Research Article

On the Generalized Ulam-Gavruta-Rassias Stability of Mixed-Type Linear and Euler-Lagrange-Rassias Functional Equations

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Received 18 April 2007; Revised 22 April 2007; Accepted 23 April 2007

In this paper, the mixed-type linear and Euler-Lagrange-Rassias functional equations introduced by J. M. Rassias is generalized to the following $n$-dimensional functional equation:

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1 \leq i < j \leq n} f(x_i - x_j)$$

when $n > 2$. We prove the general solutions and investigate its generalized Ulam-Gavruta-Rassias stability.

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1. Introduction

In 1940, Ulam [1] proposed the famous Ulam stability problem of linear mappings. In 1941, Hyers [2] considered the case of approximately additive mappings $f : E \to E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies Hyers inequality $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)$ exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying $\|f(x) - L(x)\| \leq \varepsilon$. In 1982–1998, Rassias [3–9] generalized the result to include the following theorem.

**Theorem 1.1.** Let $X$ be a real-normed linear space and let $Y$ be a real-complete-normed linear space. Assume in addition that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p, q \in \mathbb{R}$ such that $r = p + q \neq 1$, and $f$ satisfies the Cauchy-Gavruta-Rassias inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^q$$

for all $x, y \in X$. Then, there exists a unique additive mapping $L : X \to Y$ satisfying

$$f(x) - L(x) \leq \frac{\theta}{|2^r - 2|} \|x\|^r \quad \forall \ x \in X.$$  

(1.2)
If in addition $f : X \to Y$ is a mapping such that the transformation $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then $L$ is $\mathbb{R}$-linear mapping.

In 2002, Rassias [10] established the Ulam stability of the following mixed-type functional equation:

$$f\left(\sum_{i=1}^{3} x_i\right) + \sum_{i=1}^{3} f(x_i) = \sum_{1 \leq i < j \leq 3} f(x_i + x_j) \quad (1.3)$$

on restricted domains. In this paper, we will generalize Rassias’ work to the following $n$-dimensional mixed-type functional equation:

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1 \leq i < j \leq n} f(x_i + x_j) \quad (1.4)$$

when $n > 2$, and will investigate its generalized Ulam-Gavruta-Rassias stability.

2. The general solution

Theorem 2.1. Let $n > 2$ be a positive integer, and let $X$ and $Y$ be vector spaces.

A function $f : X \to Y$ satisfies the functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1 \leq i < j \leq n} f(x_i + x_j) \quad (2.1)$$

if and only if the even part of $f$, defined by $f_e(x) = (1/2)(f(x) + f(-x))$ for all $x \in X$, satisfies the classical quadratic functional equation, which is also a special Euler-Lagrange-Rassias equation [7, 9],

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad (2.2)$$

and the odd part of $f$, defined by $f_o(x) = (1/2)(f(x) - f(-x))$ for all $x \in X$, satisfies the Cauchy functional equation

$$f(x + y) = f(x) + f(y). \quad (2.3)$$

Proof. For the if part of the proof, suppose that $f : X \to Y$ satisfies (2.1), we can uniquely express $f$ as $f(x) = f_e(x) + f_o(x)$ for all $x \in X$, where the even part, $f_e$, and the odd part, $f_o$, are defined as in the theorem. We will show that $f_e$ satisfies (2.2) and $f_o$ satisfies (2.3).

Setting $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$ in (2.1), we see that $f(0) = 0$. Setting $(x_1, x_2, \ldots, x_n) = (x, y, -y, 0, 0, \ldots, 0)$ in (2.1), we get

$$f(x) + (n-2)(f(x) + f(y) + f(-y)) = f(x - y) + f(x + y) \quad (2.4)$$

$$+ (n-3)(f(x) + f(y) + f(-y)),$$

which is simplified to

$$2f(x) + f(y) + f(-y) = f(x + y) + f(x - y) \quad (2.5)$$
for all \( x, y \in X \). Replacing \( x \) and \( y \) with \(-x\) and \(-y\), respectively, then taking half the sum and half the difference with (2.5), we have

\[
2f_e(x) + f_e(y) + f_e(-y) = f_e(x + y) + f_e(x - y),
\]

\[
2f_o(x) + f_o(y) + f_o(-y) = f_o(x + y) + f_o(x - y).
\] (2.6)

By the evenness of \( f_e \), we immediately see that \( f_e \) satisfies the classical quadratic functional equation given by (2.2). By the oddness of \( f_o \), we see that \( 2f_o(x) = f_o(x + y) + f_o(x - y) \) which is recognized as the Jensen functional equation. Since \( f_o(0) = 0 \), if we put \( y = x \) in the above equation, then \( f(2x) = 2f(x) \). By another substitution, \( (x, y) = ((x + y)/2, (x - y)/2) \), we derive the Cauchy functional equation \( f_o(x + y) = f_o(x) + f_o(y) \).

Now for the only if part of the proof, suppose that the even part and the odd part of \( f : X \to Y \) satisfy (2.2) and (2.3), respectively, that is, \( f_e(x + y) + f_e(x - y) = 2f_e(x) + 2f_e(y) \) and \( f_o(x + y) = f_o(x) + f_o(y) \). We will show that \( f \) satisfies (2.1). Noting that a linear combination of two solutions of (2.1) yields just another solution, we will in turn prove that each part of \( f \) satisfies (2.1).

First, consider the odd part and make use of the linearity of the Cauchy functional equation. The left-hand side of (2.1) is

\[
f_o\left(\sum_{i=1}^{n} x_i\right) + (n - 2) \sum_{i=1}^{n} f_o(x_i) = \sum_{i=1}^{n} f_o(x_i) + (n - 2) \sum_{i=1}^{n} f_o(x_i) = (n - 1) \sum_{i=1}^{n} f_o(x_i),
\] (2.7)

and the right-hand side of (2.1) is

\[
\sum_{1 \leq i < j \leq n} f_o(x_i + x_j) = \sum_{1 \leq i < j \leq n} (f_o(x_i) + f_o(x_j)) = \frac{2}{n} \binom{n}{2} \sum_{i=1}^{n} f_o(x_i) = (n - 1) \sum_{i=1}^{n} f_o(x_i).
\] (2.8)

Thus, we have established (2.1) on the odd part of \( f \).

For the even part, we will show by mathematical induction that (2.1) holds for every positive integer \( n \). For \( n = 1 \), we take \( \sum_{1 \leq i < j \leq 1} f_e(x_i + x_j) \) as 0; then \( f_e(x_1) + (1 - 2)f_e(x_1) = 0 \), which is trivially true. For \( n = 2 \), we have \( f_e(x_1 + x_2) + 0 = f_e(x_1 + x_2) \), which is again trivially true. For \( n \geq 3 \), we assume that (2.1) holds for every number of variables from 1 to \( n - 1 \), that is,

\[
f_e\left(\sum_{i=1}^{k} x_i\right) + (k - 2) \sum_{i=1}^{k} f_e(x_i) = \sum_{1 \leq i < j \leq k} f_e(x_i + x_j)
\] (2.9)

for \( k = 1, 2, \ldots, n - 1 \). For each \( i, j = 1, 2, \ldots, n \) with \( i \neq j \), we have

\[
f_e(x_i - x_j) + f_e(x_i + x_j) = 2(f_e(x_i) + f_e(x_j)).
\] (2.10)

Then,

\[
\sum_{1 \leq i < j \leq n} (f_e(x_i - x_j) + f_e(x_i + x_j)) = 2 \sum_{1 \leq i < j \leq n} (f_e(x_i) + f_e(x_j)) = \frac{4}{n} \binom{n}{2} \sum_{i=1}^{n} f_e(x_i).
\] (2.11)
Thus,

\[ \sum_{1 \leq i < j \leq n} (f_e(x_i - x_j) + f_e(x_i + x_j)) = 2(n - 1) \sum_{i=1}^{n} f_e(x_i). \] (2.12)

For each \( j, k = 1, 2, \ldots, n \) with \( j \neq k \), we have

\[ f_e\left( \sum_{i=1}^{n} x_i - 2x_j \right) + f_e\left( \sum_{i=1}^{n} x_i - 2x_k \right) = 2 f_e\left( \sum_{i=1}^{n} x_i - x_j - x_k \right) + 2 f_e(x_j - x_k). \] (2.13)

Write down the above equation for every possible pair \((j, k)\) and note that there are \( \binom{n}{2} \) such pairs; so each \( f_e(\sum_{i=1}^{n} x_i - 2x_j) \) appears \( n - 1 \) times in all \( \binom{n}{2} \) equations. Adding up the equations, we get

\[ (n - 1) \sum_{j=1}^{n} f_e\left( \sum_{i=1}^{n} x_i - 2x_j \right) = 2 \sum_{1 \leq j < k \leq n} f_e\left( \sum_{i=1}^{n} x_i - x_j - x_k \right) + 2 \sum_{1 \leq j < k \leq n} f_e(x_j - x_k). \] (2.14)

For each \( j = 1, 2, \ldots, n \), we have

\[ f_e\left( \sum_{i=1}^{n} x_i \right) + f\left( \sum_{i=1}^{n} x_i - 2x_j \right) = 2 f_e\left( \sum_{i=1}^{n} x_i - x_j \right) + 2 f_e(x_j). \] (2.15)

Sum the above equation for all \( j \)'s and substitute the result from (2.12) and (2.14), then rearrange the resulting equation

\[ n f_e\left( \sum_{i=1}^{n} x_i \right) + \frac{2}{n - 1} \sum_{1 \leq j < k \leq n} f_e\left( \sum_{i=1}^{n} x_i - x_j - x_k \right) \]
\[ = 2 \sum_{j=1}^{n} f_e\left( \sum_{i=1}^{n} x_i - x_j \right) + \frac{2}{n - 1} \sum_{1 \leq i < j \leq n} f_e(x_i + x_j) - 2 \sum_{i=1}^{n} f_e(x_i). \] (2.16)

Note that \( \sum_{j=1}^{n} f_e(\sum_{i=1}^{n} x_i - x_j) \) is the sum of \( f \) of \( x_i \)'s taken \( n - 1 \) variables at a time, and \( \sum_{1 \leq j < k \leq n} f_e(\sum_{i=1}^{n} x_i - x_j - x_k) \) is the sum of \( f \) of \( x_i \)'s taken \( n - 2 \) variables at a time. From the induction assumption, (2.1) holds for \( n - 1 \) and \( n - 2 \) variables, that is,

\[ \sum_{j=1}^{n} f_e\left( \sum_{i=1}^{n} x_i - x_j \right) + (n - 1)(n - 3) \sum_{i=1}^{n} f_e(x_i) = (n - 2) \sum_{1 \leq i < j \leq n} f_e(x_i + x_j), \]
\[ \sum_{1 \leq j < k \leq n} f_e\left( \sum_{i=1}^{n} x_i - x_j - x_k \right) + \frac{(n - 1)(n - 2)(n - 4)}{2} \sum_{i=1}^{n} f_e(x_i) \]
\[ = \frac{(n - 2)(n - 3)}{2} \sum_{1 \leq i < j \leq n} f_e(x_i + x_j). \] (2.17)
Substitute (2.17) into (2.16) and simplify, we will finally establish (2.1) on the even part of \( f \). Thus, \( f \) satisfies (2.1) and the proof is complete. □

3. The Ulam-Gavruta-Rassias stability

Rassias [10] established the Ulam stability of (2.1) in the special case when \( n = 3 \) on restricted domains. The following theorem provides a general condition for which a true general solution discussed in Theorem 2.1 exists near an approximate solution. For convenience, we define

\[
Df(x_1,x_2,\ldots,x_n) = f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) - \sum_{1\leq i<j\leq n} f(x_i + x_j). \tag{3.1}
\]

From now on, we will refer to the even part and the odd part of a function by subscripts \( e \) and \( o \), respectively.

**Theorem 3.1.** Let \( n > 2 \) be a positive integer, let \( X \) be a real vector space, let \( Y \) be a Banach space, let \( \phi : X^n \to [0, \infty) \) be an even function. Define \( \phi(x) = \phi(x,x,-x,0,\ldots,0) \) for all \( x \in X \). If

\[
\sum_{i=0}^{\infty} 2^{-i} \phi(2^i x) \text{ converges, } \lim_{m \to \infty} 2^{-m} \phi(2^m x_1,\ldots,2^m x_n) = 0 \tag{3.2}
\]

or

\[
\sum_{i=1}^{\infty} 4^i \phi(2^{-i} x) \text{ converges, } \lim_{m \to \infty} 4^m \phi(2^{-m} x_1,\ldots,2^{-m} x_n) = 0 \tag{3.3}
\]

for all \( x_1,x_2,\ldots,x_n \in X \), and a function \( f : X \to Y \) satisfies \( f(0) = 0 \) and

\[
\|Df(x_1,x_2,\ldots,x_n)\| \leq \phi(x_1,x_2,\ldots,x_n) \tag{3.4}
\]

for all \( x_1,x_2,\ldots,x_n \in X \), then there exists a unique function \( T : X \to Y \) that satisfies functional equation (2.1) and, if condition (3.2) holds,

\[
\|f_e(x) - Te(x)\| \leq \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i} \phi(2^i x), \quad \|f_o(x) - To(x)\| \leq \frac{1}{2} \sum_{i=0}^{\infty} 2^{-i} \phi(2^i x) \tag{3.5}
\]

or, if condition (3.3) holds,

\[
\|f_e(x) - Te(x)\| \leq \frac{1}{4} \sum_{i=1}^{\infty} 4^i \phi(2^{-i} x), \quad \|f_o(x) - To(x)\| \leq \frac{1}{2} \sum_{i=1}^{\infty} 2^i \phi(2^{-i} x). \tag{3.6}
\]
The function $T$ is given by

$$T(x) = \begin{cases} 
\lim_{m \to \infty} 4^{-m} f_c(2^m x) + 2^{-m} f_0(2^m x) & \text{if condition (3.2) holds,} \\
\lim_{m \to \infty} 4^m f_c(2^{-m} x) + 2^m f_0(2^{-m} x) & \text{if condition (3.3) holds}
\end{cases}$$

for all $x \in X$.

**Proof.** We will prove the theorem for a function $\phi$ satisfying condition (3.2) and accordingly inequality (3.5). A proof for conditions (3.3) and (3.6) can be reproduced in a similar manner. Setting $(x_1, x_2, \ldots, x_n) = (x, x, -x, 0, 0, \ldots, 0)$ in (3.4) and simplifying, we have $\|3 f(x) + f(-x) - f(2x)\| \leq \phi(x)$. Replacing $x$ by $-x$, we have $\|3 f(-x) + f(x) - f(-2x)\| \leq \phi(-x) = \phi(x)$. Then,

$$\|4 f_c(x) - f_c(2x)\| = \frac{1}{2} \| (3 f(x) + f(-x) - f(2x)) + (3 f(-x) + f(x) - f(-2x)) \| \leq \frac{1}{2} \| 3 f(x) + f(-x) - f(2x) \| + \frac{1}{2} \| 3 f(-x) + f(x) - f(-2x) \| \leq \frac{1}{2} \phi(x) + \frac{1}{2} \phi(x) = \phi(x),$$

$$\|2 f_0(x) - f_0(2x)\| = \frac{1}{2} \| (3 f(x) + f(-x) - f(2x)) - (3 f(-x) + f(x) - f(-2x)) \| \leq \frac{1}{2} \| 3 f(x) + f(-x) - f(2x) \| + \frac{1}{2} \| 3 f(-x) + f(x) - f(-2x) \| \leq \frac{1}{2} \phi(x) + \frac{1}{2} \phi(x) = \phi(x).$$

Rewrite the inequality on $f_c$ as $\| f_c(x) - 4^{-1} f_c(2x) \| \leq 4^{-1} \phi(x)$ for all $x \in X$. Suppose that $\| f_c(x) - 4^{-m} f_c(2^m x) \| \leq (1/4) \sum_{i=0}^{m-1} 4^{-i} \phi(2^i x)$ for a positive integer $m$. Then,

$$\| f_c(x) - 4^{-(m+1)} f_c(2^{m+1} x) \| \leq \| f_c(x) - 4^{-m} f_c(2^m x) \| + \| 4^{-m} f_c(2^m x) - 4^{-(m+1)} f_c(2^{m+1} x) \| \leq \| f_c(x) - 4^{-m} f_c(2^m x) \| + 4^{-m} \| f_c(2^m x) - 4^{-1} f_c(2 \cdot 2^m x) \| \leq \frac{1}{2} \phi(x) + 4^{-m} \phi(2^m x) + 4^{-m} \phi(2^m x) = \frac{1}{2} \sum_{i=0}^{m-1} 4^{-i} \phi(2^i x).$$

Hence, $\| f_c(x) - 4^{-m} f_c(2^m x) \| \leq (1/4) \sum_{i=0}^{m-1} 4^{-i} \phi(2^i x)$ for every positive integer $m$.

If we rewrite the inequality for $f_0$ as $\| f_0(x) - 2^{-1} f_0(2x) \| \leq 2^{-1} \phi(x)$ and repeat the same steps as in the case of $f_c$, we will have $\| f_0(x) - 2^{-m} f_0(2^m x) \| \leq (1/2) \sum_{i=0}^{m-1} 2^{-i} \phi(2^i x)$ for every positive integer $m$. 
The convergence of the sequence \( \{4^{-m} f_e(2^m x)\} \) can be settled as follows. For every positive integer \( p \),

\[
\left\|4^{-m+p} f_e(2^{m+p} x) - 4^{-m} f_e(2^m x)\right\| = 4^{-m} \left\|4^{-p} f_e(2^p \cdot 2^m x) - f_e(2^m x)\right\|
\leq 4^{-m} \cdot \frac{1}{4} \sum_{i=0}^{p-1} 4^{-i} \phi(2^i \cdot 2^m x)
\leq \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i} \phi(2^{i+m} x).
\]

By the definition of \( \phi \) and condition (3.2), the right-hand side approaches 0 as \( m \) goes to infinity, hence, we have a Cauchy sequence in a Banach space. Let \( T_e(x) = \lim_{m \to \infty} 4^{-m} f_e(2^m x) \) for all \( x \in X \), and thus \( \|f_e(x) - T_e(x)\| \leq (1/4) \sum_{i=0}^{\infty} 4^{-i} \phi(2^i x) \). We can similarly show that \( \{2^{-m} f_o(2^m x)\} \) converges; so let \( T_o(x) = \lim_{m \to \infty} 2^{-m} f_o(2^m x) \) for all \( x \in X \), and thus \( \|f_o(x) - T_o(x)\| \leq (1/2) \sum_{i=0}^{\infty} 2^{-i} \phi(2^i x) \). Define \( T(x) = T_e(x) + T_o(x) \) for all \( x \in X \).

In order to show that \( T \) satisfies (2.1), we will in turn show that \( T_e \) and \( T_o \) satisfy (2.1). For convenience, define \( Df_e \) and \( Df_o \) as the even part and the odd part of \( Df \) in (3.1), respectively. For \( T_e \), consider

\[
4^{-m} \left\|Df_e(2^m x_1, \ldots, 2^m x_n)\right\|
= 4^{-m} \cdot \frac{1}{2} \left\|Df(2^m x_1, \ldots, 2^m x_n) + Df(-2^m x_1, \ldots, -2^m x_n)\right\|
\leq 4^{-m} \phi(2^m x_1, \ldots, 2^m x_n).
\]

As \( m \) tend to infinity, the left-hand side approaches \( \|DT_e(x_1, \ldots, x_n)\| \) and, by condition (3.2), the right-hand side approaches 0. Thus,

\[
DT_e(x_1, x_2, \ldots, x_n) = T_e \left( \sum_{i=1}^{n} x_i \right) + (n-2) \sum_{i=1}^{n} T_e(x_i) - \sum_{1 \leq i < j \leq n} T_e(x_i + x_j) = 0,
\]

which shows that \( T_e \) satisfies (2.1).

We can similarly show that \( T_o \) satisfies (2.1) by considering

\[
2^{-m} \left\|Df_o(2^m x_1, \ldots, 2^m x_n)\right\|
= 2^{-m} \cdot \frac{1}{2} \left\|Df(2^m x_1, \ldots, 2^m x_n) - Df(-2^m x_1, \ldots, -2^m x_n)\right\|
\leq 2^{-m} \phi(2^m x_1, \ldots, 2^m x_n),
\]

and take the limit as \( m \to \infty \). Hence, \( T = T_e + T_o \) satisfies (2.1) as desired.
To prove the uniqueness of $T$, suppose that there exists another function $S : X \to Y$ such that $S$ satisfies (2.1) and satisfies the inequality (3.5) with $T$ replaced by $S$. Then,

$$\|S(x) - T(x)\| \leq \|S(x) - f(x)\| + \|T(x) - f(x)\|$$

$$\leq \|S_e(x) - f_e(x)\| + \|S_o(x) - f_o(x)\|$$

$$+ \|T_e(x) - f_e(x)\| + \|T_o(x) - f_o(x)\|. \quad (3.14)$$

It is straightforward to show that every solution of the quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ has the quadratic property $f(nx) = n^2 f(x)$ and every solution of the linear functional equation $f(x + y) = f(x) + f(y)$ has the linear property $f(nx) = nf(x)$ for every positive integer $n$ and for every $x$ in the domain. We thus obtain

$$\|S(x) - T(x)\| \leq 4^{-m}\|S_e(2^m x) - f_e(2^m x)\| + 2^{-m}\|S_o(2^m x) - f_o(2^m x)\|$$

$$+ 4^{-m}\|T_e(2^m x) - f_e(2^m x)\| + 2^{-m}\|T_o(2^m x) - f_o(2^m x)\|$$

$$\leq 2 \left( 4^{-m} \cdot \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i} \varphi(2^i \cdot 2^m x) + \frac{1}{2^m} \cdot \frac{1}{2} \sum_{i=0}^{\infty} 2^{-i} \varphi(2^i \cdot 2^m x) \right) \quad (3.15)$$

$$= \frac{1}{2} \sum_{i=0}^{\infty} 4^{-(i+m)} \varphi(2^{i+m} x) + \sum_{i=0}^{\infty} 2^{-(i+m)} \varphi(2^{i+m} x)$$

for all $x \in X$. As $m$ goes to infinity, the right-hand side approaches 0, and $S(x) = T(x)$ for all $x \in X$. This completes the proof.

The following corollary proves the Hyers-Ulam stability of (2.1).

**Corollary 3.2.** If a function $f : X \to Y$ satisfies $f(0) = 0$ and the functional equation

$$\|Df(x_1, x_2, \ldots, x_n)\| \leq \varepsilon \quad (3.16)$$

for some $\varepsilon > 0$ and for all $x_1, x_2, \ldots, x_n \in X$, then there exists a unique function $T : X \to Y$ that satisfies functional equation (2.1) and, for all $x \in X$,

$$\|f_e(x) - T_e(x)\| \leq \frac{\varepsilon}{3}, \quad \|f_o(x) - T_o(x)\| \leq \varepsilon. \quad (3.17)$$

**Proof.** Let $\varphi(x_1, x_2, \ldots, x_n) = \varepsilon$, then condition (3.2) in Theorem 3.1 holds. Hence, it follows from the theorem that there exists a unique function $T : X \to Y$ such that

$$\|f_e(x) - T_e(x)\| \leq \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i} \cdot \varepsilon = \frac{\varepsilon}{3}, \quad \|f_o(x) - T_o(x)\| \leq \frac{1}{2} \sum_{i=0}^{\infty} 2^{-i} \varepsilon = \varepsilon. \quad (3.18)$$

The following corollary proves the Hyers-Ulam-Rassias stability of (2.1).
Corollary 3.3. Let \( p \) be a positive real number with \( 0 < p < 1 \) or \( p > 2 \). If a function \( f : X \to Y \) satisfies the inequality
\[
||Df(x_1, x_2, \ldots, x_n)|| \leq \varepsilon \sum_{i=1}^{n} ||x_i||^p
\]  
(3.19)
for some \( \varepsilon > 0 \) and for all \( x_1, x_2, \ldots, x_n \in X \), then there exists a unique function \( T : X \to Y \) that satisfies functional equation (2.1) and, for all \( x \in X \),
\[
||f_{\varepsilon}(x) - T_{\varepsilon}(x)|| \leq \frac{3\varepsilon}{4|1 - 2p^{-2}|} ||x||^p,
\]
(3.20)
\[
||f_{0}(x) - T_{0}(x)|| \leq \frac{3\varepsilon}{2|1 - 2p^{-1}|} ||x||^p.
\]
(3.21)

Proof. Substituting \( x_1 = x_2 = \cdots = x_n = 0 \) into (3.19), we get
\[
f(0) + (n - 2) \cdot nf(0) = \binom{n}{2} f(0).
\]
(3.22)
Since \( n > 2 \), it follows that \( 1 + n(n - 2) > \binom{n}{2} \), hence, \( f(0) = 0 \).

Let \( \phi(x_1, x_2, \ldots, x_n) = \varepsilon \sum_{i=1}^{n} ||x_i||^p \). If \( 0 < p < 1 \), then condition (3.2) in Theorem 3.1 holds and it follows that
\[
||f_{\varepsilon}(x) - T_{\varepsilon}(x)|| \leq \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i} (3\varepsilon \cdot 2^i ||x||^p) = \frac{3\varepsilon}{4(1 - 2^{-p^{-2}})} ||x||^p,
\]
(3.23)
\[
||f_{0}(x) - T_{0}(x)|| \leq \frac{1}{2} \sum_{i=0}^{\infty} 2^{-i} (3\varepsilon \cdot 2^i ||x||^p) = \frac{3\varepsilon}{2(1 - 2^{-1})} ||x||^p.
\]
(3.24)

If \( p > 1 \), we apply Theorem 3.1 with condition (3.3) to get a similar result. \( \square \)

The following corollary proves the Ulam-Gavruta-Rassias stability of (2.1).

Corollary 3.4. Let \( p_1, p_2, \ldots, p_n \) be nonnegative real numbers and \( r = \sum_{i=1}^{n} p_i \) with \( 0 < r < 1 \) or \( r > 2 \). If a function \( f : X \to Y \) satisfies the inequality
\[
||Df(x_1, x_2, \ldots, x_n)|| \leq \varepsilon \prod_{i=1}^{n} ||x_i||^{p_i}
\]  
(3.25)
for some \( \varepsilon > 0 \) and for all \( x_1, x_2, \ldots, x_n \in X \), then there exists a unique function \( T : X \to Y \) that satisfies functional equation (2.1) and, for \( n = 3 \),
\[
||f_{\varepsilon}(x) - T_{\varepsilon}(x)|| \leq \frac{\varepsilon}{4|1 - 2^{-r^{-2}}|} ||x||^r,
\]
(3.26)
\[
||f_{0}(x) - T_{0}(x)|| \leq \frac{\varepsilon}{2|1 - 2^{-r-1}|} ||x||^r
\]
(3.27)
for all \( x \in X \).

Proof. We can show that \( f(0) = 0 \) by the same substitution used in the proof of Corollary 3.3. Let \( \phi(x_1, x_2, \ldots, x_n) = \varepsilon \prod_{i=1}^{n} ||x_i||^{p_i} \). According to Theorem 3.1, if \( 0 < r < 1 \), then condition (3.2) holds, and if \( r > 2 \), then condition (3.3) holds. If \( n > 3 \), then the desired result
immediately follows. However, for $n = 3$, we have

$$\| f_e(x) - T_e(x) \| \leq \frac{1}{4} \sum_{i=0}^{\infty} 4^{-i} (\varepsilon \cdot 2^{ir} \|x\| r) = \frac{\varepsilon}{4(1 - 2^{-2})} \|x\|^r,$$

(3.25)

$$\| f_o(x) - T_o(x) \| \leq \frac{1}{2} \sum_{i=0}^{\infty} 2^{-i} (\varepsilon \cdot 2^{ir} \|x\| r) = \frac{\varepsilon}{2(1 - 2^{-1})} \|x\|^r$$

when $0 < r < 1$, and a similar result when $r > 1$.

Acknowledgment

The author would like to thank the referee for providing valuable comments and helping in improving the content of this paper.

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