Research Article

Generalizations of Morphic Group Rings

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An element $a$ in a ring $R$ is called left morphic if there exists $b \in R$ such that $1_R(a) = Rb$ and $1_R(b) = Ra$. $R$ is called left morphic if every element of $R$ is left morphic. An element $a$ in a ring $R$ is called left $\pi$-morphic (resp., left $G$-morphic) if there exists a positive integer $n$ such that $a^n$ (resp., $a^n \neq 0$) is left morphic. $R$ is called left $\pi$-morphic (resp., left $G$-morphic) if every element of $R$ is left $\pi$-morphic (resp., left $G$-morphic). In this paper, the $G$-morphic problem and $\pi$-morphic problem of group rings are studied.

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1. Introduction

An element $a$ in a ring $R$ is said to be left morphic if $R/Ra \cong 1_R(a)$, which is equivalent to that there exists $b \in R$ such that $1_R(a) = Rb$ and $1_R(b) = Ra$, where $1_R(a)$ denotes the left annihilator of $a$ in $R$. $R$ is called left morphic if every element of $R$ is left morphic. Right morphic elements and rings are defined analogously. Nicholson and Sánchez Campos introduced and investigated left morphic rings in [1] (see also [2–4] for more detailed discussion).

Left morphic rings are generalized to left $\pi$-morphic rings and left $G$-morphic rings by Huang and Chen [5]. An element $a \in R$ is called left $\pi$-morphic (resp., left $G$-morphic) if there exists a positive integer $n$ such that $a^n$ (resp., $a^n \neq 0$) is left morphic. $R$ is called left $\pi$-morphic (resp., left $G$-morphic) if every element of $R$ is left $\pi$-morphic (resp., left $G$-morphic). $R$ is called $\pi$-morphic (resp., $G$-morphic) if it is left and right $\pi$-morphic (resp., left and right $G$-morphic). Moreover, they find examples which show that left $\pi$-morphic rings are proper generalizations of left morphic rings, and left $G$-morphic elements need not be left morphic.

Example 1.1 [5, Example 2.13]. Let $R = F[x, \sigma]/(x^2) = \{a + xb \mid a, b \in F\}$, where $F$ is a field with an isomorphism $\sigma$ from $F$ to a subfield $\overline{F} \neq F$ and $cx = x\sigma(c)$ for all $c \in F$. 

S = R ⊕ R, then \( \lambda = (1, xb) \in S \) (where \( b \in F \), but \( b \notin \overline{F} \)) is left \( G \)-morphic, but not left morphic.

The question of when a group ring is morphic was studied by Chen et al. [6]. In this paper, we investigate when a group ring is \( \pi \)-morphic (resp., \( G \)-morphic). In Section 2, several general results about \( \pi \)-morphic and \( G \)-morphic group rings are obtained. In Section 3, necessary and sufficient conditions for \( RG \) to be left \( G \)-morphic are also given, where \( R = \mathbb{Z}_n \), \( G \) is a finite Abelian group. In particular, we prove that if \( G \) is a finite Abelian group or a finite \( p \)-group, \( r \geq 1 \), then \( \mathbb{Z}_p^r G \) is \( \pi \)-morphic.

All rings in this paper are associative rings with identity. Let \( R \) be a ring and let \( G \) be a group. We denote by \( RG \) the group ring of \( G \) over \( R \). The following concepts in group rings play very important roles in our discussion and will be used frequently later. For any element \( u = \sum a_i g_i \in RG \), where \( a_i \in R \), \( g_i \in G \), the augmentation of \( u \), denoted by \( \epsilon(u) \), is defined by \( \epsilon(u) = \sum a_i \). The augmentation ideal of \( RG \), denoted by \( \Delta(G) \), is defined by \( \Delta(G) = \{ u \in RG \mid \epsilon(u) = 0 \} \). If \( G \) is a cyclic group generated by \( g \), then \( \Delta(G) = RG(1 − g) \).

For any finite subgroup \( H \) of \( G \), \( \hat{H} \) is defined to be \( \hat{H} = \sum_{h \in H} h \). When \( H \) is a normal subgroup, \( \hat{H} \) is a central element in \( RG \). For any group element \( g \in G \) of finite order, define \( \hat{g} \) by \( \hat{g} = 1 + g + \ldots + g^{o(g)−1} \), where \( o(g) \) is the order of \( g \). It is not hard to verify that if \( o(g) < \infty \), then \( I_{RG}(1 − g) = R G \hat{g} \), and if \( |G| < \infty \), then \( I_{RG}(\hat{G}) = \Delta(G) \). So if \( G \) is a finite cyclic group, then \( \hat{G} \) is always left morphic in \( RG \). For more background knowledge about group rings, we refer readers to [7, 8].

2. General results

In this section, several general results about \( \pi \)-morphic and \( G \)-morphic group rings are given.

Theorem 2.1. Let \( R \) be a ring and let \( G \) be a locally finite group. If \( RG \) is left \( \pi \)-morphic (resp., left \( G \)-morphic), then \( R \) is left \( \pi \)-morphic (resp., left \( G \)-morphic).

Proof. For any \( a \in R \), since \( a \) is left \( \pi \)-morphic (resp., left \( G \)-morphic) in \( RG \), there exist a positive integer \( n \) (resp., \( a^n \neq 0 \)) and \( u \in RG \) such that \( I_{RG}(a^n) = RGu \) and \( I_{RG}(u) = R Ga^n \).

Let \( u = \sum_{i=1}^{n} a_i g_i \) and \( H = \langle g_1, \ldots, g_n \rangle \). Since \( G \) is a locally finite group, \( H \) is a finite group. Since \( a^n u = ua^n = 0 \), we have \( a^n \epsilon(u) = \epsilon(a^n u) = 0 \) and \( \epsilon(u) a^n = \epsilon(ua^n) = 0 \), where \( \epsilon(u) \) is the augmentation of \( u \). Thus \( Rb \subseteq I_R(a^n) \) and \( Ra^n \subseteq I_R(b) \), where \( b = \epsilon(u) \). Next we show that in fact, \( Rb = I_R(a^n) \) and \( Ra^n = I_R(b) \). So \( a \) is left \( \pi \)-morphic (resp., left \( G \)-morphic) in \( R \), and thus \( R \) is left \( \pi \)-morphic (resp., left \( G \)-morphic).

Let \( x \in I_R(a^n) \). Then \( x \in I_{RG}(a^n) = RGu \), so \( x = vu \), \( v \in RG \). Taking the augmentation on both sides, we obtain \( x = \epsilon(x) = \epsilon(vu) = \epsilon(v) \epsilon(u) = \epsilon(v) b \in Rb \). Therefore, \( I_R(a^n) \subseteq Rb \), and thus \( I_R(a^n) = Rb \). Next, let \( y \in I_R(b) \). Then \( yb = 0 \). Let \( \hat{H} = \sum_{h \in H} h \). Since \( u \in RH \), we have \( \hat{H}u = \epsilon(u) \hat{H} = b \hat{H} \). Thus \( y \hat{H}u = yb \hat{H} = 0 \), so \( y \hat{H} \in I_{RG}(u) = R Ga^n \). Hence \( y \hat{H} = \sum_{k \in K} ka^n \). Comparing the coefficients of the identity on both sides, we obtain that \( y = a_k a^n \in Ra^n \), and so \( I_R(b) \subseteq Ra^n \). This implies that \( I_R(b) = Ra^n \). Therefore, \( a \) is left \( \pi \)-morphic (resp., left \( G \)-morphic) and so is \( R \).

Corollary 2.2. If \( G = H \times K \) is a locally finite group and \( RG \) is left \( \pi \)-morphic (resp., left \( G \)-morphic), then \( RH \) and \( RK \) are both left \( \pi \)-morphic (resp., left \( G \)-morphic).
Proof. Note that $RG = R(H \times K) \cong (RH)K$. By Theorem 2.1, $RH$ is left $\pi$-morphic (resp., left $G$-morphic). Similarly $RK$ is left $\pi$-morphic (resp., left $G$-morphic).

Theorem 2.3. Let $G$ be a locally finite group. If $RH$ is left $\pi$-morphic (resp., left $G$-morphic) for every finite subgroup $H$ of $G$, then $RG$ is left $\pi$-morphic (resp., left $G$-morphic).

Proof. Let $u = \sum_{i=1}^{n} a_i g_i$. Now we show that $u$ is left $\pi$-morphic (resp., left $G$-morphic) in $RG$. Denote $H = \langle g_1, \ldots, g_n \rangle$. Since $G$ is locally finite, $H$ is a finite group. By the assumption, $RH$ is left $\pi$-morphic (resp., left $G$-morphic). Since $u \in RH$, there exist a positive integer $n$ (resp., $u^n \neq 0$) and $c \in RH$ such that $I_{RH}(u^n) = RHc$ and $I_{RH}(c) \subseteq RHu^n$. Since $u^n c = cu^n = 0$, we have $RGc \subseteq I_{RG}(u^n)$ and $RGu^n \subseteq I_{RG}(c)$. We next show that the other inclusions also hold.

Let $v \in I_{RG}(u^n)$ and let $\{1, g_1', g_2', \ldots\}$ be a left coset representative of $H$ in $G$. That is, $G = H \cup g_1' H \cup g_2' H \cup \cdots$. Now $v$ can be written as $v = \sum g'_i b_i$, where $b_i \in RH$. Since $0 = vu^n = \sum g'_i (b_i u^n)$ and $b_i u^n \in RH$, we obtain that $b_i u^n = 0$ for all $i$. So $b_i \in I_{RH}(u^n) = RHc$, and thus $b_i = c_i c$ for some $c_i \in RH$. It follows that $v = \sum g'_i b_i = \sum (g'_i c_i) c \in RGc$, so $I_{RG}(u^n) \subseteq RGc$, and thus $I_{RG}(c) = RGc$. Similarly, we can prove that $I_{RG}(c) = RGu^n$. This shows that $u$ is left $\pi$-morphic (resp., left $G$-morphic) in $RG$, and therefore $RG$ is left $\pi$-morphic (resp., left $G$-morphic).

Recall that a group $G$ is called a semidirect product of $H$ by $K$, denoted by $G = H \rtimes K$, if $H, K$ are subgroups of $G$ such that (1) $H \subseteq G$; (2) $HK = G$; (3) $H \cap K = 1$.

Theorem 2.4. Let $G = H \rtimes K$, $|H| < \infty$. If $RG$ is left $\pi$-morphic (resp., left $G$-morphic), then $RK$ is also left $\pi$-morphic (resp., left $G$-morphic).

Proof. We show that for any $a \in RK$, $a$ is left $\pi$-morphic (resp., left $G$-morphic) in $RK$. Since $a$ is left $\pi$-morphic (resp., left $G$-morphic) in $RG$, there exist a positive integer $n$ (resp., $a^n \neq 0$) and $u \in RG$ such that $I_{RG}(a^n) = RGu$ and $I_{RG}(u) = RGa^n$. Let $u = \sum u_i k_i$, where $u_i \in RH, k_i \in K$ (since $G = H \rtimes K$, the expression of $u$ is unique) and $a^n = \sum a_i k_i$ where $a_i \in R$. Denote $b = \sum c(u_i) k_i$, so $b \in RK$. We will show that $I_{RK}(a^n) = RKb$ and $I_{RK}(b) = RKA^n$. So $a$ is left $\pi$-morphic (resp., left $G$-morphic) in $RK$, and thus $RK$ is left $\pi$-morphic (resp., left $G$-morphic).

Let $\omega : G \to G/H$ be the natural group homomorphism. We extend $\omega$ to a ring homomorphism (still denote it by $\omega$). That is, $\omega : RG \to R(G/H)$ defined by $\omega(\sum a_i g_i) = \sum a_i \omega(g_i)$. Clearly, ker$(\omega) \cap RK = \{0\}$ and $\omega(v) = \varepsilon(v)$ for all $v \in RH$. Since $0 = a^nu, we have $0 = \omega(a^n) \omega(u) = \omega(a^n) \omega(\sum u_i k_i) = \omega(a^n) \sum \varepsilon(u_i) \omega(k_i) = \omega(a^n) \sum \varepsilon(u_i) \omega(k_i) = \omega(a^n)$. Since $a^n b \in RK$, we conclude that $a^n b = 0$. Similarly, $ba^n = 0$. This shows that $RKb \subseteq I_{RK}(a^n)$ and $RKA^n \subseteq I_{RK}(b)$. We next show that the other inclusions also hold.

Let $x \in I_{RK}(a^n)$. Then $x \in I_{RG}(a^n) = RGu$. So $x = vu$. Let $v = \sum v_j k_j$ and $c = \sum \varepsilon(v_j) k_j$, where $v_j \in RH, k_j \in K$. Then $\omega(x) = \omega(v) \omega(u) = \sum \varepsilon(v_j) \omega(k_j) \sum \varepsilon(u_i) \omega(k_i) = \omega(cb)$. Thus $x - cb \in \ker \omega \cap RK = \{0\}$. Therefore $x = cb \in RKb$. This shows that $I_{RK}(a^n) \subseteq RKb$, and thus $I_{RK}(a^n) = RKb$.

Let $y \in I_{RK}(b)$. Then $yb = 0$. Since $H \subseteq G, \hat{H} = \sum_{h \in H} h$ is central in $RG$. Now we have $y\hat{H}u = y\hat{H} \sum u_i k_i = y\sum \hat{H} \varepsilon(u_i) k_i = y\hat{H} b = yb\hat{H} = 0$. So $y\hat{H} \in I_{RG}(u) = RGa^n$. Thus $\hat{H}y = y\hat{H} = wa^n$, where $w = \sum h_j u_j, h_j \in H, u_j \in RK$. Hence
Theorem 2.7, this is impossible. So

\[ \sum h_j y = \hat{H} y = wa^u = \sum h_j (u_j a^u). \]  

(2.1)

Since \( H \cap K = \{1\} \), the expression of \( wa^u \) is unique. Comparing the coefficients of the identity \( h_0 = e \) in (2.1), we obtain \( y = u_0 a^u \in RPCa^u \). Thus \( I_{RK}(b) \subseteq RPCa^u \), and therefore \( I_{RK}(b) = RPCa^u \). \hfill \Box

From now on, we always assume that \( G \) is a finite group.

**Proposition 2.5.** Assume that \( p \) is a prime number and \( r > 1 \). If \( \mathbb{Z}_{p^r} G \) is left \( G \)-morphic, then \( p \) does not divide \( |G| \).

**Proof.** Assume that \( p \mid |G| \). Then there exists \( g \in G \) such that \( o(g) = p \). Let \( u = p^{r-1} \hat{G} \), where \( \hat{G} = \sum g \in G g \). Since \( u \) is left \( G \)-morphic in \( \mathbb{Z}_{p^r} G \), there exists a positive integer \( n \) such that \( u^n \) is left morphic in \( \mathbb{Z}_{p^r} G \). Since \( u^2 = 0 \), \( u \) is left morphic in \( \mathbb{Z}_{p^r} G \). By Chen et al. [6, Theorem 2.7], this is impossible. So \( p \nmid |G| \).

**Theorem 2.6.** Assume that \( p \) is a prime number and \( G \) is a finite \( p \)-group. \( \mathbb{Z}_{p^r} G \) is left \( G \)-morphic if and only if \( G \) is a cyclic group and \( r = 1 \).

**Proof.** “⇒” It follows from Proposition 2.5 that \( r = 1 \). Since \( R = \mathbb{Z}_p \) is a field and \( G \) is a finite \( p \)-group, \( RG \) is a local ring by Nicholson theorem [9]. Because \( RG \) is left Artinian, the Jacobson radical \( J(RG) \) is nilpotent. Since \( RG \) is left \( G \)-morphic, \( RG \) is left special by Huang and Chen [5, Theorem 2.8]. So it is left morphic. According to Chen et al. [6, Theorem 2.9], \( G \) is a cyclic group.

“⇐” If \( G = \langle g \rangle \), clearly \( \mathbb{Z}_p G \) is a special ring. Therefore it is left \( G \)-morphic. \hfill \Box

**Theorem 2.7.** Assume that \( p \) is a prime number and \( G \) is a finite \( p \)-group, \( r \geq 1 \), then \( \mathbb{Z}_{p^r} G \) is \( \pi \)-morphic.

**Proof.** Since \( R = \mathbb{Z}_{p^r} \) is local and \( G \) is a finite \( p \)-group, \( RG \) is a local ring by Nicholson’s theorem [9]. Because \( R \) is Artinian and \( G \) is a finite group, \( RG \) is Artinian by Connell [10, Theorem 1], and so the Jacobson radical \( J(RG) \) is nilpotent. According to Huang and Chen [5, Lemma 2.10], every element of \( RG \) is either nilpotent or invertible. So \( RG \) is \( \pi \)-morphic. \hfill \Box

**Remark 2.8.** By Theorem 2.6, when \( r > 1 \) and \( G \) is a finite \( p \)-group, \( \mathbb{Z}_{p^r} G \) is not left \( G \)-morphic, but by the above theorem, it is \( \pi \)-morphic.

### 3. Abelian group rings

In this section, we discuss when an Abelian group ring \( RG \) is left \( \pi \)-morphic (resp., left \( G \)-morphic).

**Lemma 3.1** [6, Lemma 3.1]. \((R_1 \oplus R_2 \oplus \cdots \oplus R_s)G \cong \oplus_{i=1}^s R_i G.\)

**Lemma 3.2.** If \( R = R_1 \oplus R_2 \oplus \cdots \oplus R_s \) is left \( \pi \)-morphic (resp., left \( G \)-morphic), then each \( R_i \) is left \( \pi \)-morphic (resp., left \( G \)-morphic).

**Proof.** For any \( r_i \in R_i, r = (0, \ldots, 0, r_i, 0, \ldots, 0) \in R \). Since \( R \) is left \( \pi \)-morphic (resp., left \( G \)-morphic), there exist \( u = (u_1, \ldots, u_{i-1}, u_i, \ldots, u_s) \in R \), where \( u_k \in R_k, k = 1, \ldots, s \), and
a positive integer $n$ (resp., $r^n \neq 0$) such that $I_R(u) = Rr^n$ and $I_R(r^n) = Ru$, so we have $I_{R_i}(u_i) = R_ir^n_i$ and $I_{R_i}(r^n_i) = R_iu_i$. Then $r_i$ is left $\pi$-morphically (resp., left $G$-morphically) in $R_i$, and thus $R_i$ is left $\pi$-morphically (resp., left $G$-morphically).

\[ \square \]

**Lemma 3.3.** Let $D$ be a division ring and $s \geq 2$. The following statements are equivalent:

1. $D(C_m \times \cdots \times C_m)$ is left $G$-morphically.
2. $D(C_m \times C_m)$ is left $G$-morphically for any $1 \leq i \neq j \leq s$.
3. At most one of $m_1, m_2, \ldots, m_s$ is not invertible in $D$.

**Proof.** We will prove $(3) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3)$.

$(3) \Rightarrow (1)$ We may assume that $m_1, \ldots, m_{s-1}$ are invertible in $D$. So $|C_{m_1} \times \cdots \times C_{m_{s-1}}| = m_1 \times \cdots \times m_{s-1}$ is invertible in $D$. By Maschke's theorem, $D(C_m \times \cdots \times C_m)$ is semisimple. It follows from [6, Lemma 3.5] that $D(C_m \times \cdots \times C_m)$ is strongly morphic, so it is $G$-morphically and (2.1) holds.

$(1) \Rightarrow (2)$ Note that $D(C_m \times \cdots \times C_m) \cong D(C_m \times C_m)(\prod_{i \neq j} C_{m_i})$ for any $1 \leq i \neq j \leq s$. It follows from Theorem 2.1 that $D(C_m \times C_m)$ is left $G$-morphically.

$(2) \Rightarrow (3)$ We prove it by contradiction. We may assume that $m_1, m_2$ are not invertible in $D$. Let char$(D) = p > 0$. By assumption, $p$ divides both $m_1$ and $m_2$. So we have $m_i = p^r_i t_i$, where $(t_i, p) = 1$, $r_i \geq 1$, $i = 1, 2$.

Note that $C_{m_1} \times C_{m_2} \cong (C_{p_1} \times C_{p_2}) \times (C_{t_1} \times C_{t_2})$, so $D(C_{m_1} \times C_{m_2}) \cong D(C_{p_1} \times C_{p_2}) \times (C_{t_1} \times C_{t_2})$. Since $D(C_{m_1} \times C_{m_2})$ is left $G$-morphically, $D(C_{p_1} \times C_{p_2})$ is left $G$-morphically by Theorem 2.1. Because $C_{p_1} \times C_{p_2}$ is a finite $p$-group, $D(C_{p_1} \times C_{p_2})$ is a local Artinian ring, so the Jacobson radical of this group ring is nilpotent. This ring is a left special ring, and then it is left morphically by Huang and Chen [5, Theorem 2.8]. Thus $C_{p_1} \times C_{p_2}$ must be cyclic, a contradiction.

\[ \square \]

**Proposition 3.4.** Let $G$ be a finite Abelian group and $r > 1$. Then $\mathbb{Z}_{p^r} G$ is $G$-morphically if and only if $(p, |G|) = 1$.

**Proof.** “$\Rightarrow$” By Chen et al. [6, Corollary 3.13], if $(p, |G|) = 1$, $\mathbb{Z}_{p^r} G$ is morphic, so it is $G$-morphically.

“$\Rightarrow$” By Proposition 2.5, if $r > 1$ and $\mathbb{Z}_{p^r} G$ is $G$-morphically, then $p \nmid |G|$, that is, $(p, |G|) = 1$.

\[ \square \]

**Theorem 3.5.** Let $G$ be a finite Abelian group. $\mathbb{Z}_n G$ is $G$-morphically if and only if for each prime number $p$ if $p \mid (n, |G|)$, then $p^2 \nmid n$ and the Sylow $p$-subgroup $G_p$ of $G$ is cyclic.

**Proof.** Let $G = C_{q_1} \times \cdots \times C_{q_m}$, $t_i \geq 1$ be a finite Abelian group and let $\alpha = q_1 \cdots q_m$. Suppose that $\mathbb{Z}_n G$ is $G$-morphically. Let $(n, |G|) = p_1^t_1 \cdots p_i^{t_i}$. If $r_i > 1$ for some $i$ (i.e., $p_i^2 \mid n$), then $n = p_i^{s_i} n_1$, where $s_i \geq r_i > 1$ and $(n_1, p_i) = 1$. Thus $\mathbb{Z}_n G \cong \mathbb{Z}_{p_i^{s_i}} G \oplus \mathbb{Z}_{n_1} G$. Since $\mathbb{Z}_{n_1} G$ is $G$-morphically, $\mathbb{Z}_{p_i^{s_i}} G$ is also $G$-morphically by Lemma 3.2. By Proposition 3.4, $(p_1, |G|) = 1$. However, $p_1 \mid (n, |G|)$. This leads to a contradiction. Thus $r_i \leq 1$ for all $i$. Next we show that $p_i^2 \nmid \alpha$. Otherwise, assume that $p_i^2 \mid \alpha$. There exists $k \neq l$ such that $q_k = q_l = p_i$. Hence $G \cong C_{q_k} \times C_{q_l} \times H$. Since $p_1 \mid n$ and $p_1^2 \mid n$, we have $n = p_1 n_1$ with $(p_1, n_1) = 1$. So $\mathbb{Z}_n G \cong$
\[ \mathbb{Z}_p \oplus \mathbb{Z}_{n_i} \mathbb{G}. \] By Lemma 3.2, \( \mathbb{Z}_p \mathbb{G} \) is \( \mathbb{G} \)-morphic. Since \( \mathbb{Z}_p \mathbb{G} \cong \mathbb{Z}_p \mathbb{G}(C_{q_i}^k \times C_{q_i}^l)H \), we conclude that \( \mathbb{Z}_p \mathbb{G}(C_{q_i}^k \times C_{q_i}^l) = \mathbb{Z}_p \mathbb{G}(C_{p_i}^r \times C_{p_i}^s) \) is \( \mathbb{G} \)-morphic. This contradicts the result of Theorem 2.6. Therefore, \( p^2 \nmid \alpha \), and thus \( \mathbb{G}_{p_i} \) is cyclic. \( \square \)

**Remark 3.6.** According to Proposition 3.4 and Theorem 3.5, the following group rings are not \( \mathbb{G} \)-morphic:

\[
\begin{align*}
\mathbb{Z}_4 \mathbb{C}_2, & \quad \mathbb{Z}_4 \mathbb{C}_4, \\
\mathbb{Z}_4(\mathbb{C}_2 \times \mathbb{C}_2), & \quad \mathbb{Z}_2(\mathbb{C}_2 \times \mathbb{C}_2), \\
\mathbb{Z}_2(\mathbb{C}_2 \times \mathbb{C}_4). &
\end{align*}
\] (3.1)

But by Theorem 2.7, the above group rings are all \( \pi \)-morphic.

**Lemma 3.7.** Let \( R \) be a ring and let \( \mathbb{G} \) be a group. If \( a \in R \) is left morphic in \( R \), then \( a \) is left morphic in \( RG \).

**Proof.** If \( a \in R \) is left morphic, there exists \( b \in R \) such that \( I_R(a) = Rb \) and \( I_R(b) = Ra \). Since \( ba = ab = 0 \), we have \( RG = I_{RG}(a) \) and \( RG = I_{RG}(b) \). We next show that the other inclusions also hold.

Let \( x \in I_{RG}(a), x = \sum r_j g_j \), where \( r_j \in R, g_j \in G \). Then \( \sum r_j g_j a = 0 = \sum (r_j a) g_j = 0 \), so all \( r_j a = 0 \). Thus \( r_j \in Rb \) and \( r_j = r'_j b, r'_j \in R \). Therefore, \( x = \sum (r'_j b) g_j = \sum r'_j g_j b \in RGb \). This shows that \( I_{RG}(a) \subseteq RGb \), and thus \( I_{RG}(a) = RGb \).

Using a similar proof, we can show that \( I_{RG}(b) \subseteq RGA \), and thus \( I_{RG}(b) = RGA \). So \( a \) is left morphic in \( RG \). \( \square \)

Recall that if \( n = p^\mu n_1, (n_1, p) = 1 \), we denote that \( p^\mu \| n \).

**Lemma 3.8.** Let \( p \) be a prime number, \( r \geq 1, p^r \| m, \) and \( 1 \leq n \leq m \).

1. If \((p, n) = 1\), then \( p^r \mid C_m^n \).
2. If \( p^r \| n, r \geq t, \) then \( p^{r-t} \mid C_m^n \).

**Proof.** Let \( m = m_1 p^r, (m_1, p) = 1 \). Then

\[
C_m^n = \frac{m(m-1) \cdots (m-(n-1))}{1 \cdots (n-1)n} = \frac{m}{n} C_{m-1}^{n-1} = \frac{m_1 p^r}{n_1} C_{m-1}^{n-1}.
\] (3.2)

(1) If \((p, n) = 1\), then \((p^r, n) = 1\), so \( p^r \mid C_m^n \).
(2) If \( p^r \| n, t \leq r\), then \( n = n_1 p^t\), where \((p, n_1) = 1\), so

\[
C_m^n = \frac{m_1 p^r}{n} C_{m-1}^{n-1} = \frac{m_1 p^r}{n_1 p^t} C_{m-1}^{n-1} = \frac{m_1 p^{r-t}}{n_1} C_{m-1}^{n-1}.
\] (3.3)

We have \( p^{r-t} \mid C_m^n n_1 \). Since \((p, n_1) = 1\), \((p^{r-t}, n_1) = 1\), so \( p^{r-t} \mid C_m^n \). \( \square \)

**Proposition 3.9.** Let \( p \) be a prime number and let \( \mathbb{G} \) be a finite Abelian group. If for some \( r, t \geq 1, x \in \mathbb{Z}_{p^r}(C_{p^t} \times \mathbb{G}) = \mathbb{Z}_{p^r}((g) \times \mathbb{G}), \) then \( x^{p^t} \in \mathbb{Z}_{p^r}(C_{p^{t-1}} \times \mathbb{G}) = \mathbb{Z}_{p^r}((g^p) \times \mathbb{G}) \).
Proof. For $x \in \mathbb{Z}_{p^r}(C_{p^r} \times G) = (\mathbb{Z}_{p^r}G)C_{p^r} = (\mathbb{Z}_{p^r}G)\langle g \rangle$, $x = r_0 + r_1g + \ldots + r_{p^r-1}g^{p^r-1}$, where $r_i \in \mathbb{Z}_{p^r}G$. Since

$$
(x_1 + x_2 + \ldots + x_i)^k
= \sum_{k_{i-2}=0}^{k_{i-2}} \sum_{k_{i-1}=0}^{k_{i-1}} \cdots \sum_{k_1=0}^{k_1} \sum_{k_0=0}^{k_0} C_{k_1}^k C_{k_2}^{k_1} \cdots C_{k_{i-1}}^{k_{i-2}} x_1^{k_1} x_2^{k_2} \cdots x_{i-2}^{k_{i-2}} x_{i-1}^{k_{i-1}} x_i^k,
$$

(3.4)

$$
x^{p^r} = (r_0 + r_1g + \ldots + r_{p^r-1}g^{p^r-1})^{p^r}
= \sum_{n_1=0}^{p^r-1} \cdots \sum_{n_{p^r-1}=0}^{p^r-1} C_{n_1}^{n_1} \cdots C_{n_{p^r-1}}^{n_{p^r-1}} r_0^{p^r-n_1} (r_1g)^{n_1-n_2} \cdots (r_{p^r-1}g^{p^r-1})^{n_{p^r-1}}.
\]
morphic in $\mathbb{Z}_{p^r} H$. Thus $x^n$ is morphic in $\mathbb{Z}_{p^r} G$ by Lemma 3.7. Hence $x$ is $\pi$-morphic in $\mathbb{Z}_{p^r} G$. □

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