Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 63739, 12 pages
doi:10.1155/2007/63739

Research Article

Extensions of Some Parametric Families of $D(16)$-Triples

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Received 7 September 2006; Revised 24 November 2006; Accepted 29 November 2006

Recommended by Dihua Jiang

Let n be an integer. A set of m positive integers is called a $D(n)$-m-tuple if the product of any two of them increased by n is a perfect square. In this paper, we consider extensions of some parametric families of $D(16)$-triples. We prove that if $\{k-4, k+4, 4k, d\}$, for $k \geq 5$, is a $D(16)$-quadruple, then $d = k^3 - 4k$. Furthermore, if $\{k-4, 4k, 9k-12\}$, for $k > 5$, is a $D(16)$-quadruple, then $d = 9k^3 - 48k^2 + 76k - 32$. But for $k = 5$, this statement is not valid. Namely, the $D(16)$-triple $\{1, 20, 33\}$ has exactly two extensions to a $D(16)$-quadruple: $\{1, 20, 33, 105\}$ and $\{1, 20, 33, 273\}$.

1. Introduction

Let n be an integer. A set of m positive integers $\{a_1, a_2, \ldots, a_m\}$ is called a Diophantine m-tuple with the property $D(n)$ or simply $D(n)$-m-tuple, if $a_i a_j + n$ is a perfect square for $1 \leq i < j \leq m$.

This problem was first studied by Diophantus for the case $n = 1$. He found a set of four positive rationals with the property $D(1)$. It was the set $\{1/16, 33/16, 17/4, 105/16\}$. However, the first $D(1)$-quadruple, $\{1, 3, 8, 120\}$, was found by Fermat. Euler was able to add the fifth positive rational, 777 480/8 288 641, to the Fermat set (see [1, pages 103-104, 232]). Recently, Gibbs [2] found some examples of sets of six positive rationals with the property $D(1)$. The conjecture is that there does not exist a $D(1)$-quintuple. In 1969, Baker and Davenport [3] proved that the Fermat set cannot be extended to a $D(1)$-quintuple. More precisely, they proved that if $\{1, 3, 8, d\}$ is a $D(1)$-quadruple, then $d = 120$. Dujella gave one generalization to this result in [4], that only extension of $D(1)$-triple $\{k-1, k+1, 4k\}$, for an integer $k \geq 2$, to a $D(1)$-quadruple $\{k-1, k+1, 4k, d\}$, is given by $d = 16k^3 - 4k$. Fujita (see [5]) obtained a result for the case $n = 4$, which can be regarded as a generalization of the result from [4]. He proved that only extension of $D(4)$-triple $\{k-2, k+2, 4k\}$, for an integer $k \geq 3$, to a $D(4)$-quadruple $\{k-2, k+2, 4k, d\}$, is

given by $d = 4k^3 - 4k$. We will prove a result for the case $n = 16$, which generalizes results from [4, 5]. Those results support the strong $D(1)$-quadruple and $D(4)$-quadruple conjecture, which state that in the $D(1)$-quadruple, respectively, $D(4)$-quadruple $\{a,b,c,d\}$, such that $a < b < c < d$, element d is uniquely determined with a, b, and c.

However, in the case $n = 16$, it is not true. We will prove that only extension of $D(16)$-triple of the form $\{k-4, 4k, 9k-12\}$, for an integer $k > 5$, to a $D(16)$-quadruple $\{k-4, 4k, 9k-12, d\}$, is given by $d = 9k^3 - 48k^2 + 76k - 32$. But for $k = 5$, we will prove that $D(16)$-triple $\{1,20,33\}$ can be extended to exactly two $D(16)$-quadruples $\{1,20,33,105\}$ and $\{1,20,33,273\}$. Perhaps we could say that these results suggest that the strong $D(16)$-quadruple conjecture might be valid for a, b, c sufficiently large.

In the proofs, we will use the same strategy and the methods from [4, 5].

2. Extension of $D(16)$-triples of the form $\{k-4, k+4, 4k\}$

2.1. System of Pellian equations. Assume that the $D(16)$-triple $\{k-4, k+4, 4k\}$, for $k \geq 5$, can be extended to a quadruple, $\{k-4, k+4, 4k, d\}$. We note that because of Fujita's result [5] of extension of $D(4)$-triples of the form $\{k-2, k+2, 4k\}$, it is enough to consider odd k. Then there exist positive integers x, y, z, such that

$$(k-4)d + 16 = x^2, \qquad (k+4)d + 16 = y^2, \qquad 4kd + 16 = 4z^2. \tag{2.1}$$

Eliminating d, we get the following system of Pellian equations:

$$(k-4)z^2 - kx^2 = -12k - 16, \tag{2.2}$$

$$(k+4)z^2 - ky^2 = -12k + 16. \tag{2.3}$$

From the theory of Pellian equations, we know that if (z,x) is solution to (2.2), then there exists an integer $m \geq 0$ such that

$$z\sqrt{k-4} + x\sqrt{k} = \left(z_0^{(i)}\sqrt{k-4} + x_0^{(i)}\sqrt{k}\right)\left(\frac{k-2+\sqrt{k(k-4)}}{2}\right)^m, \tag{2.4}$$

where $\{(z_0^{(i)}, x_0^{(i)}) : i = 1, \ldots, i_0\}$ is a finite set of fundamental solutions of (2.2).

Indeed, let (z,x) be a solution of (2.2) in positive integers. Consider all pairs of integers (z^*, x^*) of the form

$$z^*\sqrt{k-4} + x^*\sqrt{k} = \left(z\sqrt{k-4} + x\sqrt{k}\right)\left(\frac{k-2+\sqrt{k(k-4)}}{2}\right)^m, \tag{2.5}$$

for $m \in \mathbb{Z}$. It is easy to see that (z^*, x^*) is a solution to (2.2) in integers. From $z^*\sqrt{k-4} + x^*\sqrt{k} > 0$ and $|x^*\sqrt{k}| > |z^*\sqrt{k-4}|$, we conclude that x^* is a positive integer. Among all the pairs (z^*, x^*), we chose the one for which x^* is minimal, and we denote it by (z_0, x_0). Define integers z' and x' by

$$z'\sqrt{k-4} + x'\sqrt{k} = \left(z_0\sqrt{k-4} + x_0\sqrt{k}\right)\left(\frac{k-2-\varepsilon\sqrt{k(k-4)}}{2}\right), \tag{2.6}$$

where $\varepsilon = 1$, if $z_0 \geq 0$, and $\varepsilon = -1$, if $z_0 < 0$. Then from minimality of x_0 we get

$$x' = \frac{1}{2}\left(x_0(k-2) - \varepsilon z_0(k-4)\right) \geq x_0, \tag{2.7}$$

which implies, after some calculation, $x_0^2 \leq 3k+4$.

Moreover from $x_0^2 \equiv 16(\mathrm{mod}(k-4))$, we get the following possibilities for $x_0^2 : x_0^2 = 16$, $k+12$, $2k+8$, $3k+4$. Inserting that in (2.2) we get the value for z_0^2. Then we conclude that only two possibilities are $x_0 = 4$, $z_0 = \pm 2$, or $x_0^2 = z_0^2 = 3k+4$, because in other cases fundamental solutions are not integers. At the end it is easy to see that $m \geq 0$.

In the exactly same way we conclude that the solutions (z, y) of (2.3) are given by

$$z\sqrt{k+4} + y\sqrt{k} = \left(z_1^{(j)}\sqrt{k+4} + y_1^{(j)}\sqrt{k}\right)\left(\frac{k+2+\sqrt{k(k+4)}}{2}\right)^n, \tag{2.8}$$

for some integer $n \geq 0$, and $\{(z_1^{(j)}, y_1^{(j)}) : j = 1, \ldots, j_0\}$ is finite set of fundamental solutions of (2.3). In this case (omitting the index j), we get only one possibility $y_1 = 4$, $z_1 = \pm 2$, except in the case $k = 5$.

In the case $k = 5$, we get Pellian equation $9x^2 - y^2 = 128$. It is easy to see that its all solutions are given by $x = 4, 6, 11$. From that we conclude that the only extensions of the $D(16)$-pair $\{1, 9\}$ to a triple are $\{1, 9, 20\}$ and $\{1, 9, 105\}$, which implies that only extension of $D(16)$-triple $\{1, 9, 20\}$ to $D(16)$-quadruple is $\{1, 9, 20, 105\}$. So from now on, we can assume that $k \geq 7$ and k is odd.

We have proved the following lemma.

Lemma 2.1. *All solutions (z, x) of (2.2) are given by*

$$z\sqrt{k-4} + x\sqrt{k} = \left(z_0\sqrt{k-4} + x_0\sqrt{k}\right)\left(\frac{k-2+\sqrt{k(k-4)}}{2}\right)^m, \tag{2.9}$$

where m is a nonnegative integer and $(z_0, x_0) \in \{(\pm 2, 4), (\pm\sqrt{3k+4}, \sqrt{3k+4})\}$.

All solutions (z, y) of (2.3) are given by

$$z\sqrt{k+4} + y\sqrt{k} = \left(\pm 2\sqrt{k-4} + 4\sqrt{k}\right)\left(\frac{k+2+\sqrt{k(k+4)}}{2}\right)^n, \tag{2.10}$$

where n is a nonnegative integer.

2.2. Congruences. From (2.4) we conclude that $z = v_m$, for some $m \geq 0$, where the sequence $(v_m)_{m \geq 0}$ is defined by

$$v_0 = z_0, \qquad v_1 = \frac{1}{2}\left(z_0(k-2) + x_0 k\right), \qquad v_{m+2} = (k-2)v_{m+1} - v_m. \tag{2.11}$$

From (2.8) we conclude that $z = w_n$, for some $n \geq 0$, where $(w_n)_{n\geq 0}$ is defined by

$$w_0 = z_1, \qquad w_1 = \frac{1}{2}(z_1(k+2) + y_1 k), \qquad w_{n+2} = (k+2)w_{n+1} - w_n. \tag{2.12}$$

We have now transformed the system of (2.2) and (2.3) to finitely many Diophantine equations of the form $v_m = w_n$. By induction, from (2.11) and (2.12), we get $v_m \equiv (-1)^m z_0 (\operatorname{mod} k)$ and $w_n \equiv z_1 (\operatorname{mod} k)$. Then we get $z_0^2 = 3k+4$ is possible only in the case $k = 7$. And that case will be considered at the end of this section. From now we will assume $z_0 = \pm 2$, $z_1 = \pm 2$. We will consider the equation $z = v_m = w_n$ for $m, n \geq 6$, since for the remaining values of m and n it is easy to check if for some k the equality can hold.

From (2.11) and (2.12) we get the following lemma by induction.

Lemma 2.2. *If $m, n \geq 6$, then*

$$\begin{aligned} (k+2)(k-3)^{m-1} &< v_m < (3k-2)(k-2)^{m-1}, \\ (k-2)(k+1)^{n-1} &< w_n < (3k+2)(k+2)^{n-1}. \end{aligned} \tag{2.13}$$

Lemma 2.3. *If $v_m = w_n$, and $m, n \geq 6$, then $n \leq m < 1.73n$.*

Proof. From Lemma 2.2, $v_m = w_n$ implies

$$(k+2)(k-3)^{m-1} < (3k+2)(k+1)^{n-1}, \tag{2.14}$$

and $(k-3)^m < 3(k+2)^n$. Then

$$\frac{m}{n} < \frac{\log 3}{n \log(k-3)} + \frac{\log(k+2)}{\log(k-3)}, \tag{2.15}$$

which implies $m < 0.8 + 1.59n < 1.73n$. Lemma 2.2 also implies

$$(k-2)(k+1)^{n-1} < (3k-2)(k-2)^{m-1}, \tag{2.16}$$

and $(k+1)^{n-2} < 3(k-2)^{m-2}$. Then

$$\frac{n-2}{m-2} < \frac{\log 3}{(m-2)\log(k+1)} + \frac{\log(k-2)}{\log(k+1)}, \tag{2.17}$$

which implies $n - 2 < 0.53 + m - 2$, and $n < m + 1$, which proves the statement. □

The following lemma can also be proved by induction.

Lemma 2.4.

(i) *If $z_0 = 2$, then $v_m \equiv (-1)^{m+1}(m^2+2m)k + (-1)^m \cdot 2 (\operatorname{mod} k^2)$.*
(ii) *If $z_0 = -2$, then $v_m \equiv (-1)^m (m^2-2m)k + (-1)^{m+1} \cdot 2 (\operatorname{mod} k^2)$.*
(iii) *If $z_1 = 2$, then $w_n \equiv (n^2+2n)k + 2 (\operatorname{mod} k^2)$.*
(iv) *If $z_1 = -2$, then $w_n \equiv -(n^2-2n)k - 2 (\operatorname{mod} k^2)$.*

Lemma 2.5 gives us the lower bound of m, depending on k.

Lemma 2.5. *If $v_m = w_n$, and $m, n \geq 6$, then $m > \sqrt{k/3}$.*

Proof. Assume $v_m = w_n$, for $m,n \geq 6$ and $m \leq \sqrt{k/3}$. Then using Lemma 2.4, when we consider the congruence relations, absolute values of both hand sides are less than k^2, so we actually have the equalities. In the case $z_0 = z_1 = 2$, we have

$$(m^2+2m)k = k^2 - (n^2+2n)k, \tag{2.18}$$

and $m^2+2m+n^2+2n = k$, which is obviously impossible because the left-hand side is less than k (we also use Lemma 2.3). On the same way, we get the contradiction in the remaining three cases. □

2.3. Large parameters. In this section we prove that, for $k > 2.67 \cdot 10^7$, the equation $v_m = w_n$, for $n,m \geq 6$, has no solution. First we have to estimate $\log z$, where $z = v_m = w_n$.

Lemma 2.6. *Let $z = v_m = w_n$, $n,m \geq 6$. Then*

$$\log z > \sqrt{\frac{k}{3}}\log(k-3) - \log 3. \tag{2.19}$$

Proof. Let $z = v_m$. We can now consider both cases at the same time, if we define $z = |v'_m|$, where $(v'_m)_{m\in\mathbb{Z}}$ is a sequence defined by

$$v'_0 = 2, \qquad v'_1 = 3k-2, \qquad v'_{m+2} = (k-2)v'_{m+1} - v'_m, \quad m \in \mathbb{Z}. \tag{2.20}$$

If $\varphi = (k-2+\sqrt{k(k-4)})/2$, it is not hard to see that for $m \geq 0$ we have $v'_m \geq \varphi^m$, and $|v'_m| \geq (1/3)\varphi^{|m|}$.

Then if $z = v_m$, and $m \geq 6$, we have

$$z \geq \frac{1}{3}\varphi^m = \frac{1}{3}\left(\frac{k-2+\sqrt{k(k-4)}}{2}\right)^m > \frac{1}{3}(k-3)^m > \frac{1}{3}(k-3)^{\sqrt{k/3}}. \tag{2.21}$$

The last inequality follows from Lemma 2.5. □

We will now apply Bennett's theorem [6, Theorem 3.2], to obtain the upper bound for $\log z$. Let us first define $\theta_1 = \sqrt{(k-4)/k}$, $\theta_2 = \sqrt{(k+4)/k}$.

Lemma 2.7. *Let x, y, z be positive solutions of the system of (2.2) and (2.3). Then*

$$\max\left\{\left|\theta_1 - \frac{x}{z}\right|, \left|\theta_2 - \frac{y}{z}\right|\right\} < 11z^{-2}. \tag{2.22}$$

Proof. Using (2.2) and (2.3) we get

$$\begin{aligned}
\left|\theta_1 - \frac{x}{z}\right| &= \left|\frac{k-4}{k} - \frac{x^2}{z^2}\right| \cdot \left|\sqrt{\frac{k-4}{k}} + \frac{x}{z}\right|^{-1} < \frac{1}{kz^2}|-12k-16|\left(2\sqrt{1-\frac{4}{k}}\right)^{-1} < 11z^{-2}, \\
\left|\theta_2 - \frac{y}{z}\right| &= \left|\frac{k+4}{k} - \frac{y^2}{z^2}\right| \cdot \left|\sqrt{\frac{k+4}{k}} + \frac{y}{z}\right|^{-1} < \frac{1}{kz^2}|-12k+16|\left(2\sqrt{1+\frac{4}{k}}\right)^{-1} < 11z^{-2}.
\end{aligned} \tag{2.23}$$

□

Theorem 2.8 [6, Theorem 3.2]. *Let a_i, p_i, q, and N be integers for $0 \le i \le 2$ such that $a_0 < a_1 < a_2$, $a_j = 0$ for some $0 \le j \le 2$, $q \ne 0$, and $N > M^9$, where $M = \max\{|a_i| : 0 \le i \le 2\} \ge 3$. Then*

$$\max\left\{\left|\sqrt{1+\frac{a_i}{N}}-\frac{p_i}{q}\right| : 0 \le i \le 2\right\} > (130N\gamma)^{-1}q^{-\lambda}, \tag{2.24}$$

where

$$\lambda = 1+\frac{\log(32.04N\gamma)}{\log\left(1.68N^2\prod_{0\le i<j\le 2}(a_i-a_j)^{-2}\right)},$$
$$\gamma = \begin{cases} \dfrac{(a_2-a_0)^2(a_2-a_0)^2}{2a_2-a_0-a_1}, & \text{if } a_2-a_1 \ge a_1-a_0, \\ \dfrac{(a_2-a_0)^2(a_1-a_0)^2}{a_1+a_2-2a_0}, & \text{if } a_2-a_1 < a_1-a_0. \end{cases} \tag{2.25}$$

We will now apply Theorem 2.8 to the following numbers:

$$a_0 = -4, \qquad a_1 = 0, \qquad a_2 = 4, \qquad N = k, \qquad p_0 = x, \qquad p_2 = y, \qquad q = z, \tag{2.26}$$

for $k > 2.67 \cdot 10^7 > 4^9$, because then the condition of the theorem is satisfied. Then

$$\max\left\{\left|\theta_1-\frac{x}{z}\right|, \left|\theta_2-\frac{y}{z}\right|\right\} > (130k\gamma)^{-1}z^{-\lambda}, \tag{2.27}$$

where

$$\lambda = 1+\frac{\log(2734.08k)}{\log(0.0001025k^2)}, \qquad \gamma = \frac{256}{3}. \tag{2.28}$$

From Lemma 2.7, we get $(11094k)^{-1}z^{-\lambda} < 11z^{-2}$, and $\log z < \log(122034k)/(2-\lambda)$. Moreover,

$$\frac{1}{2-\lambda} = \frac{1}{1-\log(2734.08k)/\log(0.0001025k^2)} < \frac{2\log(0.01k)}{\log(3.75\cdot 10^{-8}k)}. \tag{2.29}$$

Lemma 2.6 now implies

$$\sqrt{\frac{k}{3}}\log(k-3)-\log 3 < \frac{2\log(112034k)\log(0.01k)}{\log(3.75\cdot 10^{-8}k)}. \tag{2.30}$$

Function on the left-hand side is increasing faster, and for $k > 2.67 \cdot 10^7$, the inequality is not satisfied, so we proved the following proposition.

Proposition 2.9. *If $k > 2.67 \cdot 10^7$, then $v_m = w_n$ has no solution for $m, n \ge 6$.*

2.4. Linear form in logarithms. In this section we will prepare everything for Baker-Davenport reduction. We want to prove that the statement of Proposition 2.9 holds for $k \le 2.67 \cdot 10^7$.

Lemma 2.10. *If $v_m = w_n$, $m,n \ge 6$, then*

$$0 < m\log\alpha_1 - n\log\alpha_2 + \log\alpha_3 < 20\alpha_2^{-2n}, \tag{2.31}$$

where

$$\alpha_1 = \frac{k-2+\sqrt{k(k-4)}}{2}, \qquad \alpha_2 = \frac{k+2+\sqrt{k(k+4)}}{2}, \qquad \alpha_3 = \frac{\sqrt{k+4}(2\sqrt{k}\pm\sqrt{k-4})}{\sqrt{k-4}(2\sqrt{k}\pm\sqrt{k+4})}. \tag{2.32}$$

Proof. From relations (2.4) and (2.8) we get

$$\begin{aligned} v_m &= \frac{2\sqrt{k}\pm\sqrt{k-4}}{\sqrt{k-4}}\left(\frac{k-2+\sqrt{k(k-4)}}{2}\right)^m - \frac{2\sqrt{k}\mp\sqrt{k-4}}{\sqrt{k-4}}\left(\frac{k-2-\sqrt{k(k-4)}}{2}\right)^m, \\ w_n &= \frac{2\sqrt{k}\pm\sqrt{k+4}}{\sqrt{k+4}}\left(\frac{k+2+\sqrt{k(k+4)}}{2}\right)^n - \frac{2\sqrt{k}\mp\sqrt{k+4}}{\sqrt{k+4}}\left(\frac{k+2-\sqrt{k(k+4)}}{2}\right)^n. \end{aligned} \tag{2.33}$$

Then if we define

$$P = \frac{2\sqrt{k}\pm\sqrt{k-4}}{\sqrt{k-4}}\left(\frac{k-2+\sqrt{k(k-4)}}{2}\right)^m, \qquad Q = \frac{2\sqrt{k}\pm\sqrt{k+4}}{\sqrt{k+4}}\left(\frac{k+2+\sqrt{k(k+4)}}{2}\right)^n, \tag{2.34}$$

then $v_m = w_n$, $m,n \ge 6$, implies $P - ((3k+4)/(k-4))P^{-1} = Q - ((3k-4)/(k+4))Q^{-1}$. Obviously $P,Q > 1$. Furthermore,

$$P - Q = \frac{3k+4}{k-4}P^{-1} - \frac{3k-4}{k+4}Q^{-1} > \frac{3k+4}{k-4}(Q-P)P^{-1}Q^{-1}, \tag{2.35}$$

implies $P > Q$. We also have

$$Q \ge \frac{2\sqrt{k}-\sqrt{k+4}}{\sqrt{k+4}}\left(\frac{k+2+\sqrt{k(k+4)}}{2}\right)^6 > 0.59(k+1)^6. \tag{2.36}$$

From $P > Q - (3k-4)/(k+4)$, we conclude $P^{-1} < ((1-(3k-4)/(k+4))Q^{-1})^{-1}Q^{-1}$. Hence

$$\begin{aligned} P - Q &= \frac{3k+4}{k-4}P^{-1} - \frac{3k-4}{k+4}Q^{-1} < \frac{3k+4}{k-4}\left(1-\frac{3k-4}{k+4}Q^{-1}\right)^{-1}Q^{-1} - \frac{3k-4}{k+4}Q^{-1} \\ &< \frac{3k+4}{k-4}\left(1-\frac{3k-4}{k+4}\cdot\frac{1}{0.59(k+1)^6}\right)^{-1}Q^{-1} - \frac{3k-4}{k+4}Q^{-1} < 6.8Q^{-1}. \end{aligned} \tag{2.37}$$

We have $0 < (P-Q)/P < 6.8P^{-1}Q^{-1} < 6.8Q^{-2}$. It implies

$$\begin{aligned} 0 < \log\frac{P}{Q} &= -\log\left(1-\frac{P-Q}{P}\right) < 6.8Q^{-2} + \left(6.8Q^{-2}\right)^2 \\ &< 6.8Q^{-2}\left(1+6.8\cdot\frac{1}{0.59^2(k+1)^{12}}\right) < 7Q^{-2}. \end{aligned} \tag{2.38}$$

The statement of the lemma follows from

$$7Q^{-2} \le 7\left(\frac{\sqrt{k+4}}{2\sqrt{k}-\sqrt{k+4}}\right)^2 \alpha_2^{-2n} < 20\alpha_2^{-2n}. \tag{2.39}$$

□

2.5. Reduction. To complete reduction, we will use Baker-Wüstholz theorem from [7].

Theorem 2.11 [7]. *Let $\Lambda = b_1\log\alpha_1 + \cdots + b_l\log\alpha_l \neq 0$ be a linear form of l logarithms of algebraic numbers $\alpha_1,\ldots,\alpha_l$ with integer coefficients $b_1,\ldots,b_l$. Then*

$$\log\Lambda > -18(l+1)!l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log B, \tag{2.40}$$

where $B = \max\{|b_j| : 1 \le j \le l\}$, d is a degree of the extension of algebraic number field generated by $\alpha_1,\ldots,\alpha_l$, and $h'(\alpha) = (1/d)\max\{h(\alpha), \log|\alpha|, 1\}$. $h(\alpha)$ denotes standard logarithmic Weil height of α.

In the notation of the last theorem, we have $d = 4$, $l = 3$, $B = m$, and minimal polynomials of α_1, α_2, α_3 are given by

$$\begin{gathered} \alpha_1^2 - (k-2)\alpha_1 + 1 = 0, \qquad \alpha_2^2 - (k+2)\alpha_2 + 1 = 0, \\ (3k-4)^2(k-4)^2\alpha_3^4 \pm 4(k+4)(3k-4)(k-4)^2\alpha_3^3 - 6\left(k^2-16\right)\left(7k^2+16\right)\alpha_3^2 \\ \pm 4(k-4)(3k+4)(k+4)^2\alpha_3 + (k+4)^2(3k+4)^2 = 0. \end{gathered} \tag{2.41}$$

Then we have the estimates

$$\begin{aligned} h'(\alpha_1) &= \frac{1}{2}\log\alpha_1 < \frac{1}{2}\log(k-1), \qquad h'(\alpha_2) = \frac{1}{2}\log\alpha_2 < \frac{1}{2}\log(k+2), \\ h'(\alpha_3) &= \frac{1}{4}\left\{\log(3k-4)^2(k-4)^2 + \log\frac{(3k+4)(k+4)\sqrt{k+4}\left(2\sqrt{k}+\sqrt{k-4}\right)}{(3k-4)(k-4)\sqrt{k-4}\left(2\sqrt{k}-\sqrt{k+4}\right)}\right\} \\ &< \frac{1}{4}\left\{\log(3k-4)^2(k-4)^2 + \log\frac{5.54(3k+4)(k+4)}{(3k-4)(k-4)}\right\} \\ &= \frac{1}{4}\log 5.54(3k-4)(k-4)(3k+4)(k+4) < \frac{1}{4}\log 49.86k^4. \end{aligned} \tag{2.42}$$

If we apply this to Lemma 2.10, and use $m < 2n$, we get

$$\frac{m}{\log m} < 2.4\cdot 10^{14}\log(k-1)\log 49.86k^4. \tag{2.43}$$

Using $k < 2.67\cdot 10^7$, we conclude $m < 2\cdot 10^{19}$.

Now we need one version of Baker-Davenport lemma (see [3, 8]).

Lemma 2.12. *Let M be a positive integer and let p/q be a convergent of the continued fraction expansion of the number κ such that $q > 6M$. Furthermore, let $\varepsilon = \|\mu q\| - M\|\kappa q\|$, where $\|\cdot\|$ denotes the distance to the nearest integer. If $\varepsilon > 0$, then the inequality*

$$0 < m\kappa - n + \mu < AB^{-m} \tag{2.44}$$

in integers m and n has no solution for

$$\frac{\log(Aq/\varepsilon)}{\log B} \le m \le M. \tag{2.45}$$

In the notation of the Lemma 2.12, we have

$$\kappa = \frac{\log\alpha_1}{\log\alpha_2}, \qquad \mu = \frac{\log\alpha_3}{\log\alpha_2}, \qquad A = \frac{20}{\log\alpha_2}, \qquad B = \alpha_2, \qquad M = 2\cdot 10^{19}. \tag{2.46}$$

We have implemented this method in Mathematica 5.0, and we got $m \le 6$ for all $k < 2.67\cdot 10^7$. Now we have to see what is happening for $m \le 6$. But it is easy to check, because we get polynomial equations, that it gives us two extensions $d = 0$, and $d = k^3 - 4k$. Before we formulate our main result, we have to check what is happening in the case $k = 7$ for the other fundamental solution. But using the same methods, only it is easier this time, we do not get anything new, actually we get one "extension" $d = 3$, which will not give us the $D(16)$-quadruple because in this case $k - 4 = 7 - 4 = 3$.

Theorem 2.13. *The only extension of the $D(16)$-triple $\{k-4, k+4, 4k\}$, for integer $k \ge 5$, to a $D(16)$-quadruple $\{k-4, k+4, 4k, d\}$, is given by $d = k^3 - 4k$.*

3. Extension of $D(16)$-triples of the form $\{k-4, 4k, 9k-12\}$

3.1. System of Pellian equations. In this section we will not give all the details, because we will use the same method as in the case of previous parametric family. Actually one Pellian equation will be exactly the same.

Assume that $D(16)$-triple $\{k-4, 4k, 9k-12\}$, for $k \ge 5$, can be extended to a quadruple $\{k-4, 4k, 9k-12, d\}$. Then there exist positive integers x, y, z, such that

$$(k-4)d + 16 = x^2, \qquad 4kd + 16 = 4y^2, \qquad (9k-12)d + 16 = z^2. \tag{3.1}$$

Eliminating d, we get the following system of simultaneous Pellian equations:

$$(k-4)y^2 - kx^2 = -12k - 16, \tag{3.2}$$

$$kz^2 - (9k-12)y^2 = -20k + 48. \tag{3.3}$$

Lemma 3.1. *All solutions (z, x) of (3.2) are given by*

$$y\sqrt{k-4} + x\sqrt{k} = \left(y_0\sqrt{k-4} + x_0\sqrt{k}\right)\left(\frac{k-2+\sqrt{k(k-4)}}{2}\right)^m, \tag{3.4}$$

where m is nonnegative integer, and $(y_0, x_0) \in \{(\pm 2, 4), (\pm\sqrt{3k+4}, \sqrt{3k+4})\}$.

All solutions (z, y) of (3.3) are given by

$$z\sqrt{k} + y\sqrt{9k-12} = \left(z_1\sqrt{k} + y_1\sqrt{9k-12}\right)\left(\frac{3k-2+\sqrt{k(9k-12)}}{2}\right)^n, \tag{3.5}$$

where n is nonnegative integer, and $(z_1, y_1) \in \{(\pm 4, 2), (\pm\sqrt{9k+4}, \sqrt{k+4})\}$.

It is easy to see that it is enough to consider solutions of (3.3) such that $y_1^2 \equiv 4 (\operatorname{mod} k)$. Otherwise we will not get the extension with integer. Then we get the following possibilities for y_1^2: $y_1^2 = 4$, $k+4$. We have two cases $y_1 = 2$, $z_1 = \pm 4$, or $y_1^2 = k+4$, $z_1^2 = 9k+4$. Obviously the second one is possible only if $k = 5$ and we will consider that case at the end of the section.

3.2. Congruences. From Lemma 3.1, we get $y = v_m$, for some $m \geq 0$, where $(v_m)_{m\geq 0}$ is defined by

$$v_0 = y_0, \qquad v_1 = \frac{1}{2}(y_0(k-2) + x_0 k), \qquad v_{m+2} = (k-2)v_{m+1} - v_m. \tag{3.6}$$

On the other hand, $y = w_n$, for some $n \geq 0$, where $(w_n)_{n\geq 0}$ is defined by

$$w_0 = y_1, \qquad w_1 = \frac{1}{2}(y_1(3k-2) + z_1 k), \qquad w_{n+2} = (3k-2)w_{n+1} - w_n. \tag{3.7}$$

We again transform our system of (3.2) and (3.3) to finitely many equations of the form $v_m = w_n$. From (3.6) and (3.7), we get by induction $v_m \equiv (-1)^m y_0 (\operatorname{mod} k)$ and $w_n \equiv (-1)^n y_1 (\operatorname{mod} k)$, that is, it is enough to consider only such solutions because otherwise we will not get the extension of our $D(16)$-triple. Then we have that only possibility for $y_0^2 = 3k+4$ is in the case $k = 7$, and we will consider that case separately, but again it will give us the "extension" $d = 3$.

The following two lemmas can be proven as similar as the lemmas in the previous section.

Lemma 3.2. *If $m, n \geq 6$, then*

$$\begin{gathered}(k+2)(k-3)^{m-1} < v_m < (3k-2)(k-2)^{m-1}, \\ (k-2)(3k-3)^{n-1} < w_n < (5k-2)(3k-2)^{n-1}.\end{gathered} \tag{3.8}$$

Lemma 3.3. *If $v_m = w_n$, $m, n \geq 6$, then $n < m < 3.71n$.*

We get the following lemma by induction. Remember we only have to consider sequences such that $v_m \equiv (-1)^m y_0 (\operatorname{mod} k)$ and $w_n \equiv (-1)^n y_1 (\operatorname{mod} k)$.

Lemma 3.4.

(i) *If $y_0 = 2$, then $v_m \equiv (-1)^{m+1}(m^2+2m)k + (-1)^m \cdot 2 (\operatorname{mod} k^2)$.*
(ii) *If $y_0 = -2$, then $v_m \equiv (-1)^m(m^2-2m)k + (-1)^{m+1} \cdot 2 (\operatorname{mod} k^2)$.*
(iii) *If $z_1 = 4$, then $w_n \equiv (-1)^{n+1}(3n^2+2n)k + (-1)^n \cdot 2 (\operatorname{mod} k^2)$.*
(iv) *If $z_1 = -4$, then $w_n \equiv (-1)^{n+1}(3n^2-2n)k + (-1)^n \cdot 2 (\operatorname{mod} k^2)$.*

From Lemma 3.4, we obtain the lower bound for m, depending on k.

Lemma 3.5. *If $v_m = w_n$, $m,n \geq 6$, then $m > \sqrt{k/5}$.*

3.3. Large parameters. We now prove that for $k > 4^9$, equation $v_m = w_n$, for $n,m \geq 6$, has no solution. First, we will get the lower bound for $\log y$, where $y = v_m = w_n$.

Lemma 3.6. *Let $y = v_m = w_n$, $n,m \geq 6$. Then*

$$\log y > \sqrt{\frac{k}{5}}\log(k-3) - \log 3. \tag{3.9}$$

Proof. The statement follows from Lemmas 2.6 and 3.5. □

Let us define

$$\theta_1 = \sqrt{\frac{k-4}{k}}, \qquad \theta_2 = \sqrt{\frac{9k-12}{k}}. \tag{3.10}$$

Lemma 3.7. *Let x, y, z be positive solutions of the system of (3.2) and (3.3). Then*

$$\max\left\{\left|\theta_1 - \frac{x}{y}\right|, \left|\theta_2 - \frac{z}{y}\right|\right\} < 17y^{-2}. \tag{3.11}$$

Now we will again apply the same Bennett's theorem. First we see that

$$\left|\sqrt{\frac{k-4}{k}} - \frac{x}{y}\right| = \left|\sqrt{1-\frac{12}{3k}} - \frac{3x}{3y}\right|, \qquad \left|\sqrt{\frac{9k-12}{k}} - \frac{z}{y}\right| = 3\left|\sqrt{1-\frac{4}{3k}} - \frac{z}{3y}\right|. \tag{3.12}$$

We apply Bennett's theorem (Theorem 2.8), actually we can make slight modification in special case similar to [5, Theorem 4.1.] to get better constants to the following numbers:

$$a_0 = -12, \quad a_1 = -4, \quad a_2 = 0, \quad N = 3k, \quad p_0 = 3x, \quad p_1 = z, \quad q = 3y, \tag{3.13}$$

for $k > 4.34 \cdot 10^8$, and like for the parametric family from Section 2, we get the following proposition.

Proposition 3.8. *If $k > 4.34 \cdot 10^8$, then equation $v_m = w_n$ has no solution for $m,n \geq 6$.*

3.4. Linear form in logarithms and reduction.

Lemma 3.9. *If $v_m = w_n$, $m,n \geq 6$, then*

$$0 < m\log\alpha_1 - n\log\alpha_2 + \log\alpha_3 < 244\alpha_2^{-2n}, \tag{3.14}$$

where

$$\alpha_1 = \frac{k-2+\sqrt{k(k-4)}}{2}, \qquad \alpha_2 = \frac{3k-2+\sqrt{k(9k-12)}}{2},$$
$$\alpha_3 = \frac{\sqrt{9k-12}\left(2\sqrt{k} \pm \sqrt{k-4}\right)}{\sqrt{k-4}\left(\sqrt{9k-12} \pm 2\sqrt{k}\right)}. \tag{3.15}$$

Using Baker-Wüstholz theorem (Theorem 2.11), we get $m < 3 \cdot 10^{19}$. Then by Baker-Davenport reduction (Lemma 2.12), we obtain $m \leq 10$. So we have to check what is happening for $m \leq 10$. But again we have polynomial equalities that are easy to solve in integers. We get two extensions with $d = 0$, and $d = 9k^3 - 48^2 + 76k - 32$. In the case $k = 5$, we get one more extension with the element $d = 105$.

Theorem 3.10. *The only extension of the $D(16)$-triple $\{k-4, 4k, 9k-12\}$, for an integer $k > 5$, to a $D(16)$-quadruple $\{k-4, 4k, 9k-12, d\}$, is given by $d = 9k^3 - 48k^2 + 76k - 32$. Furthermore, all extensions of the $D(16)$-triple $\{1, 20, 33\}$ to a $D(16)$-quadruple are $\{1, 20, 33, 105\}$ and $\{1, 20, 33, 273\}$.*

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