that

$$\langle X\lambda_\tau(z), A\lambda_\tau(y)\rangle_\tau = \langle A^+\lambda_\tau(z), X\lambda_\tau(y)\rangle_\tau. \tag{4.14}$$

Hence, $X \in \pi_\tau(\mathcal{M})''_{\sigma\sigma}$. This shows that

$$\rho_\tau(\mathcal{M})'_c \subset \pi_\tau(\mathcal{M})''_{\sigma\sigma}. \tag{4.15}$$

By Lemma 4.8,

$$\rho_\tau(\mathcal{M}) \subset \pi_\tau(\mathcal{M})'_\sigma, \tag{4.16}$$

whence

$$\pi_\tau(\mathcal{M})''_{\sigma c} \subset \rho_\tau(\mathcal{M})'_c. \tag{4.17}$$

Combining (4.15) and (4.17) yields

$$\pi_\tau(\mathcal{M})''_{\sigma c} \subset \rho_\tau(\mathcal{M})'_c \subset \pi_\tau(\mathcal{M})''_{\sigma\sigma} \subseteq \pi_\tau(\mathcal{M})''_{c\sigma}, \tag{4.18}$$

which is (ii) of the theorem. From this relation, by intersecting with $L^+(\mathcal{D}_\tau) \cap \mathrm{B}(\mathcal{H}_\tau)$, we get

$$\pi_\tau(\mathcal{M})''_{\sigma c} \subset \rho_\tau(\mathcal{M})'_c \subset \pi_\tau(\mathcal{M})''_{c\sigma} \cap L^+(\mathcal{D}_\tau) \cap \mathrm{B}(\mathcal{H}_\tau) = \pi_\tau(\mathcal{M})''_{cc}, \tag{4.19}$$

whence

$$\pi_\tau(\mathcal{M})''''_{cc\sigma} \subset \rho_\tau(\mathcal{M})''_{c\sigma} \subset \pi_\tau(\mathcal{M})''''_{\sigma c\sigma}. \tag{4.20}$$

This is (iii) of the theorem.

Similarly, the relation

$$\pi_\tau(\mathcal{M}) \subset \rho_\tau(\mathcal{M})'_\sigma \tag{4.21}$$

obtained in Lemma 4.8 implies that

$$\rho_\tau(\mathcal{M})'_{\sigma c} \subset \pi_\tau(\mathcal{M})'_c \tag{4.22}$$

while (4.15) implies that

$$\pi_\tau(\mathcal{M})'_\sigma \subset \rho_\tau(\mathcal{M})''_{c\sigma}. \tag{$*$}$$

Since $\pi_\tau(\mathcal{M})'_c \subset \pi_\tau(\mathcal{M})'_\sigma$, $(*)$ implies that

$$\pi_\tau(\mathcal{M})'_c \subset \rho_\tau(\mathcal{M})''_{c\sigma}. \tag{4.23}$$

Combining (4.22) and (4.23) yields

$$\rho_\tau(\mathcal{M})''_{\sigma c} \subset \pi_\tau(\mathcal{M})'_c \subset \rho_\tau(\mathcal{M})''_{c\sigma}, \tag{4.24}$$

which is (iv) of the theorem. From this relation, by intersecting with $L^+(\mathcal{D}_\tau) \cap \mathrm{B}(\mathcal{H}_\tau)$, we get

$$\rho_\tau(\mathcal{M})''_{\sigma c} \subset \pi_\tau(\mathcal{M})'_c \subset \rho_\tau(\mathcal{M})''_{c\sigma} \cap L^+(\mathcal{D}_\tau) \cap \mathrm{B}(\mathcal{H}_\tau) = \rho_\tau(\mathcal{M})''_{cc}, \tag{4.25}$$

whence

$$\rho_\tau(\mathcal{M})''''_{cc\sigma} \subset \pi_\tau(\mathcal{M})''_{c\sigma} \subset \rho_\tau(\mathcal{M})''''_{\sigma c\sigma}. \tag{4.26}$$

This is (v) of the theorem, thus completing proof of the theorem. □

Definition 4.11. A unital partial O^*-algebra $\mathcal{M}$ on $\mathcal{D}$ will be called a *partial W^*-algebra* if $\mathcal{M}$ is t_s^*-closed and $\mathcal{M}''_{c\sigma} = \mathcal{M}$.

Remark 4.12. (i) The notion of a *partial W^*-algebra* is a generalization of the notion of a *W^*-algebra*. In view of the existence of a multiplicity of commutants and higher commutants of a given partial O^*-algebra, several generalizations of the notion of *W^*-algebra* are possible. For example, in [5, 16], a *partial GW^*-algebra* $\mathcal{M}$ is defined as a partial O^*-algebra $\mathcal{M}$ on $\mathcal{D}$ which satisfies $\mathcal{M}''_{w\sigma} = \mathcal{M}$ (and some other conditions), where $\mathcal{M}'_w$ is the *weak bounded commutant* [5] of $\mathcal{M}$. Every partial GW^*-algebra is a quasi-$*$-algebra [5] because it always contains a W^*-algebra that is t_s^*-dense in it.

The notion of a partial W^*-algebra introduced in this paper appears natural, as it is not merely a quasi-$*$-algebra [5]. A detailed study of the connection between partial W^*-algebras and partial GW^*-algebras is ongoing.

(ii) Partial W^*-algebras may be classified as follows by means of the type of bitraces that are defined on them.

Definition 4.13. A partial W^*-algebra $\mathcal{M}$ on $\mathcal{D}$ will be called

(i) *finite* if there is a faithful, normal, regular finite bitrace on $\mathcal{M}$;

(ii) *semifinite* if there is a faithful, normal, regular semifinte bitrace on $\mathcal{M}$;

(iii) *properly infinite* if there is no nonzero normal regular finite bitrace on $\mathcal{M}$;

(iv) *purely infinite* if there is no nonzero normal regular semifinite bitrace on $\mathcal{M}$.

Remark 4.14. A systematic study of these classes of partial W^*-algebra will be pursued elsewhere.

Acknowledgments

This work was done during my visit as a Senior Associate to the *Abdus Salam International Centre for Theoretical Physics* (ICTP), Trieste, Italy. I am grateful to the *Swedish International Development Cooperation Agency* for providing the funds for my visit, and also to the Director of the ICTP, Professor K. R. Sreenivasan, the Head of the Mathematics Section of the ICTP, Professor Le Dung Trang, and other staff of the ICTP, for their hospitality. Finally, I thank the referees for helpful and stimulating comments.

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