Research Article

An L^p - L^q -Version of Morgan's Theorem for the n -Dimensional Euclidean Motion Group

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We establish an L^p - L^q -version of Morgan's theorem for the group Fourier transform on the n -dimensional Euclidean motion group $M(n)$.

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1. Introduction

An aspect of uncertainty principle in real classical analysis asserts that a function f and its Fourier transform $\hat{f}$ cannot decrease simultaneously very rapidly at infinity. As illustrations of this, one has Hardy's theorem [1], Morgan's theorem [2], and Beurling-Hörmander's theorem [3–5]. These theorems have been generalized to many other situations; see, for example, [6–10].

In 1983, Cowling and Price [11] have proved an L^p - L^q -version of Hardy's theorem. An L^p - L^q -version of Morgan's theorem has been also proved by Ben Farah and Mokni [7].

To state the L^p - L^q -versions of Hardy's and Morgan's theorems more precisely, we propose the following.

Let $a, b > 0$, $p, q \in [1, +\infty]$, $\alpha \geq 2$, and β such that $1/\alpha + 1/\beta = 1$.

If we consider measurable functions f on $\mathbb{R}$ such that

$$e^{a|x|^\alpha} f \in L^p(\mathbb{R}), \quad e^{b|y|^\beta} \hat{f} \in L^q(\mathbb{R}), \quad (1.1)$$

we obtain the following.

- (i) If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin(\pi/2)(\beta - 1))^{1/\beta}$, then $f = 0$ a.e.
- (ii) If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} \leq (\sin(\pi/2)(\beta - 1))^{1/\beta}$, then one has infinitely many such f .

The case $\alpha = \beta = 2$, $p = q = +\infty$ corresponds to Hardy's theorem.

The case $\alpha = \beta = 2$, $1 \leq p, q < +\infty$ corresponds to the Cowling-Price theorem.

The case $\alpha > 2$, $p = q = +\infty$ corresponds to Morgan's theorem.

The case $\alpha > 2$, $1 \leq p, q < +\infty$ corresponds to the Ben Farah-Mokni theorem.

We remark that for each one of those cases there are further requirements for f if $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} = (\sin(\pi/2)(\beta - 1))^{1/\beta}$.

In this paper, we give an L^p - L^q -version of Morgan's theorem for the n -dimensional Euclidean motion group $M(n)$, $n \geq 2$.

We can note that for the motion group, theorems of Beurling and Hardy have been studied by Sarkar and Thangavelu [12]. For example, the condition in Theorem 1.1 below for $f = 0$ a.e. for the case $\alpha = 2$ follows from their work.

The motion group $M(n)$ is the semidirect product of $\mathbb{R}^n$ with $K = \text{SO}(n)$. As a set $M(n) = \mathbb{R}^n \times K$, and the group law is given by

$$(x, k)(x', k') = (x + k \cdot x', kk'), \quad (1.2)$$

here $k \cdot x'$ is the naturel action of K on $\mathbb{R}^n$. The Haar measure of $M(n)$ is $dx dk$, where dx is the Lebesgue measure on $\mathbb{R}^n$ and dk is the normalized Haar measure on K .

Denote by $\widehat{M}(n)$ the unitary dual of the motion group. The abstract Plancherel theorem asserts that there is a unique measure μ on $\widehat{M}(n)$ such that for all $f \in L^1(M(n)) \cap L^2(M(n))$,

$$\int_{M(n)} |f(x, k)|^2 dx dk = \int_{\widehat{M}(n)} \text{tr}(\pi(f)\pi(f)^*) d\mu(\pi), \quad (1.3)$$

where $\pi(f) = \int_{M(n)} f(x, k)\pi(x, k)dx dk$ is the group Fourier transform of f at $\pi \in \widehat{M}(n)$.

It is well known that μ is supported by the set of infinite-dimensional elements of $\widehat{M}(n)$, which is parametrized by $(r, \lambda) \in]0, \infty[\times \widehat{U}$, where $U = \text{SO}(n-1)$ is the subgroup of $\text{SO}(n)$ leaving fixed $\varepsilon_n = (0, \dots, 0, 1)$ in $\mathbb{R}^n$. As such an element $\pi_{r, \lambda}$ is realized in a Hilbert space H_λ , we note that for $f \in L^1(M(n)) \cap L^2(M(n))$, $\pi_{r, \lambda}(f)$ is a Hilbert-Schmidt operator on H_λ , moreover the restriction of the Plancherel measure on the part $]0, \infty[\times \{\lambda\}$ is given up to a constant depending only on n , by $r^{n-1}dr$.

For the analogue of Morgan's theorem on $M(n)$ we propose the following version, where we use the notation $\widehat{f}(r, \lambda) = \pi_{r, \lambda}(f)$.

THEOREM 1.1. *Let $p, q \in [1, +\infty]$, $a, b \in]0, +\infty[$, and α, β positive real numbers satisfying $\alpha > 2$ and $1/\alpha + 1/\beta = 1$.*

Suppose that f is in $L^2(M(n))$ such that

(i) $e^{a\|x\|^\alpha} f(x, k) \in L^p(M(n))$,

(ii) $e^{br^\beta} \|\widehat{f}(r, \lambda)\|_{\text{HS}} \in L^q(\mathbb{R}^+, r^{n-1}dr)$ for all fixed λ in $\widehat{U}$.

If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} > (\sin(\pi/2)(\beta - 1))^{1/\beta}$, then f is null a.e.

If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} \leq (\sin(\pi/2)(\beta - 1))^{1/\beta}$, then there are infinitely many such f .

This paper is organized as follows.

In Section 2, we give a description of the unitary dual of the n -dimensional Euclidean motion group $M(n)$. Section 3 is devoted to the above version of Morgan's theorem for $M(n)$.

2. Description of the unitary dual of $M(n)$

We are going to describe the infinite-dimensional elements of $\widehat{M}(n)$, which are sufficient for the Plancherel formula. We start by some notations.

For any integer m , let $\langle \cdot, \cdot \rangle$ denote the Hermitian (resp., Euclidian) product on $\mathbb{C}^m$ (resp., on $\mathbb{R}^m$) and let $\| \cdot \|$ be the corresponding norm. For $y \neq 0$ in $\mathbb{R}^n$ let U_y be the stabilizer of y in K under its natural action on $\mathbb{R}^n$. U_y is conjugate to the subgroup $U = \text{SO}(n-1)$ of $\text{SO}(n)$ leaving fixed $\varepsilon_n = (0, \dots, 0, 1)$ in $\mathbb{R}^n$.

We remark that $\widehat{\mathbb{R}}^n$, the set of unitary characters of $\mathbb{R}^n$, is identified with $\mathbb{R}^n$. In fact any such character is of the form χ_y , $y \in \mathbb{R}^n$, and is defined for all $x \in \mathbb{R}^n$ by $\chi_y(x) = e^{i\langle x, y \rangle}$. The trivial character corresponds to $y = 0$.

To construct an infinite-dimensional irreducible unitary representation of the motion group $M(n)$, we use the following steps.

Step 1. Take a nontrivial element χ_y in $\widehat{\mathbb{R}}^n$. It is stabilized under the action of K by U_y .

Step 2. Take $\lambda \in \widehat{U}_y$ and consider $\chi_y \otimes \lambda$ as a representation of the semidirect product of $\mathbb{R}^n$ by U_y denoted by $\mathbb{R}^n \ltimes U_y$.

Step 3. Induce $\chi_y \otimes \lambda$ from $\mathbb{R}^n \ltimes U_y$ to $M(n)$ to obtain a representation $T_{y,\lambda}$ of $M(n)$.

We have then the following properties (see [13, 14] for details).

- (a) For $y \neq 0$ and any $\lambda \in \widehat{U}_y$, the representation $T_{y,\lambda}$ is unitary and irreducible.
- (b) Every infinite-dimensional irreducible unitary representation of $M(n)$ is equivalent to $T_{y,\lambda}$ for some y and λ as above.
- (c) The representations T_{y_1,λ_1} and T_{y_2,λ_2} are equivalent if and only if $\|y_1\| = \|y_2\|$ and λ_1 is equivalent to λ_2 under the obvious identification of U_{y_1} with U_{y_2} .

In particular, when $\|y\| = r > 0$, $T_{y,\lambda}$ is equivalent to $T_{r\varepsilon_n,\lambda}$, so the different classes of infinite-dimensional representations of $M(n)$ can be parametrized by $(r, \lambda) \in]0, \infty[\times \widehat{U}$. We use the notation $\pi_{r,\lambda}$ for $T_{r\varepsilon_n,\lambda}$ and for its equivalence class in $\widehat{M}(n)$. Let us make this representation explicit.

λ is an irreducible unitary representation of $U = \text{SO}(n-1)$, it is of finite dimension d_λ and acts on $\mathbb{C}^{d_\lambda}$. Let H_λ be the vector space of all measurable function $\psi : K \rightarrow \mathbb{C}^{d_\lambda}$ such that $\int_K \|\psi(k)\|^2 dk < \infty$ and $\psi(uk) = \lambda(u)(\psi(k))$ for all $u \in U, k \in K$. H_λ is a Hilbert space with respect to the inner product defined by

$$(\psi_1 | \psi_2) = d_\lambda \int_K \langle \psi_1(k), \psi_2(k) \rangle dk. \quad (2.1)$$

$\pi_{r,\lambda}$ acts on H_λ via

$$[\pi_{r,\lambda}(a, k)\psi](k_0) = e^{i\langle k_0^{-1} \cdot r\varepsilon_n, a \rangle} \psi(k_0 k), \quad \psi \in H_\lambda, \quad (2.2)$$

for $a \in \mathbb{R}^n, k, k_0 \in K$.

The Plancherel measure μ is then supported by the subset of $\widehat{M}(n)$ given by $\{\pi_{r,\lambda} : \lambda \in \widehat{U}, r \in \mathbb{R}^+\}$, and on each “piece” $\{\pi_{r,\lambda} : r \in \mathbb{R}^+\}$ with λ fixed in $\widehat{U}$, it is given by $C_n r^{n-1} dr$, where C_n is a constant depending only on n .

The Fourier transform of a function f in $L^1(M(n))$ is denoted as above by $\hat{f}$. It is defined for $(r, \lambda) \in]0, \infty[\times \hat{U}$ by

$$\hat{f}(r, \lambda) = \pi_{r, \lambda}(f) = \int_{\mathbb{R}^n} \int_K f(a, k) \pi_{r, \lambda}(a, k) dk da \quad (2.3)$$

(the integral being interpreted suitably, see [15]).

By the Plancherel theorem we know that for $f \in L^1(M(n)) \cap L^2(M(n))$, $\hat{f}(r, \lambda)$ is a Hilbert-Schmidt operator. Let $\|\hat{f}(r, \lambda)\|_{HS}$ be its Hilbert-Schmidt norm.

3. Morgan's theorem for the motion group

Before giving Morgan's theorem for the motion group $M(n)$, we state the following complex analysis lemma proved by Ben Farah and Mokni [7]. This lemma plays a crucial role in the proof of our main theorem.

LEMMA 3.1. Suppose $\rho \in]1, 2[$, $q \in [1, +\infty]$, $\sigma > 0$, and $B > \sigma \sin(\pi/2)(\rho - 1)$.

If g is an entire function on $\mathbb{C}$ satisfying the conditions

$$\begin{aligned} |g(x + iy)| &\leq \text{const } e^{\sigma|y|^\rho} \quad \text{for any } x, y \in \mathbb{R}, \\ e^{B|x|^\rho} g|_{\mathbb{R}} &\in L^q(\mathbb{R}), \end{aligned} \quad (3.1)$$

then $g = 0$.

We now give the L^p - L^q -version of Morgan's theorem.

THEOREM 3.2. Let $p, q \in [1, +\infty]$, $a, b \in]0, +\infty[$, and α, β positive real numbers satisfying $\alpha > 2$ and $1/\alpha + 1/\beta = 1$.

Suppose that f is a measurable function on $M(n)$ such that

- (i) $e^{a\|x\|^\alpha} f(x, k) \in L^p(M(n))$,
- (ii) $e^{br^\beta} \|\hat{f}(r, \lambda)\|_{HS} \in L^q(\mathbb{R}^+, r^{n-1} dr)$ for all fixed λ in $\hat{U}$.

If $(a\alpha)^{1/\alpha} (b\beta)^{1/\beta} > (\sin(\pi/2)(\beta - 1))^{1/\beta}$, then f is null a.e.

Proof. To prove that $f = 0$, we are going to prove that $\hat{f}(r, \lambda) = 0$. For this, it suffices to show that for fixed $\lambda \in \hat{U}$ and for any fixed K -finite vectors φ and ψ in H_λ , the condition $(a\alpha)^{1/\alpha} (b\beta)^{1/\beta} > (\sin(\pi/2)(\beta - 1))^{1/\beta}$ implies that $(\hat{f}(r, \lambda)\varphi | \psi) \equiv 0$ as a function of r and λ .

Let $\lambda \in \hat{U}$ and let φ, ψ be K -finite vectors in H_λ . We note that φ and ψ are continuous on K and thus bounded. On the other hand, for $r \in \mathbb{R}$,

$$(\hat{f}(r, \lambda)\varphi | \psi) = \int_K \int_{\mathbb{R}^n} f(x, k) (\pi_{r, \lambda}(x, k)\varphi | \psi) dx dk. \quad (3.2)$$

Let $\Phi_r(x, k) = (\pi_{r,\lambda}(x, k)\varphi \mid \psi)$ for $r \in \mathbb{R}$ and $(x, k) \in M(n)$. Then, by definition of $\pi_{r,\lambda}$, we have

$$\begin{aligned}\Phi_r(x, k) &= d_\lambda \int_K \langle (\pi_{r,\lambda}(x, k)\varphi)(k_0), \psi(k_0) \rangle dk_0 \\ &= d_\lambda \int_K e^{i(k_0^{-1} \cdot r \varepsilon_n x)} \langle \varphi(k_0 k), \psi(k_0) \rangle dk_0 \\ &= d_\lambda \int_K e^{i(r \varepsilon_n, k_0 x)} \langle \varphi(k_0 k), \psi(k_0) \rangle dk_0.\end{aligned}\quad (3.3)$$

Note that the integral on the right-hand side makes sense even if $r \in \mathbb{C}$. Hence, with (x, k) fixed, the function $\Phi_r(x, k)$ of the variable r extends to the whole complex plane. One can easily see that for fixed (x, k) , $z \mapsto \Phi_z(x, k)$ is an entire function on $\mathbb{C}$. Moreover, for $z \in \mathbb{C}$,

$$|\Phi_z(x, k)| \leq d_\lambda \int_K |e^{i(z \varepsilon_n, k_0 a)}| \cdot |\varphi(k_0 k)| \cdot |\psi(k_0)| dk_0. \quad (3.4)$$

Then

$$|\Phi_z(x, k)| \leq A \int_K e^{-\langle (\operatorname{Im} z) \varepsilon_n, k_0 x \rangle} dk_0, \quad (3.5)$$

where A is a constant depending only on λ , φ , and ψ . (Note that φ and ψ are continuous functions on K and hence are bounded.)

Using the fact that dk_0 is a normalized measure on K , we obtain

$$|(\Phi_z(x, k))| \leq A e^{|\operatorname{Im} z| \cdot \|x\|}. \quad (3.6)$$

By definition of $\Phi_z(x, k)$, we have

$$(\hat{f}(z, \lambda)\varphi \mid \psi) = \int_K \int_{\mathbb{R}^n} f(x, k) \Phi_z(x, k) dx dk. \quad (3.7)$$

Since f satisfies hypothesis (i) of Theorem 3.2 and $|(\Phi_z(x, k))| \leq A e^{|\operatorname{Im} z| \cdot \|x\|}$, we conclude that the function $r \mapsto (\hat{f}(r, \lambda)\varphi \mid \psi)$ can be extended to the whole of $\mathbb{C}$ and indeed it can be proved that the function

$$z \mapsto (\hat{f}(z, \lambda)\varphi \mid \psi) \quad \text{is an entire function.} \quad (3.8)$$

Further, from (3.6) and (3.7), we deduce that

$$|(\hat{f}(z, \lambda)\varphi \mid \psi)| \leq A \int_K \int_{\mathbb{R}^n} |f(x, k)| e^{|\operatorname{Im} z| \cdot \|x\|} dx dk. \quad (3.9)$$

Let $I =](b\beta)^{-1/\beta}(\sin(\pi/2)(\beta - 1))^{1/\beta}, (a\alpha)^{1/\alpha}[$, and $C \in I$. Applying the convex inequality $|ty| \leq (1/\alpha)|t|^\alpha + (1/\beta)|y|^\beta$ to the positive numbers $C\|x\|$ and $|\operatorname{Im} z|/C$, we obtain

$$|\operatorname{Im} z| \cdot \|x\| \leq \frac{C^\alpha}{\alpha} \|x\|^\alpha + \frac{1}{\beta C^\beta} |\operatorname{Im} z|^\beta, \quad (3.10)$$

thus

$$|(\hat{f}(z, \lambda)\varphi | \psi)| \leq Ae^{(1/\beta C^\beta)|\text{Im}z|^\beta} \int_K \int_{\mathbb{R}^n} |f(x, k)| e^{(C^\alpha/\alpha)\|x\|^\alpha} dx dk. \quad (3.11)$$

Then

$$|(\hat{f}(z, \lambda)\varphi | \psi)| \leq Ae^{(1/\beta C^\beta)|\text{Im}z|^\beta} \int_K \int_{\mathbb{R}^n} e^{a\|x\|^\alpha} |f(x, k)| e^{(C^\alpha/\alpha - a)\|x\|^\alpha} dx dk. \quad (3.12)$$

Using this inequality, hypothesis (i), the fact that dk is a normalized measure, and the inequality $a > c^\alpha/\alpha$, we obtain

$$|(\hat{f}(z, \lambda)\varphi | \psi)| \leq \text{const} e^{(1/\beta C^\beta)|\text{Im}z|^\beta}. \quad (3.13)$$

On the other hand, since $\pi_{-r, \lambda}$ and $\pi_{r, \lambda}$ are equivalent as representations of $M(n)$,

$$\|\hat{f}(-r, \lambda)\|_{HS} = \|\hat{f}(r, \lambda)\|_{HS}. \quad (3.14)$$

Hypothesis (ii) of Theorem 3.2 and the inequality (3.14) imply that the function

$$r \mapsto e^{br^\beta} \|\hat{f}(r, \lambda)\|_{HS} \quad \text{belongs to } L^q(\mathbb{R}), \quad (3.15)$$

thus

$$r \mapsto e^{br^\beta} (\hat{f}(r, \lambda)\varphi | \psi)_{L^2(H_\lambda)} \quad \text{belongs to } L^q(\mathbb{R}). \quad (3.16)$$

It is clear from (3.8), (3.13), (3.16) that the function $z \mapsto (\hat{f}(z, \lambda)\varphi, \psi)$ satisfies the hypothesis of Lemma 3.1, and so

$$(\hat{f}(z, \lambda)\varphi | \psi) \equiv 0 \quad (3.17)$$

as a function of z .

Since φ , ψ , and λ are arbitrary, then $\hat{f}(r, \lambda) \equiv 0$ for all $r \in \mathbb{R}_+$ and $\lambda \in \hat{U}$. Hence, by the Plancherel formula, we get that $f = 0$ a.e. This completes the proof of the theorem. $\square$

In order to prove that our version respects the analogy with Morgan's theorem, let us now establish the sharpness of the condition

$$(a\alpha)^{1/\alpha} (b\beta)^{1/\beta} > (\sin(\pi/2)(\beta - 1))^{1/\beta} \quad (3.18)$$

in Theorem 3.2.

PROPOSITION 3.3. *Let $p, q \in [1, +\infty]$, $a, b \in]0, +\infty[$, and α, β positive real numbers satisfying $\alpha > 2$ and $1/\alpha + 1/\beta = 1$.*

If $(a\alpha)^{1/\alpha} (b\beta)^{1/\beta} \leq (\sin(\pi/2)(\beta - 1))^{1/\beta}$, then there are infinitely many measurable functions on $M(n)$ satisfying

- (i) $e^{a\|x\|^\alpha} f(x, k) \in L^p(M(n))$,
- (ii) $e^{br^\beta} \|\hat{f}(r, \lambda)\|_{HS} \in L^q(\mathbb{R}^+, r^{n-1} dr)$ for any λ fixed in $\hat{U}$.

To prove this proposition, we use the following lemma for a, b, α, β as above.

LEMMA 3.4. *If $(a\alpha)^{1/\alpha}(b\beta)^{1/\beta} = (\sin(\pi/2)(\beta - 1))^{1/\beta}$, then for all $m \in \mathbb{R}$ and $m' = (2m + d(2 - \alpha))/(2\alpha - 2)$, there exists a nonzero measurable function on $M(n)$ satisfying*

- (i) $(1 + \|x\|)^{-m} e^{a\|x\|^\alpha} f \in L^\infty(M(n))$,
- (ii) $(1 + r)^{-m'} e^{br^\beta} \|\hat{f}(r, \lambda)\|_{HS} \in L^\infty(\mathbb{R}^+, r^{n-1} dr)$ for any fixed λ in $\hat{U}$.

Proof. We put for (x, k) in $M(n)$

$$f(x, k) = -i \int_C z^\nu e^{z^\eta - qA\|x\|^2 z} dz, \quad (3.19)$$

where $q = \alpha/(\alpha - 2)$, $A^\alpha = (1/4)((\alpha - 2)a)^2$, $\nu = (2m + 4 - \alpha)/2(\alpha - 2)$, and C is the path which lies in the half-plane $\operatorname{Re} z > 0$, and goes to infinity, in the directions $\arg z = \pm\theta_0$, $\pi/2q < \theta_0 < \pi/q$.

According to Morgan (see [2, page 190]), for $\|x\| \rightarrow \infty$, we have

$$f(x, k) \sim (\alpha - 2) \left(\frac{(\alpha - 2)a}{2} \right)^{m/\alpha} \sqrt{\left(\frac{\pi}{\alpha} \right)} \|x\|^m e^{-a\|x\|^\alpha}. \quad (3.20)$$

On the other hand, for λ fixed in $\hat{U}$, $(\hat{f}(r, \lambda)\varphi \mid \psi)$ is equal to

$$-id_\lambda \int_K \int_{\mathbb{R}^n} \int_C \int_K z^\nu e^{z^\eta - qA\|a\|^2 z} e^{i\langle r\varepsilon_n, k_0 a \rangle} \langle \varphi(k_0 k), \psi(k_0) \rangle dk_0 dz da dk, \quad (3.21)$$

which by a change of variables $x = k_0^{-1}a$ is equal to

$$-id_\lambda \int_K \int_{\mathbb{R}^n} \int_C \int_K z^\nu e^{z^\eta - qA\|x\|^2 z} e^{i\langle r\varepsilon_n, x \rangle} \langle \varphi(k_0 k), \psi(k_0) \rangle dk_0 dz dx dk. \quad (3.22)$$

Using this equality and Fubini's theorem, we obtain the following expression for $(\hat{f}(r, \lambda)\varphi \mid \psi)$:

$$-id_\lambda \left(\iint_K \langle \varphi(k_0 k), \psi(k_0) \rangle dk_0 dk \right) \int_C \int_{\mathbb{R}^n} z^\nu e^{z^\eta - qA\|x\|^2 z} e^{i\langle r\varepsilon_n, x \rangle} dx dz. \quad (3.23)$$

Since

$$\int_{\mathbb{R}^n} e^{-qA\|x\|^2 z} e^{i\langle k_0^{-1} r\varepsilon_n, x \rangle} dx = \left(\frac{\pi}{qAz} \right)^{n/2} e^{-r^2/4qaz}, \quad (3.24)$$

we deduce that

$$(\hat{f}(r, \lambda)\varphi \mid \psi) = -id_\lambda \left(\frac{\pi}{qA} \right)^{n/2} \left(\iint_K \langle \varphi(k_0 k), \psi(k_0) \rangle dk_0 dk \right) \int_C z^{\nu - n/2} e^{z^\eta - r^2/4aqz} dz. \quad (3.25)$$

Now, we fix an orthonormal basis $\{e_j; j \in \mathbb{N}\}$ of H_λ . Taking into account that $\hat{f}(r, \lambda)$ is a Hilbert-Schmidt operator, we then replace φ by e_i , ψ by e_j and take the sum on $i, j \in \mathbb{N}$ to

obtain

$$\|\hat{f}(r, \lambda)\|_{HS} = \text{const.} \left| \int_C z^{y-n/2} e^{z^q - r^2/4aqz} dz \right| \quad a.e. \quad (3.26)$$

Adapting the method of Morgan (see [2, page 191]), we obtain

$$\|\hat{f}(r, \lambda)\|_{HS} = O(r^{m'} e^{-br^\beta}) \quad (3.27)$$

with $m' = (2m + n(2 - \alpha))/(2\alpha - 2)$. We conclude by using the estimations (3.20) and (3.27). $\square$

Proof of Proposition 3.3. It suffices to prove the proposition for

$$(a\alpha)^{1/\alpha} (b\beta)^{1/\beta} = \left(\sin \frac{\pi}{2} (\beta - 1) \right)^{1/\beta}, \quad (3.28)$$

and the rest is a deduction. Let m be a real number verifying

$$m < \min \left(-\frac{n}{p}, \frac{n(1-\alpha)}{q} + \frac{n(\alpha-2)}{2} \right) \quad (3.29)$$

with the convention $1/r = 0$ when $r = \infty$. If $m' = (2m + n(2 - \alpha))/(2\alpha - 2)$, then $m' < -n/q$.

For fixed λ in $\hat{U}$, Lemma 3.4 gives a nonzero measurable function f on $M(n)$ satisfying the inequalities

$$\begin{aligned} e^{a\|x\|^\alpha} |f(x, k)| &\leq \text{const.} (1 + \|x\|)^m, \\ e^{br^\beta} \|\hat{f}(r, \lambda)\|_{HS} &\leq \text{const.} (1 + r)^{m'}. \end{aligned} \quad (3.30)$$

The conditions $m < -n/p$ and $m' < -n/q$ and the fact that dk is a normalized measure imply that $e^{a\|x\|^\alpha} f$ belongs to $L^p(M(n))$ and $e^{br^\beta} \|\hat{f}(r, \lambda)\|_{HS}$ belongs to $L^q(\mathbb{R}^+, C_n r^{n-1} dr)$ for fixed λ in $\hat{U}$. $\square$

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