Research Article

Viscosity Approximation Methods for Nonexpansive Nonself-Mappings in Hilbert Spaces

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Viscosity approximation methods for nonexpansive nonself-mappings are studied. Let C be a nonempty closed convex subset of Hilbert space H , P a metric projection of H onto C and let T be a nonexpansive nonself-mapping from C into H . For a contraction f on C and $\{t_n\} \subseteq (0, 1)$, let x_n be the unique fixed point of the contraction $x \mapsto t_n f(x) + (1 - t_n)(1/n) \sum_{j=1}^n (PT)^j x$. Consider also the iterative processes $\{y_n\}$ and $\{z_n\}$ generated by $y_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n)(1/(n+1)) \sum_{j=0}^n (PT)^j y_n, n \geq 0$, and $z_{n+1} = (1/(n+1)) \sum_{j=0}^n P(\alpha_n f(z_n) + (1 - \alpha_n)(TP)^j z_n), n \geq 0$, where $y_0, z_0 \in C, \{\alpha_n\}$ is a real sequence in an interval $[0, 1]$. Strong convergence of the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ to a fixed point of T which solves some variational inequalities is obtained under certain appropriate conditions on the real sequences $\{\alpha_n\}$ and $\{t_n\}$.

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1. Introduction

Throughout this paper, we denote the set of all nonnegative integers by $\mathbb{N}$. Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let C be a closed convex subset of H , and T a nonself-mapping from C into H . We denote the set of all fixed points of T by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$. T is said to be *nonexpansive mapping* if

$$\|Tx - Ty\| \leq \|x - y\| \quad (1.1)$$

for all $x, y \in C$. From condition on C , there is a mapping P from H onto C which satisfies

$$\|x - P_C x\| = \min_{y \in C} \|x - y\| \quad (1.2)$$

for all $x \in C$. This mapping P is said to be the *metric projection* from H onto C . We know that the metric projection is nonexpansive. Recall that a self-mapping $f : C \rightarrow C$ is a *contraction* on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| \quad \forall x, y \in C. \quad (1.3)$$

We use Π_C to denote the collection of all contractions on C . That is,

$$\Pi_C = \{f : f : C \rightarrow C \text{ a contraction}\}. \quad (1.4)$$

Note that each $f \in \Pi_C$ has a unique fixed point in C .

Given a real sequence $\{t_n\} \subseteq (0, 1)$ and a contraction $f \in \Pi_C$, define another mapping $T_n : C \rightarrow C$ by

$$T_n x = t_n f(x) + (1 - t_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x \quad \forall n \geq 1. \quad (1.5)$$

It is not hard to see that T_n is a contraction on C . Indeed, for $x, y \in C$, we have

$$\begin{aligned} \|T_n x - T_n y\| &= \left\| t_n (f(x) - f(y)) + (1 - t_n) \frac{1}{n} \left(\sum_{j=1}^n (PT)^j x - \sum_{j=1}^n (PT)^j y \right) \right\| \\ &\leq t_n \|f(x) - f(y)\| + (1 - t_n) \frac{1}{n} \sum_{j=1}^n \|(PT)^j x - (PT)^j y\| \\ &\leq t_n \alpha \|x - y\| + (1 - t_n) \|x - y\| \\ &= (1 - t_n(1 - \alpha)) \|x - y\|. \end{aligned} \quad (1.6)$$

For each n , let $x_n \in C$ be the unique fixed point of T_n . Thus x_n is the unique solution of fixed point equation

$$x_n = t_n f(x_n) + (1 - t_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n \quad \forall n \geq 1. \quad (1.7)$$

One of the purposes of this paper is to study the convergence of $\{x_n\}$ when $t_n \rightarrow 0$ as $n \rightarrow \infty$ in Hilbert spaces. Fix $u \in C$ and define a contraction S_n on C by

$$S_n x = t_n u + (1 - t_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x \quad \forall n \geq 1. \quad (1.8)$$

Let $s_n \in C$ be the unique fixed point of S_n . Thus

$$s_n = t_n u + (1 - t_n) \frac{1}{n} \sum_{j=1}^n (PT)^j s_n \quad \forall n \geq 1. \quad (1.9)$$

Shimizu and Takahashi [1] studied the strong convergence of the sequence $\{s_n\}$ defined by (1.9) for asymptotically nonexpansive mappings in Hilbert spaces.

We also study the convergence of the following iteration schemes: for $y_0, z_0 \in C$, compute the sequences $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$y_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n, \quad n \geq 0, \quad (1.10)$$

$$z_{n+1} = \frac{1}{n+1} \sum_{j=0}^n P(\alpha_n f(z_n) + (1 - \alpha_n)(TP)^j z_n), \quad n \geq 0, \quad (1.11)$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$, $f : C \rightarrow C$ is a contraction mapping on C , and P is the metric projection of H onto C . The first special case of (1.10) was considered by Shimizu and Takahashi [2] who introduced the following iterative process:

$$y_{n+1} = \alpha_n y + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n T^j y_n, \quad n \geq 0, \quad (1.12)$$

where y, y_0 are arbitrary (but fixed) and $\{\alpha_n\} \subseteq [0, 1]$ and then they proved the following theorem.

THEOREM 1.1 [2]. *Let C be a nonempty closed convex subset of a Hilbert space H , let T be a nonexpansive self-mapping of C such that $F(T)$ is nonempty, and let $P_{F(T)}$ be the metric projection from C onto $F(T)$. Let $\{\alpha_n\}$ be a real sequence which satisfies $0 \leq \alpha_n \leq 1$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let y and y_0 be element of C and let $\{y_n\}$ be the sequence defined by (1.12). Then $\{y_n\}$ converges strongly to $P_{F(T)}y$.*

The second special case of (1.10) and (1.11) was considered by Matsushita and Kuroiwa [3] who introduced the following iterative process:

$$y_{n+1} = \alpha_n y + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n, \quad n \geq 0, \quad (1.13)$$

$$z_{n+1} = \frac{1}{n+1} \sum_{j=0}^n P(\alpha_n z + (1 - \alpha_n)(TP)^j z_n), \quad n \geq 0,$$

where y, z, y_0, z_0 are arbitrary (but fixed) in C and $\{\alpha_n\} \subseteq [0, 1]$. More precisely, they proved the following theorem.

THEOREM 1.2 [3]. *Let H be a Hilbert space, C a closed convex subset of H , P the metric projection of H onto C , and let T be a nonexpansive nonself-mapping from C into H such that $F(T)$ is nonempty, and $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\{y_n\}$ and $\{z_n\}$ are defined by (1.13), respectively. Then $\{y_n\}$ and $\{z_n\}$ converge strongly to $P_{F(T)}y$ and $P_{F(T)}z$ in $F(T)$, respectively, where $P_{F(T)}$ is the metric projection from C onto $F(T)$.*

The purpose of this paper is twofold. First, we study the convergence of the sequence $\{x_n\}$ defined by (1.7) in Hilbert spaces. Second, we prove the strong convergence of the iteration schemes $\{y_n\}$ and $\{z_n\}$ defined by (1.10) and (1.11), respectively, in Hilbert

spaces. Our results extend and improve the corresponding ones announced by Shimizu and Takahashi [2], Matsushita and Kuroiwa [3], and others.

2. Preliminaries

For the sake of convenience, we restate the following concepts and results.

LEMMA 2.1. *Let H be a real Hilbert space, C a closed convex subset of H , and $P_C : H \rightarrow C$ the metric (nearest point) projection. Given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C. \quad (2.1)$$

Definition 1. A mapping $T : C \rightarrow H$ is said to satisfy *nowhere normal outward (NNO) condition* if and only if for each $x \in C$, $Tx \in S_x^C$, where $S_x = \{y \in H : y \neq x, Py = x\}$ and P is the metric projection from H onto C .

The following results were proved by Matsushita and Kuroiwa [4].

LEMMA 2.2 (see [4, Proposition 2, page 208]). *Let H be a Hilbert space, C a nonempty closed convex subset of H , P the metric projection of H onto C , and $T : C \rightarrow H$ a nonexpansive nonself-mapping. If $F(T)$ is nonempty, then T satisfies NNO condition.*

LEMMA 2.3 (see [4, Proposition 1, page 208]). *Let H be a Hilbert space, C a nonempty closed convex subset of H , P the metric projection of H onto C , and $T : C \rightarrow H$ a nonself-mapping. Suppose that T satisfies NNO condition. Then $F(PT) = F(T)$.*

LEMMA 2.4 (see [4]). *Let H be a Hilbert space, C a closed convex subset of H , and $T : C \rightarrow C$ a nonexpansive self-mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C such that $\{x_{n+1} - (1/(n+1)) \sum_{i=1}^{n+1} T^i x_n\}$ converges strongly to 0 as $n \rightarrow \infty$ and let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to x . Then x is a fixed point of T .*

Finally, the following two lemmas are useful for the proof of our main theorems.

LEMMA 2.5 (see [5]). *Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ that satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{a_n\}$ be a sequence of nonnegative real numbers such that for all $\epsilon > 0$, there exists an integer $N \geq 1$ such that for all $n \geq N$,*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \epsilon. \quad (2.2)$$

Then $\lim_{n \rightarrow \infty} a_n = 0$.

LEMMA 2.6 (see [5]). *Let H be a Hilbert space, C a nonempty closed convex subset of H , and $f : C \rightarrow C$ a contraction with coefficient $\alpha < 1$. Then*

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq (1 - \alpha)\|x - y\|^2, \quad x, y \in C. \quad (2.3)$$

Remark 2.7. As in Lemma 2.6, if f is a nonexpansive mapping, then

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq 0 \quad \forall x, y \in C. \quad (2.4)$$

3. Main results

THEOREM 3.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H , P the metric projection of H onto C , and $T : C \rightarrow H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{t_n\}$ be sequence in $(0, 1)$ which satisfies $\lim_{n \rightarrow \infty} t_n = 0$. Then for a contraction mapping $f : C \rightarrow C$ with coefficient $\alpha \in (0, 1)$, the sequence $\{x_n\}$ defined by (1.7) converges strongly to z , where z is the unique solution in $F(T)$ to the variational inequality*

$$\langle (I - f)z, x - z \rangle \geq 0, \quad x \in F(T), \quad (3.1)$$

or equivalently $z = P_{F(T)}f(z)$, where $P_{F(T)}$ is a metric projection mapping from H onto $F(T)$.

Proof. Since $F(T)$ is nonempty, it follows that T satisfies NNO condition by Lemma 2.2. We first show that $\{x_n\}$ is bounded. Let $q \in F(T)$. We note that

$$\begin{aligned} \|x_n - q\| &= \left\| t_n f(x_n) + (1 - t_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n - q \right\| \\ &\leq \left\| t_n (f(x_n) - q) + (1 - t_n) \frac{1}{n} \sum_{j=1}^n ((PT)^j x_n - (PT)^j q) \right\| \\ &\leq t_n \|f(x_n) - q\| + (1 - t_n) \|x_n - q\| \quad \forall n \geq 1. \end{aligned} \quad (3.2)$$

So we get

$$\begin{aligned} \|x_n - q\| &\leq \|f(x_n) - q\| \leq \|f(x_n) - f(q)\| + \|f(q) - q\| \\ &\leq \alpha \|x_n - q\| + \|f(q) - q\| \quad \forall n \geq 1. \end{aligned} \quad (3.3)$$

Hence

$$\|x_n - q\| \leq \frac{1}{1 - \alpha} \|f(q) - q\| \quad \forall n \geq 1. \quad (3.4)$$

This shows that $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{(1/n) \sum_{j=1}^n (PT)^j x_n\}$. Further, we note that

$$\begin{aligned} \left\| x_n - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n \right\| &= \left\| t_n f(x_n) + (1 - t_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n \right\| \\ &= t_n \left\| f(x_n) - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n \right\| \\ &\leq t_n \left(\|f(x_n)\| + \left\| \frac{1}{n} \sum_{j=1}^n (PT)^j x_n \right\| \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.5)$$

Thus $\{x_n - (1/n) \sum_{j=1}^n (PT)^j x_n\}$ converges strongly to 0. Since $\{x_n\}$ is a bounded sequence, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $z \in C$. By Lemmas 2.3 and 2.4, we have $z \in F(T)$. For each $n \geq 1$, since

$$x_n - z = t_n(f(x_n) - z) + (1 - t_n) \frac{1}{n} \sum_{j=1}^n ((PT)^j x_n - z), \quad (3.6)$$

we get

$$\begin{aligned} \|x_n - z\|^2 &= (1 - t_n) \left\langle \frac{1}{n} \sum_{j=1}^n ((PT)^j x_n - z), x_n - z \right\rangle + t_n \langle f(x_n) - z, x_n - z \rangle \\ &\leq (1 - t_n) \|x_n - z\|^2 + t_n \langle f(x_n) - z, x_n - z \rangle. \end{aligned} \quad (3.7)$$

Hence

$$\begin{aligned} \|x_n - z\|^2 &\leq \langle f(x_n) - z, x_n - z \rangle \\ &= \langle f(x_n) - f(z), x_n - z \rangle + \langle f(z) - z, x_n - z \rangle \\ &\leq \alpha \|x_n - z\|^2 + \langle f(z) - z, x_n - z \rangle. \end{aligned} \quad (3.8)$$

This implies that

$$\|x_n - z\|^2 \leq \frac{1}{1 - \alpha} \langle x_n - z, f(z) - z \rangle. \quad (3.9)$$

In particular, we have

$$\|x_{n_j} - z\|^2 \leq \frac{1}{1 - \alpha} \langle x_{n_j} - z, f(z) - z \rangle. \quad (3.10)$$

Since $x_{n_j} \rightharpoonup z$, it follows that

$$x_{n_j} \longrightarrow z \quad \text{as } j \longrightarrow \infty. \quad (3.11)$$

Next we show that $z \in C$ solves the variational inequality (3.1). Indeed, we note that

$$x_n = t_n f(x_n) + (1 - t_n) \frac{1}{n} \sum_{j=1}^n (PT)^j x_n \quad \forall n \geq 1, \quad (3.12)$$

we have

$$(I - f)x_n = -\frac{1 - t_n}{t_n} \left(x_n - \frac{1}{n} \sum_{j=1}^n (PT)^j x_n \right). \quad (3.13)$$

Thus for any $q \in F(T)$, we infer by Remark 2.7 that

$$\begin{aligned} \langle (I - f)x_n, x_n - q \rangle &= -\frac{1 - t_n}{t_n} \left\langle \left(I - \frac{1}{n} \sum_{j=1}^n (PT)^j \right) x_n, x_n - q \right\rangle \\ &= -\frac{1 - t_n}{t_n} \left\langle \left(I - \frac{1}{n} \sum_{j=1}^n (PT)^j \right) x_n - \left(I - \frac{1}{n} \sum_{j=1}^n (PT)^j \right) z, x_n - q \right\rangle \\ &\leq 0 \quad \forall n \geq 1. \end{aligned} \quad (3.14)$$

In particular

$$\langle (I - f)x_{n_j}, x_{n_j} - q \rangle \leq 0 \quad \forall j \geq 1. \quad (3.15)$$

Taking $j \rightarrow \infty$, we obtain

$$\langle (I - f)z, z - q \rangle \leq 0 \quad \forall q \in F(T), \quad (3.16)$$

or equivalent to $z = P_{F(T)}f(z)$ as required. Finally, we will show that the whole sequence $\{x_n\}$ converges strongly to z . Let another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ be such that $x_{n_k} \rightarrow z' \in C$ as $k \rightarrow \infty$. Then $z' \in F(T)$, it follows from the inequality (3.16) that

$$\langle (I - f)z, z - z' \rangle \leq 0. \quad (3.17)$$

Interchange z and z' to obtain

$$\langle (I - f)z', z' - z \rangle \leq 0. \quad (3.18)$$

Adding (3.17) and (3.18) and by Lemma 2.6, we get

$$(1 - \alpha)\|z - z'\|^2 \leq \langle z - z', (I - f)z - (I - f)z' \rangle \leq 0. \quad (3.19)$$

This implies that $z = z'$. Hence $\{x_n\}$ converges strongly to z . This completes the proof. $\square$

THEOREM 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H , P the metric projection of H onto C , and $T : C \rightarrow H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ which satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then for a contraction mapping $f : C \rightarrow C$ with coefficient $\alpha \in (0, 1)$, the sequence $\{y_n\}$ defined by (1.10) converges strongly to z , where z is the unique solution in $F(T)$ of the variational inequality (3.1).*

Proof. Since $F(T)$ is nonempty, it follows that T satisfies *NNO* condition by Lemma 2.2. We first show that $\{y_n\}$ is bounded. Let $q \in F(T)$. We note that

$$\begin{aligned}
 \|y_{n+1} - q\| &= \left\| \alpha_n f(y_n) + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n - q \right\| \\
 &\leq \alpha_n \|f(y_n) - q\| + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n \|(PT)^j y_n - q\| \\
 &\leq \alpha_n \|f(y_n) - f(q)\| + \alpha_n \|f(q) - q\| + (1 - \alpha_n) \|y_n - q\| \quad (3.20) \\
 &\leq \alpha_n \alpha \|y_n - q\| + \alpha_n \|f(q) - q\| + (1 - \alpha_n) \|y_n - q\| \\
 &= (1 - \alpha_n(1 - \alpha)) \|y_n - q\| + \alpha_n \|f(q) - q\| \\
 &\leq \max \left\{ \|y_n - q\|, \frac{1}{1 - \alpha} \|f(q) - q\| \right\} \quad \forall n \geq 1.
 \end{aligned}$$

So by induction, we get

$$\|y_n - q\| \leq \max \left\{ \|y_0 - q\|, \frac{1}{1 - \alpha} \|f(q) - q\| \right\}, \quad n \geq 0. \quad (3.21)$$

This shows that $\{y_n\}$ is bounded, so are $\{f(y_n)\}$ and $\{(1/(n+1)) \sum_{j=0}^n (PT)^j y_n\}$. We observe that

$$\begin{aligned}
 \left\| y_{n+1} - \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n \right\| &= \left\| \alpha_n f(y_n) + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n - \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n \right\| \\
 &= \alpha_n \left\| f(y_n) - \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n \right\| \\
 &\leq \alpha_n \left(\|f(y_n)\| + \left\| \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n \right\| \right). \quad (3.22)
 \end{aligned}$$

Hence $\{y_{n+1} - (1/(n+1)) \sum_{j=0}^n (PT)^j y_n\}$ converges strongly to 0. We next show that

$$\limsup_{n \rightarrow \infty} \langle z - y_n, z - f(z) \rangle \leq 0. \quad (3.23)$$

Let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$ such that

$$\lim_{j \rightarrow \infty} \langle z - y_{n_j}, z - f(z) \rangle = \limsup_{n \rightarrow \infty} \langle z - y_n, z - f(z) \rangle, \quad (3.24)$$

and $y_{n_j} \rightarrow q \in C$. It follows by Lemmas 2.3 and 2.4 that $q \in F(PT) = F(T)$. By the inequality (3.1), we get

$$\limsup_{n \rightarrow \infty} \langle z - y_n, z - f(z) \rangle = \langle z - q, z - f(z) \rangle \leq 0 \quad (3.25)$$

as required. Finally, we will show that $y_n \rightarrow z$. For each $n \geq 0$, we have

$$\begin{aligned}
 \|y_{n+1} - z\|^2 &= \|y_{n+1} - z + \alpha_n(z - f(z)) - \alpha_n(z - f(z))\|^2 \\
 &\leq \|y_{n+1} - z + \alpha_n(z - f(z))\|^2 + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle \\
 &= \left\| \alpha_n f(y_n) + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n (PT)^j y_n - (\alpha_n f(z) + (1 - \alpha_n)z) \right\|^2 \\
 &\quad + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle \\
 &= \left\| \alpha_n (f(y_n) - f(z)) + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n ((PT)^j y_n - z) \right\|^2 \\
 &\quad + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle \\
 &\leq \left[\alpha_n \|f(y_n) - f(z)\| + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n \|(PT)^j y_n - z\| \right]^2 \\
 &\quad + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle \\
 &\leq \left[\alpha_n \alpha \|y_n - z\| + (1 - \alpha_n) \frac{1}{n+1} \sum_{j=0}^n \|y_n - z\| \right]^2 \\
 &\quad + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle \\
 &= (1 - \alpha_n(1 - \alpha))^2 \|y_n - z\|^2 + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle \\
 &\leq (1 - \alpha_n(1 - \alpha)) \|y_n - z\|^2 + 2\alpha_n \langle y_{n+1} - z, f(z) - z \rangle.
 \end{aligned} \tag{3.26}$$

Now, let $\epsilon > 0$ be arbitrary. Then, by the fact (3.23), there exists a natural number N such that

$$\langle z - y_n, z - f(z) \rangle \leq \frac{\epsilon}{2} \quad \forall n \geq N. \tag{3.27}$$

From (3.26), we get

$$\|y_{n+1} - z\|^2 \leq (1 - \alpha_n(1 - \alpha)) \|y_n - z\|^2 + \alpha_n \epsilon. \tag{3.28}$$

By Lemma 2.5, the sequence $\{y_n\}$ converges strongly to a fixed point z of T . This completes the proof. $\square$

By using the same arguments and techniques as those of Theorem 3.2, we have also the following main theorem.

THEOREM 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H , P the metric projection of H onto C , and $T : C \rightarrow H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be sequence in $[0, 1]$ which satisfies $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then for a contraction mapping $f : C \rightarrow C$ with coefficient $\alpha \in (0, 1)$, the sequence $\{z_n\}$ defined by (1.11) converges strongly to z , where z is the unique solution in $F(T)$ of the variational inequality (3.1).*

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