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Research Article

Multismoothness in Banach Spaces

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In this paper, motivated by the results published by R. Khalil and A. Saleh in 2005, we study the notion of k-smooth points and the notion of k-smoothness, which are dual to the notion of k-rotundity. Generalizing these notions and combining smoothness with the recently introduced notion of unitary, we study classes of Banach spaces for which the vector space, spanned by the state space corresponding to a unit vector, is a closed set.

1. Introduction

Let X be a real or complex Banach space. Let $S(X)$ denote the unit sphere. $x_0 \in S(X)$ is said to be a smooth point if there is a unique $x^* \in S(X^*)$ such that $x^*(x_0) = 1$. Specific description of smooth points is known for classical function spaces and spaces of operators; see [1, 2]. It is also known that certain Calkin algebras (quotient spaces of bounded operators by the space of compact operators) fail to have smooth points; see [3, 4], and also [5, Section VI.5].

When x_0 is not a smooth point, consider the state space $S_{x_0} = \{x^* \in S(X^*) : x^*(x_0) = 1\}$, equipped with the weak*-topology. An interesting question in analysis is to study points for which S_{x_0} is a "large set." One such notion is that of a unitary. Analogous to the corresponding notion from the theory of C^*-algebras, x_0 is said to be a unitary if $\operatorname{span} S_{x_0} = X^*$. This notion was introduced and extensively studied in [6, 7]. A unitary is an extreme point and continues to be a unitary in X^{**} under the canonical embedding. Also if $x \in X$ is a unitary in X^{**}, then it is a unitary in X.

In [8] the authors study another notion of largeness of S_{x_0} by defining x_0 to be a multismooth point of order k if there exists a linearly independent set of cardinality k in S_{x_0}.

It should be noted that in all the results in [8] it is assumed that there are exactly k independent vectors in S_{x_0}. Thus when there are exactly k independent vectors (we will call such a set an exact independent set) in S_{x_0}, x_0 is a multismooth point of order k. We will call them as k-smooth points or smooth points of finite order without referring to k. It is easy to generate k-smooth points from smooth points. If $x \in X$ is a smooth point, then $(x,\dots,x)$ is a k-smooth point in the ℓ^∞ direct sum $\oplus_k^1 X$ and conversely if $(x,\dots,x)$ is a k-smooth point, then x is a smooth point.

In Section 2 of the paper, we study k-smooth points and relate it to the notion of k-rotundity from [9]. It turns out that for a nonreflexive space, if the fourth dual $X^{(4)}$ is k-rotund, then every k-smooth point of $S(X^*)$ attains its norm. Following the lead of [10] we give a characterization of k-smoothness in terms of unbounded-nested sequences of closed balls. We also discuss the stability of k-smoothness as one passes from a subspace to the whole space, with particular emphasis on the situation $X \subset X^{**}$ (we always consider the canonical embedding). We extend [8, Theorem 1.4] that describes k-smooth points of $C(K,X)$, the space of X-valued continuous functions on the compact set K, to the injective tensor product space $X \otimes_\epsilon Y$ when one of the summands is an L^1-predual space, that is, the dual of the space is isometric to $L^1(\mu)$ for a positive measure μ.

In Section 3, we consider a more general situation by calling a unit vector x_0 an ω-smooth point, if $\operatorname{span} S_{x_0}$ is a closed set. Thus this notion incorporates both the notion of k-smoothness and the extreme case of being a unitary. We show that for an L^1-predual space X, every unit vector is ω-smooth. For a large class of Banach spaces, that includes the ℓ^p $(1 < p < \infty)$ spaces, we show that the Calkin space $\mathcal{L}(X,Y) \mid \mathcal{K}(X,Y)$ has no k-smooth points, extending [8, Theorem 3.2] with a correct proof. We also show that in $\mathcal{K}(\ell^2)$ any ω-smooth point is of finite order. In the last section, we point out the relationship between ω-smooth points and the problem in convexity theory dealing with the linear span of a closed face being closed.

Our notation and terminology are fairly standard and can be found in [5, 11]. For a convex set K we denote the set of extreme points by $\partial_e K$ and for a Banach space X, X_1 denotes the closed unit ball.

2. k-smooth points

The following proposition and its corollary illustrate the relationship between k-smooth points and unitaries. As a consequence we get another proof of Theorem 4.1 from [8].

Proposition 2.1. *A unit vector x_0 is a k-smooth point if and only if* $\operatorname{span} S_{x_0}$ *is k-dimensional. Also the basis vectors can be taken to be the extreme points of S_{x_0}.*

Proof. Suppose x_0 is a k-smooth point. Let $f_1,\dots,f_m$ be linearly independent vectors in $\operatorname{span} S_{x_0}$. We may assume that for a finite set $B \subset S_{x_0}$, $\operatorname{span}\{f_1,\dots,f_m\} \subset \operatorname{span} B$. Since any generating set of a finite-dimensional space contains a basis, we have that B contains m independent vectors and hence $m \le k$. Therefore, $\operatorname{span} S_{x_0}$ is of dimension k. Now since S_{x_0} is a compact convex set in a finite-dimensional space, it is the convex hull of its extreme points. Thus the set of extreme points is also a generating set for $\operatorname{span} S_{x_0}$. Thus there are precisely k independent extreme points in S_{x_0}. □

The following corollary is now easy to prove.

Corollary 2.2. *Let x_0 be a k-smooth point of X. Let $Y = \{x \in X : f(x) = 0$ for all $f \in S_{x_0}\}$. Then $[x_0]$ is a unitary of the quotient space $X \mid Y$. In particular if x_0 is a k-smooth point of X and $k = \dim(X)$, then x_0 is a unitary. Conversely any unitary in a finite-dimensional space X is a k-smooth point with $k = \dim(X)$.*

Definition 2.3. A Banach space X is k-smooth if every point of $S(X)$ is a smooth point of order less than or equal to k.

Remark 2.4. Clearly any finite-dimensional space X is $k = \dim(X)$ smooth. For a compact set K, it is easy to see that $C(K)$ has a smooth point of finite order if and only if it has a smooth point and hence a G_δ set that is a singleton. Thus by taking a compact set K with only k many singleton's (e.g., by adding finitely many isolated points to $[0,1]^{[0,1]}$) that are G_δ's, we get an infinite-dimensional space whose smooth points of finite order are of order at most k.

There are known geometric conditions which put a restriction on the order of smoothness. We recall the definition of k-rotundity from [9].

Definition 2.5. A Banach space X of dimension $\dim(X) \geq k+1$ is said to be k-rotund, if given $k+1$ linearly independent points $\{x_i\}_{1\leq i\leq k+1} \subset S(X)$, $\|x_1 + \cdots + x_{k+1}\| < k+1$.

As remarked in [10], k-rotundity of X^* is equivalent to the fact that every face of X_1^* has at most k linearly independent vectors. As state spaces are faces of X_1^*, X has no n-smooth points for $n > k$ and thus X is k-smooth. The following proposition gives some consequences of higher ordered k-rotundity. We recall from [9] that k-rotundity implies $k+1$-rotundity.

Proposition 2.6. *If $X^{(4)}$ is k-rotund, then every k-smooth point of $S(X^*)$ attains its norm on $S(X)$.*

Proof. Let $x^* \in S(X^*)$ be nonnorm attaining. Let S_{x^*} be the state space in X^{**} and let S^{x^*} be the state space in $X^{(4)}$. Let $J : X \to X^{**}$ be the canonical embedding. At the next level we ignore the canonical embedding and consider $X^{**} \subset X^{(4)}$. By an observation of Dixmier (see [12]), since any $\Lambda \in S^{x^*}$ is not in $X, \Lambda \neq J^{**}(\Lambda)$. Also $J^{**}(\Lambda) \in S^{x^*}$ as $J^{**}(\Lambda) = \Lambda$ on X^*. Since J^{**} is also an embedding it maps independent vectors $\{\Lambda_i\}$ in S_{x^*} to independent vectors in S^{x^*} that are different from $\{\Lambda_i\}$.

Thus if $X^{(4)}$ is k-rotund, any k-smooth point of X^* is norm attaining. □

Remark 2.7. Let $X^{(4)}$ be a k-rotund space. Since k rotundity implies $k+1$ rotundity, k can be taken to be an even integer. Thus any vector in $S(X^*)$ that does not attain its norm can be smooth of order at most $k/2$.

As a first application of this notion, we extend the result of Beauzamy and Maurey [13], that characterizes smoothness in terms of unbounded nested sequences of closed balls. We only consider real Banach spaces. We follow the notation and terminology of [10], some of which we now recall.

Definition 2.8. A sequence $\{B_n = B(x_n, r_n)\}_{n\geq 1}$ of closed balls in X is said to be unbounded and nested if $r_n \uparrow \infty$ and $B_n \subset B_{n+1}$ for all n. It is called straight if $x_n = \lambda_n x_0$ for all n and for some $x_0 \in S(X)$.

It is known [10, Theorem 2.2] that the union of a straight unbounded nested sequence of balls is always a cone.

Theorem 2.9. *Suppose $\{B_n\}_{n\geq 1}$ is a straight unbounded nested sequence of balls in X. Suppose $X \neq B = \bigcup_1^\infty B_n$. Then the $x_0 \in S(X)$ associated with this sequence is k-smooth if and only if B is the intersection of exactly k half spaces determined by independent unit vectors that are bounded below on B.*

Proof. Since x_0 is k-smooth if and only if $-x_o$ is so, we may assume without loss of generality that $\lambda_n > 0$ for large n. Let $A = \{x^* \in S(X^*) : \inf x^*(B) > -\infty\}$. Thus as in the proof of [10, Theorem 2.2] we get that $A = S_{x_0}$. Therefore, if x_0 is k-smooth, let $\{x_i^*\}_{1\leq i\leq k} \subset A$ be a basis for $\operatorname{span} S_{x_0}$. Let $v \in X$ be such that $x_i^*(v) = \inf x_i^*(B)$ for $1 \leq i \leq k$. Now for any $x^* \in A$, it follows from [10, Lemma 2.6] that $\inf x^*(B) = \sum_1^k a_i \inf x_i^*(B) = \sum_1^k a_i x_i^*(v) = x^*(v)$ since $\sum_1^k a_i = 1$. Thus $B = \bigcap_1^k \{x \in X : x_i^*(x) > \inf x_i^*(B)\}$.

The converse part is also similarly proved by considering the comments made after the proof of [10, Theorem 2.2]. □

We recall that a closed subspace $Y \subset X$ is said to be a U-subspace if every $y^* \in Y^*$ has a unique norm-preserving extension in X^*. In particular a Banach space X is said to be Hahn-Banach smooth if X is a U-subspace of X^{**} under the canonical embedding (see [5, Chapter III]). It is well-known that $c_0 \subset \ell^\infty$ and for $1 < p < \infty$, $\mathcal{K}(\ell^p) \subset \mathcal{L}(\ell^p)$ are examples of this phenomenon.

Remark 2.10. If X is a Hahn-Banach smooth subspace, then since the state space of an $x \in S(X)$ remains the same in X^{**}, it is easy to see that x is k-smooth in X^{**} if and only if it is k-smooth point in X. We do not know a general local geometric condition to ensure that the state of a unit vector in X and its bidual remain the same.

Example 2.11. Let X be a smooth, nonreflexive Banach space such that X is an L-summand in its bidual under the canonical embedding (i.e., $X^{**} = X \oplus_1 M$, for a closed subspace M, see [5, Chapter IV]). The Hardy space H_0^1 is one such example (see [5, page 167]). Since X is nonreflexive, it is easy to see that when $X^{**} = X \oplus_1 M$, M is infinite dimensional. Now every unit vector x of X is a smooth point of X but for no k, x is a k-smooth point in X^{**}.

We next use the notion of an intersection property of balls, from [14], to establish a relation between k-smooth points in the subspace and the whole space in the case of U-subspaces. In the next two results we assume that X is a real Banach space.

Definition 2.12. Let $n \geq 3$. A closed subspace $M \subset X$ is said to have the $n \cdot X \cdot$-intersection property $(n \cdot X \cdot I \cdot P)$ if $\{B(a_i, r_i)\}_{1\leq i\leq n}$ are n closed balls in M with $\cap_1^n B(a_i, r_i) \neq \emptyset$ in X (when they are considered as closed balls in X) then $M \cap \cap_1^n B(a_i, r_i + \epsilon) \neq \emptyset$ for all $\epsilon > 0$.

We note that if X is an L^1-predual space, then for $n \geq 4$, X has the $n \cdot Y \cdot I \cdot P$ in any Y that isometrically contains X. To see this, let $\{B(a_i, r_i)\}_{1\leq i\leq n}$ be n closed balls in X with $\cap_1^n B(a_i, r_i) \neq \emptyset$ in Y. Let $\epsilon > 0$. These balls thus pairwise intersect in X. As X is an L^1-predual space, it follows from [11, Section 21, Theorem 6] that $X \cap \cap_1^n B(a_i, r_i + \epsilon) \neq \emptyset$.

Proposition 2.13. *Suppose $M \subset X$ has the $k \cdot X \cdot I \cdot P$ and M is a U-subspace. If $x \in M$ is a k-smooth point in X, then it is a k-smooth point in M.*

Proof. Let $\{x_i^*\}_{1\le i\le k} \subset S_x$ be a linearly independent set. Let $f_i = x_i^* \mid M$. Note that $\|x_i^*\| = 1 = \|f_i\|$. We claim that the f_i's are linearly independent. Suppose $\sum_1^k \alpha_i f_i = 0$ for some scalars α_i. By [14, Theorem 3.1] it follows that there exists norm preserving extensions $f_i' \in X^*$ of $\alpha_i f_i$ such that $\sum_1^k f_i' = 0$. But by the uniqueness of the extensions this implies $\sum_1^k \alpha_i x_i^* = 0$ and hence $\alpha_i = 0$ for $1 \le i \le k$. On the other hand if $\{g_i\}_{1\le i\le l}$ is any linearly independent set in the state space of x in M, the corresponding Hahn-Banach extensions are clearly linearly independent in S_x. Thus $l \le k$. □

Remark 2.14. In the absence of uniqueness of extensions, one can impose k-rotundity on X^*, to get a bound on the order of smoothness of a point of M when considered in X. Recall from [9] that under the assumption of k-rotundity, functionals in M^* have at most k-linearly independent extensions.

We next give another instance where k-smoothness is preserved between the space and its bidual. Let c denote the space of convergent sequences and c_0 the space of sequences converging to 0. We recall that since c_0 is a Hahn-Banach smooth subspace of its bidual ℓ^∞, smoothness properties are preserved.

Proposition 2.15. *Let $x = (x_1, x_2, \ldots) \in S(c)$ be a k-smooth point such that for a finite set A, $\operatorname{Sup}\{|x_i| : i \notin A\} < 1$. Then it is a k-smooth point in $c^{**} = \ell^\infty$.*

Proof. Considering $c = C(N \cup \{\infty\})$, we have from [8, Theorem 1.5] that there exists $\{l_i\}_{1\le i\le k} \subset A$ such that $|x_{l_i}| = 1$ for all i and $\sup\{|x_i| : i \notin \{l_i\}_{1\le i\le k}\} < 1$.

Now consider $x \in \ell^\infty = C(\beta(N))$, where $\beta(N)$ is the Stone-Čech compactification of N. Now for any $\tau \in \beta(N) - N$, $|\tau(x)| = \lim |x_n| < 1$, where the limit is taken along the ultrafilter τ. Therefore by [8, Theorem 1.5] again we get that x is a k-smooth point in ℓ^∞. □

Remark 2.16. We note that if $\lim x_n = 1$, then for any $\tau \in \beta(N) - N$, $\tau(x) = 1$. Thus such a point is not k-smooth in ℓ^∞.

We next consider k-smooth points in ℓ^∞-direct sums of Banach spaces.

We need the following lemmas. The first one follows from [8, Lemma 2.1].

Lemma 2.17. *Let $\{x_i\}_{1\le i\le n} \subset S(X)$ be such that $\|x_i \pm x_j\| \le 1$ for all $i \ne j$ and $\|\sum_1^n x_i\| = 1$. Then $x = \sum_1^n x_i$ is not a smooth point of order r for any $r \le n-1$.*

Lemma 2.18. *Suppose $X = M \oplus_\infty N$ and $x = (m,n) \in S(X)$ with $\|m\| = 1$ and $\|n\| < 1$. m is a k-smooth point in M if and only if x is a k-smooth point in X. If $X = \oplus_\infty X_i$ for Banach spaces $\{X_i\}_{1\le i\le n}$, then any vector $x = (x_1, \ldots, x_n) \in S(X)$ with $x_i \in S(X_i)$ is not an r-smooth point for $r \le n-1$.*

Proof. To prove the first part we only need to note that if $x^* = (m^*, n^*) \in S_x$, then $n^* = 0$. To see the second part observe that $x = e_1 + \cdots + e_n$ where $e_i = (0, \ldots, 0, x_i, 0, \ldots, 0)$ and $\|e_i \pm e_j\| \le 1$ for any $i \ne j$. □

We now consider ℓ^∞-sums over infinite index set I. We note that k-smooth points of finite order in ℓ^∞-sums over a finite index set can be characterized along the lines of [8, Theorem 1.4]. The condition in the theorem below is clearly satisfied in c_0-direct sum of Banach spaces. For any nonempty set $J \subset I$, let P_J denote the projection to the ℓ^∞-sum over J.

Theorem 2.19. *Let $\{X_i\}_{i\in I}$ be an infinite family of nonzero Banach spaces. Let $X = \oplus_\infty X_i$. A unit vector x is a k-smooth point if and only if there exists a finite set $J \subset I$ such that for $j \in J$ with $\|x_j\| = 1$, $\sup\{\|x_i\| : i \notin J\} = \|x - P_J(x)\| < 1$ and $P_J(x)$ is a k-smooth point in $\oplus_\infty\{X_j : j \in J\}$.*

Proof. Suppose $\|x - P_J(x)\| < 1$ and $\|x_j\| = 1$ for $j \in J$ and $P_J(x)$ is a k-smooth point. By decomposing X as an ℓ^∞-direct sum of sums over J and $I - J$, the conclusion follows from Lemma 2.18.

To prove the converse part, suppose there is no finite index set with the above norm condition. Then it is easy to get a partition $\{J_i\}_{1\le i\le k+1}$ of I such that $\|P_{J_i}(x)\| = 1$. Thus by Lemma 2.17, x cannot be a k-smooth point. When the norm condition is satisfied then by Lemma 2.18, we get that $P_J(x)$ is a k-smooth point in $\oplus_\infty\{X_j : j \in J\}$. □

The following theorem extends [8, Theorem 1.4] to injective tensor products by L^1-predual spaces. We recall that for a Banach space Y, $C(K, Y)$ can be identified with the injective tensor product space $C(K) \otimes_\epsilon Y$ and $C(K)^*$ is an $L^1(\mu)$-space, see [11, Chapter 7]. More generally for any Banach space X, Y let $KW^*(X^*, Y)$ denote the space of weak*-weak continuous linear operators, equipped with the operator norm. When Y is an L^1-predual, since Y^* has the metric approximation property, $X \otimes_\epsilon Y$ can be identified as the space $KW^*(X^*, Y)$, see [15]. A result [15, Theorem 4.2.1] of Ruess-Stegall identifies $\partial_e(X \otimes_\epsilon Y)_1^*$ as $\{x^* \otimes y^* : x^* \in \partial_e X_1^*, y^* \in \partial_e Y_1^*\}$. In order to make the arguments simple, in the following theorem, we assume that Y is a smooth space.

Theorem 2.20. *Let X be an L^1-predual space and Y a smooth Banach space. $T \in X \otimes_\epsilon Y = KW^*(Y^*, X)$ is a k-smooth point if and only if there exist precisely k independent vectors $\{x_i^*\}_{1\le i\le n} \subset \partial_e X_1^*$ where T^* attains its norm.*

Proof. Suppose T is a k-smooth point. Let $\{x_i^* \otimes y_i^*\}_{1\le i\le n} \subset \partial_e(X \otimes_\epsilon Y)_1^*$ be independent vectors in S_T. Since $\{x_i^*\}_{1\le i\le n} \subset \partial_e X_1^* = \partial_e L^1(\mu)_1$ are distinct, by the disjointness of the measure atoms, we conclude that this is an independent set. Thus it is easy to see that $\{y_i^*\}_{1\le i\le n}$ is also an independent set.

Now suppose $x^* \in \partial_e X_1^*$ is such that $\|T^*(x^*)\| = 1$. Let $y^* \in \partial_e Y_1^*$ be the unique vector such that $y^*(T^*(x^*)) = 1$. Thus $x^* \otimes y^* \in S_T \bigcap \partial_e(X \otimes Y)_1^*$. So by our assumption there exists scalars α_i such that $x^* \otimes y^* = \sum_1^n \alpha_i x_i^* \otimes y_i^*$. Since the L^1-norm is additive on distinct atoms, we see that $\sum_1^n |\alpha_i| = 1$. As $x^* \otimes y^*$ is an extreme point we conclude that $x^* \otimes y^* = x_i^* \otimes y_i^*$ for some i. Again since x^* and x_i^* cannot be distinct atoms, we have $x^* = x_i^*$ and thus we get $y^* = y_i^*$.

Conversely suppose that $\{x_i^*\}_{1\le i\le n} \subset \partial_e X_1^*$ is an exact independent set with $\|T^*(x_i^*)\| = 1$. Choose (unique) vectors $y_i^* \in \partial_e Y_1^*$ such that $(x_i^* \otimes y_i^*)(T) = 1$. Arguments similar to the ones given above can now be used to show that this is an exact independent set of k vectors. □

Let K be a compact convex set which is a Choquet simplex. Let $A(K,X)$ denote the space of X-valued affine continuous functions defined on K equipped with the supremum norm. As an application of the above theorem we have a description of k-smooth points of $A(K,X)$.

Corollary 2.21. *Let X be a smooth Banach space. $f \in A(K,X)$ is a k-smooth point if and only if there exist exactly k extreme points of K where f attains its norm.*

Proof. For a Choquet simplex K, $A(K)$ (real-valued functions) is an L^1-predual space (see [11, Chapter 7, Section 20]). We next note that in this case $A(K,X)$ can be identified with $A(K) \otimes_\epsilon X$. Let $\delta : K \to A(K)_1^*$ denote the canonical evaluation map. Let $\partial_e K$ denote the set of extreme points. For $f \in A(K,X)$, it is easy to see that $T = x^* \to x^* \circ f \in KW^*(X^*,A(K))$ and $\|f\| = \|T\|$. Also given $T \in KW^*(X^*,A(K))$, $f(k) = T^*(\delta(k)) \in A(K,X)$. It is well-known that $\partial_e A(K)_1^* = \delta(K) \bigcup -\delta(K)$. Thus the conclusion follows from the above theorem. □

Remark 2.22. Arguments similar to the ones given above can be used to describe smooth points of order k for complex function algebras (i.e., closed subspaces of $C(K)$ containing constants and separating points) with the help of [5, Theorem V.4.2]. It is also easy to extend such an argument to the case of $A \otimes_\epsilon X$, where A is a complex function algebra and X is a smooth space.

Remark 2.23. The authors in [8, Theorem 2.2] claim the analogue of the above theorem for $\ell^p \otimes_\epsilon \ell^q$ for $1 < p < \infty$. However, it is not clear to us how the independence of the set in $\partial_e(\ell^p \otimes_\epsilon \ell^q)_1^*$ is to be concluded from the corresponding statement about extremal (unit) vectors where the norm is attained.

3. ω-smooth points

We now introduce the notion of ω-smoothness without referring to the dimension of the state space so as to unify it with the notion of a unitary.

Definition 3.1. A unit vector x_0 is said to be ω-smooth if $\operatorname{span} S_{x_0}$ is a closed subspace of X^*. Since S_{x_0} is a weak*-compact set, this is equivalent to $\operatorname{span} S_{x_0}$ being a weak*-closed set.

In a unital C^*-algebra since the state space of a unitary spans the dual, any unitary is ω-smooth. We do not know a precise description of ω-smooth points of a C^*-algebra. We first exhibit a class of Banach spaces for which every vector in $S(X)$ is ω-smooth. Such spaces we will call ω-smooth spaces.

Proposition 3.2. *Let X be an L^1-predual space. Every vector in $S(X)$ is ω-smooth and is a unitary of a quotient of X.*

Proof. Let $x \in S(X)$. Then S_x is a weak*-closed face of X_1^*. It follows from [16, Theorem 3.3] that $\operatorname{span} S_x$ is a weak*-closed subspace. Therefore, every vector in $S(X)$ is ω-smooth. Also if $M = \{x \in X : x^*(x) = 0,\ x^* \in S_x\}$, then for the quotient space, we have $(X \mid M)^* = \operatorname{span} S_x$. Hence the conclusion follows. □

Remark 3.3. Since the second dual of an L^1-predual space X is a $C(K)$ space, we have that all the points of $S(X)$ continue to be ω-smooth in X^{**}. Also all the duals of even order of X are ω-smooth.

Using arguments similar to the ones given in the previous section, we can prove the following corollary when Y is any Banach space. An example below illustrates that pointwise ω-smooth need not imply ω-smooth.

Corollary 3.4. *Let X be an L^1-predual space and let $T \in S(KW^*(Y^*,X))$ be such that T^* attains its norm at only finitely many independent set of vectors in $\partial_e X_1^*$ and the image under T^* of these points is ω-smooth in Y. Then T is an ω-smooth point.*

Turning briefly once again to the case of vector-valued functions $C(K,X)$, the following example from [6] shows that an $f \in C(K,X)$ can be pointwise ω-smooth without being ω-smooth.

Example 3.5. Let $L = \{(\cos(\pi/n), \sin(\pi/n)) \in R^2 : n = 2,3,\dots\}$.

Let E be the two-dimensional Banach space whose unit ball is the convex hull, $\mathrm{CO}(L \cup -L)$. Let $K = N \cup \{\infty\}$ and consider $f \in C(K,X)$ defined by $f(n) = (\cos(\pi/n), \sin(\pi/n))$ and $f(\infty) = (1,0)$. As noted in [6] it is easy to see that each $f(k)$ is a 2-smooth point, for $k \in K$.

By [6, Proposition 4.5] we have that f is a vertex, that is, $\operatorname{span} S_f$ is weak* dense in $C(K,X)^*$. If f were ω-smooth, we get that $\operatorname{span} S_f = C(K,X)^*$ and hence f is a unitary. But as noted in [6, Example 4.4], f is not a unitary.

We recall from [5] that a closed subspace $M \subset X$ is said to be an M-ideal if there exists a projection $P \in \mathcal{L}(X^*)$ such that $\ker(P) = M^\perp$ and $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$ for all $x^* \in X^*$. The importance of this notion to smoothness questions is illustrated by the observations that M is a U-subspace of X and M^* is canonically identified as a subspace of X^* such that $X^* = M^* \oplus_1 M^\perp$ (ℓ^1-direct sum); see [5, Chapter I]. In particular, for any $x \in M$ since the state space remains the same in both M and X, k-smoothness is preserved.

The following lemma illustrates the role of ω-smooth points in spaces with proper M-ideals.

Lemma 3.6. *Let $M \subset X$ be an M-ideal. Suppose $X \mid M$ does not have ω-smooth points. Then $x \in X$ is an ω-smooth point if and only if $d(x,M) < 1$ and $\operatorname{span}(S_x \cap M^*)$ is weak*-closed.*

Proof. Let P be the projection with $\|P(x^*)\| + \|x^* - P(x^*)\| = \|x^*\|$ with $\ker P = M^\perp$. Thus we have $X_1^* = \mathrm{CO}(M_1^* \cup (M^\perp)_1)$. Since S_x is a face, $S_x = \mathrm{CO}(M_1^* \cap S_x \cup S_x \cap M_1^\perp)$. Also $\operatorname{span} S_x = \operatorname{span}(S_x \cap M_1^*) \oplus_1 \operatorname{span}(S_x \cap M_1^\perp)$ (ℓ^1-direct sum). Suppose x is an ω-smooth point. By the closed graph theorem, $\operatorname{span}(S_x \cap M_1^*)$ and $\operatorname{span}(S_x \cap M_1^\perp)$ are closed subspaces. If $\operatorname{span}(S_x \cap M_1^\perp)$ is a nonzero subspace then $d(x,M) = \|[x]\| = 1$ and $[x]$ is an ω-smooth point of $X \mid M$. Therefore, $d(x,M) < 1$.

Conversely suppose $d(x,M) < 1$ and $\operatorname{span}(S_x \cap M^*)$ is a closed subspace. Clearly $S_x \cap M^\perp = \emptyset$. Thus $\operatorname{span} S_x = \operatorname{span}(S_x \cap M_1^*)$ is a closed subspace. Therefore, x is ω-smooth. $\square$

Remark 3.7. Similar ideas can be used to show that if $X \mid M$ has no smooth points of finite order, then $x \in X$ is a k-smooth point if and only if $d(x,M) < 1$ and there exists precisely k independent vectors in $\partial_e M_1^*$ that attain their norm at x.

We recall from [5, Section VI.5] that for $1 < p < \infty$, X is said to be an (M_p) space if the space of compact operators $\mathcal{K}(X \oplus_p X)$ is an M-ideal in $\mathcal{L}(X \oplus_p X)$. These spaces exhibit properties similar to ℓ^p-spaces (see [5, Theorem VI.5.10]). In particular, they are reflexive spaces. It is known that the ℓ^p spaces are (M_p) spaces. We use this idea to correct and extend [8, Theorem 3.2] to this class of spaces. It may be noted that the authors in [8] only exhibit an infinite collection of pairwise independent vectors in the state space of any unit vector of the Calkin space, which does not immediately lead to the desired conclusion. We follow the arguments given during the proof of [5, Lemma VI.5.18].

Theorem 3.8. *Let X, Y be separable (M_p), (M_q) spaces, respectively. Then $\mathcal{L}(X,Y) \mid \mathcal{K}(X, Y)$ has no smooth points of finite order.*

Proof. We only need to consider the case when $\mathcal{K}(X,Y)$ is a proper M-ideal in $\mathcal{L}(X,Y)$. Let $\|[T]\| = 1$. We may assume without loss of generality that $\|T\| = 1$. As in the proof of [5, Lemma VI.5.18] we assume that there is a weak-null sequence $\{x_n\}_{n\geq 1} \subset S(X)$ with $\|[T]\| = \lim \|T(x_n)\|$. Since X is in the class (M_p), there exists a projection $P \in \mathcal{L}(X)$ whose range is the closed span of $\{x_n\}_{n\geq 1}$. For any k, we split this sequence into the union of k-disjoint subsequences and let P_i denote the canonical projection from the closed span of $\{x_n\}_{n\geq 1}$ to the closed span of the ith subsequence. Fix an i and let $M_i = \operatorname{span}\{(T - T \circ P_i \circ P), \{T \circ P_j \circ P\}_{j\neq i}, \mathcal{K}(X,Y)\}$. Note that $d(T \circ P_i \circ P, M_i) \geq \|T\| = 1$. Also if $\{x_n^i\}$ denotes the ith subsequence, for $S \in M_i$, since a compact operator maps weakly null sequences to norm null ones and taking into account the nature of the projections P and P_i, we get $\|T - S\| \geq \limsup \|T(x_n^i)\| = 1$. Thus $d(T \circ P_i \circ P, M_i) = 1$. By an application of the Hahn-Banach theorem, we get $\Lambda_i \in (M_i^{\perp})_1$ such that $\Lambda_i(T \circ P_i \circ P) = 1$. Thus $\Lambda_i \in S_{[T]}$ and by our choice they are independent. □

Remark 3.9. We note that when $X = Y$, the identity operator is a unitary of the Calkin algebra and hence an ω-smooth point. In the case of a Hilbert space since the Calkin algebra is a C^*-algebra, there are plenty of unitaries and hence ω-smooth points.

Since under the above hypothesis, $\mathcal{K}(X,Y)$ is an M-ideal in $\mathcal{L}(X,Y)$ we get from the previous remark that $T \in S(\mathcal{L}(X,Y))$ is a k-smooth point if and only if $d(T, \mathcal{K}(X,Y) < 1)$ and there exists precisely k-independent vectors of $\partial_e \mathcal{K}(X,Y)_1^* = \{x \otimes y^* : x \in \partial_e X_1, y^* \in \partial_e Y_1^*\}$ in S_T.

A Banach space X is said to be an M-embedded space if under the canonical embedding X is an M-ideal in X^{**}. For a Hilbert space H, the space $\mathcal{K}(H)$ and more generally for any X in the class (M_p) $(1 < p \leq \infty)$, the spaces $\mathcal{K}(X)$ are examples of such spaces. See [5, Chapter III] for several examples of M-embedded spaces from among classical function spaces and spaces of operators. By [5, Theorem III.4.6] any such space can be renormed to be smooth and M-embedded. Our next proposition gives a condition under which such spaces have only smooth points of finite order. Any infinite-dimensional space in the class (M_∞), its infinite-dimensional subspaces, and quotient spaces satisfy the

hypothesis of the following proposition. M-embedded spaces are also weakly compactly generated and hence have smooth points.

Proposition 3.10. *Let X be an M-embedded space without reflexive infinite-dimensional quotient spaces. Then X has only smooth points of finite order.*

Proof. Suppose $x_0 \in S(X)$ is such that $\operatorname{span} S_{x_0}$ is closed and infinite dimensional. Let $M = \{x \in X : x^*(x) = 0 \text{ for all } x^* \in S_{x_0}\}$. As noted earlier, $[x_o]$ is a unitary of the infinite-dimensional space $X \mid M$. By [5, Theorem III.1.6], $X \mid M$ is an M-embedded space. By [6, Corollary 3.6], $[x_0]$ is a strong extreme point. Therefore, by [5, Proposition II.4.2 and Theorem II.4.4] we get that $X \mid M$ is reflexive. A contradiction. Thus X has only smooth points of finite order. □

Our next result gives further examples of spaces which have only smooth points of finite order. Here we make use of the fact that the state space is contained in the set of norm attaining elements, $\mathrm{NA}(X)$. That $\mathcal{K}(\ell^2)$ satisfying the hypothesis of the following proposition was proved in [17]. We note that $\mathcal{K}(\ell^2)$ is an M-embedded space with a reflexive quotient, ℓ^2. It is easy to see that $\mathrm{NA}(c_0) = \operatorname{span}\{e_n\}_{n\geq 1}$. See [17] for results relating to c_0-direct sum of spaces with the properties described below.

Proposition 3.11. *Let X be a Banach space such that* $\mathrm{NA}(X)$ *is a vector space and has no infinite-dimensional complete subspaces. Then X has only smooth points of finite order.*

Proof. Let $x \in S(X)$. Clearly $S_x \subset \mathrm{NA}(X)$. Since $\mathrm{NA}(X)$ is a linear space, $\operatorname{span} S_x \subset \mathrm{NA}(X)$. Now if x is an ω-smooth point, then by our hypothesis, x is a smooth point of finite order. □

In the following proposition we consider ω-smooth points that are not of finite order in ℓ^1-direct sums of Banach spaces.

Proposition 3.12. *Let $\{X_i\}_{i\in I}$ be a family of Banach spaces. Let $X = \oplus_1 X_i$. Suppose $x_0 \in S(X)$ is not a smooth point of finite order. Then x_o is an ω-smooth point if and only if $x_0(i)/\|x_0(i)\|$ is an ω-smooth point of $S(X_i)$ when ever $x_0(i) \neq 0$.*

Proof. Suppose x_0 is an ω-smooth point. If $\operatorname{span} S_{x_0} = X^*$, since x_0 is a unitary and therefore an extreme point, one has $x_0(i) = 0$ for all but one $i \in I$. In any case, it is easy to see that $x^* \in S_{x_0}$ if and only if $x^*(i) \in S_{x_0(i)/\|x_0(i)\|}$ when ever $x_0(i) \neq 0$. Thus if $P_i : X^* \to X_i^*$ denotes the canonical weak*-continuous projection, then $P_i(\operatorname{span} S_{x_0}) = \operatorname{span} S_{x_0(i)/\|x_0(i)\|}$. So if x_0 is ω-smooth so is $x_0(i)/\|x_0(i)\|$.

Conversely if $\operatorname{span} S_{x_o(i)/\|x_0(i)\|}$ is weak*-closed for each i with $x_0(i) \neq 0$, then since $\operatorname{span} S_{x_0} = \oplus_\infty \operatorname{span} S_{x_0(i)/\|x_0(i)\|}$ we get that x_0 is ω-smooth. □

Remark 3.13. If Γ is an uncountable discrete set, then $\ell^1(\Gamma)$ has no smooth points of finite order and any unit vector with infinite support is an ω-smooth point. We do not know how to extend the above proposition to the case of the space of Bochner integrable functions $L^1(\mu, X)$ even when X^* is separable (in which case $L^1(\mu, X)^* = L^\infty(\mu, X^*)$) and when μ is a nonatomic measure. See [18] for the smooth case. In the case $f \in S(L^1(\mu, X))$ is a unitary, since f being an extreme point, it is a constant $x_0 \in \partial_e X_1$ on a measure atom,

one can see that x_0 is a unitary. Also if f is ω-smooth by considering extremal states in $\partial_e L^\infty(\mu, X^*)_1$, one can see that $f(w)/\|f(w)\| \in S_{f(w)}$ a.e.

4. Concluding remarks

We now relate ω-smooth points to the structure of closed faces from convexity theory [19]. Let K be a compact convex set and let $A(K)$ denote the space of real-valued affine continuous functions on K, equipped with the supremum norm. The evaluation map $K \to A(K)_1^*$ identifies K with the state space S_1. It is well known that $A(K)_1^* = \mathrm{CO}(K \cup -K)$. For any $a \in S(A(K))$, let $F = \{k \in K : a(k) = 1\}$. Such an F is called an exposed face. It is easy to see that $\mathrm{CO}(S_a \cup -S_a) = \mathrm{CO}(F \cup -F)$ so that $\operatorname{span} F = \operatorname{span} S_a$. In the case of a Choquet simplex K, since $A(K)$ is an L^1-predual space (or directly from the results in [19]), $A(K)$ is ω-smooth. For a general K, a theorem by D. A. Edwards [19, Theorem II.5.5] gives conditions in terms of bounded extension property of the affine continuous functions on F, for an $a \in S(A(K))$ to be ω-smooth. [19, Proposition II.5.26], in the notation of this paper, is an example of a compact convex set K and an $a \in S(A(K))$ that is not an ω-smooth point.

Question 1. Describe infinite-dimensional compact convex sets K for which $A(K)$ is ω-smooth.

We have not considered here the corresponding notion of weak*-k-smooth points in the dual space X^*. For $x^* \in S(X^*)$, let $F = \{x \in X_1 : x^*(x) = 1\}$. When F is nonempty we say that x^* is a weak*-k-smooth point if $\operatorname{span} F$ is of dimension k. Here the independent vectors in $\operatorname{span} F$ need not be extreme points. The extremal case, that is, $\operatorname{span} F = X$ here corresponds to the notion of weak*-unitary defined in [6], which coincides with unitaries in the case of a von Neumann algebra. Our results here show that both in the case of a commutative von Neumann algebra and $\mathscr{L}(H)$, k-smooth points are weak*-k-smooth.

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