Research Article

An Integral Representation of Standard Automorphic $L$ Functions for Unitary Groups

Yujun Qin

Received 29 May 2006; Revised 5 November 2006; Accepted 26 November 2006

Recommended by Dihua Jiang

Let $F$ be a number field, $G$ a quasi-split unitary group of rank $n$. We show that given an irreducible cuspidal automorphic representation $\pi$ of $G(\mathbb{A})$, its (partial) $L$ function $L^S(s, \pi, \sigma)$ can be represented by a Rankin-Selberg-type integral involving cusp forms of $\pi$, Eisenstein series, and theta series.

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1. Introduction

Let $F$ be a number field, $G$ the general linear group of degree $n$ defined over $F$. Let $\pi$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. In [1–3], a Rankin-Selberg-type integral is constructed to represent the $L$ function of $\pi$. That the integrals of Jacquet, Piatetski-Shapiro, and Shalika are Eulerian follows from the uniqueness of Whittaker models and the fact that cuspidal representations of $GL_n$ are always generic. For other reductive group whose cuspidal representations are not always generic, in [4], Piatetski-Shapiro and Rallis construct a Rankin-Selberg integral for symplectic group $G = \text{Sp}_{2n}$ to represent the partial $L$ function of a cuspidal representation $\pi$ of $G(\mathbb{A})$. In this paper, we apply similar method to the quasi-split unitary group of rank $n$.

Let $F$ be a number field, $E$ a quadratic field extension of $F$. Let $V$ be a $2n$-dimensional vector space over $E$ with an anti-Hermitian form

$$\eta_{2n} = \begin{pmatrix} & \end{pmatrix}$$

on it. Let $G = U(\eta_{2n})$ be the unitary group of $\eta_{2n}$. Let $\pi$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, $f$ a cusp form belonging to the isotypic space of $\pi$. The
Rankin-Selberg-type integral is defined by

$$\int_{G(F)\backslash G(A)} f(g)E(g,s)\theta(g)dg,$$  \tag{1.2}

where $E(g,s)$ is an Eisenstein series associated with a degenerate principle series, $\theta$ is a theta series defined by the Weil representation of $Sp(V \otimes W)$, where $W$ is a nondegenerate Hermitian space of dimension $n$. We show in Theorem 6.3 that (1.2) represent the standard partial $L$ function $L^\pi(s, \sigma)$ of $\pi$.

In [4], after showing the Rankin-Selberg integral has a Euler product decomposition, Piatetski-Shapiro and Rallis continued to show that if $n/2 + 1$ is a pole of partial $L$ function, then theta lifting is nonvanishing [4, Proposition on page 120]. There should be a parallel application of our paper, that is, relate the largest possible pole with nonvanishing of period integral.

2. Notations and conventions

Let $F$ be a field of characteristic 0, $E$ a commutative $F$-algebra with rank two. Let $\rho$ be an $F$-linear automorphism of $E$. We are interested in $(E, \rho)$ of the following two types:

1. $E$ is a quadratic field extension of $F$, $\rho$ is the nontrivial element of $Gal(E/F)$;
2. $E = F \oplus F$, $(x, y)^\rho = (y, x)$.

Let $tr$ be the trace of $E$ over $F$, that is, it is defined by

$$tr(z) = z + z^\rho, \quad z \in E.$$

(2.1)

Let $V$ be a left $E$-module, $\varphi : V \times V \to E$ a nonsingular $\epsilon$-Hermitian form on $V$, here $\epsilon = \pm 1$. The unitary group of $\varphi$ is

$$U(\varphi) = \{ \alpha \in GL(V, E) \mid \varphi(x\alpha, y\alpha) = \varphi(x, y), \forall x, y \in V \}.$$  \tag{2.2}

Let $\epsilon' = -\epsilon$ so that $\epsilon\epsilon' = -1$. Let $(W, \varphi')$ be a nonsingular $\epsilon'$-Hermitian space. Put

$$\mathbb{W} = V \otimes W.$$  \tag{2.3}

Then $\mathbb{W}$ is a nonsingular symplectic space over $F$ with symplectic form

$$\psi = tr(\varphi \otimes \varphi').$$  \tag{2.4}

Let $G = U(\varphi)$, $G' = U(\varphi')$ be the unitary groups corresponding to $\varphi$ and $\varphi'$, respectively. It is well known that $G \times G'$ embeds as a dual pair in $Sp(\psi)$.

We often express various objects by matrices. For a matrix $x$ with entries in $E$, put

$$x^* = x^\rho, \quad x^{-\rho} = (x^\rho)^{-1}, \quad \hat{x} = x^{-\rho},$$

(2.5)

assuming $x$ to be square and invertible if necessary. Assume that $V \cong E^\ell$ for some nonzero positive integer $\ell$. Let $\varphi_0$ be an $\ell \times \ell$ matrix satisfying $\varphi_0^* = \epsilon \varphi_0$. We can define an $\epsilon$-Hermitian form $\varphi$ on $V$ by requiring

$$\varphi(x, y) = x\varphi_0 y^*.$$  \tag{2.6}
Then the unitary group $U(\varphi)$ is isomorphic to the subgroup of $\text{GL}_\ell(E)$ consisting elements $g$ satisfying

$$g \varphi_0 g^* = \varphi_0.$$  \hspace{1cm} (2.7)

In the following we let $\varepsilon = -1$. Then $\varphi$ is a nonsingular skew-Hermitian form, hence $\ell = 2n$ for some positive integer $n$. Let $e_1, \ldots, e_{2n}$ be a basis of $V$ such that $\varphi$ is represented by

$$\eta_{2n} = \begin{pmatrix} 1_n \\ -1_n \end{pmatrix}. \hspace{1cm} (2.8)$$

Put

$$X = \oplus_{i=1}^n Ee_i, \quad Y = \oplus_{i+1}^{2n} Ee_i. \hspace{1cm} (2.9)$$

Then $X$, $Y$ are maximal isotropic spaces of $V$. Let $P$ be the maximal parabolic subgroup of $G$ preserving $Y$. Then

$$P(F) = \left\{ \begin{pmatrix} g & gu \\ g^{-1} \end{pmatrix} | g \in \text{GL}_n(E), u \in S(F) \right\}. \hspace{1cm} (2.10)$$

Here

$$S(F) = \{ b \in M_{n,n}(E) | b^* = b \} \hspace{1cm} (2.11)$$

is the set of Hermitian matrices of degree $n$. Let $N$ be the unipotent radical of $P$. Then $N(F)$ consists of elements of the following type:

$$n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \text{ with } b \in S(F). \hspace{1cm} (2.12)$$

Let

$$M = \{ g \in P | Xg \subset X, Yg \subset Y \}. \hspace{1cm} (2.13)$$

Then $M$ is a Levi subgroup of $P$. The $F$-rational points $M(F)$ of $M$ consists of elements of the following form:

$$m(a) = \begin{pmatrix} a \\ & a \end{pmatrix}, \text{ with } a \in \text{GL}_n(E). \hspace{1cm} (2.14)$$

Define an action of $\text{GL}_n(E)$ on $S(F)$ by

$$(a, b) \rightarrow aba^*, \text{ with } a \in \text{GL}_n(E), b \in S(F). \hspace{1cm} (2.15)$$

It is equivalent to the adjoint action of $M$ on $N$, since

$$m(a)n(b)m(a)^{-1} = n(aba^*). \hspace{1cm} (2.16)$$

We will say “the action of $M(F)$ on $S(F)$” if no confusion is caused.
Let $O$ be the unique open orbit of $M(F) \setminus S(F)$, then
\[ O = \{ b \in S(F) \mid \det b \neq 0 \}. \tag{2.17} \]

For $\beta \in O$, let $M_\beta$ be the stabilizer of $\beta$. Since $\beta$ is a nonsingular Hermitian matrix,
\[ M_\beta \cong U(\beta) \tag{2.18} \]
is the unitary group of $\beta$.

Let $\mathcal{Y} = Y \otimes W$. For $w \in \mathcal{Y}$, let us write
\[ w = \sum_{i=1}^{n} e_{n+i} \otimes w_i, \quad \text{with } w_i \in W, \ i = 1, \ldots, n. \tag{2.19} \]

Define the moment map $\mu : \mathcal{Y} \to S(F)$ by
\[ \mu(w) = (\varphi'(w_i, w_j))_{1 \leq i, j \leq n}. \tag{2.20} \]

It is clear that if $m = m(a) \in M(F)$, then
\[ \mu(wm) = 'a\mu(w)a'. \tag{2.21} \]

Denote the image of $\mu$ by $\mathcal{C}$, then it is invariant under $M(F)$. Let $T$ be a Hermitian matrix representing $\varphi'$. If $\dim W = n$, then $T \in \mathcal{C} = O$. In particular, from (2.18),
\[ M_T = G'. \tag{2.22} \]

### 3. Localization of various objects

Let $F$ be a number field, $E$ a quadratic field extension of $F$. Let $v$ be the set of all places of $F$, $a, f$ be the sets of Archimedean and non-Archimedean places, respectively. Then $v = a \cup f$. For $v \in v$, let $F_v$ be the $v$-completion of $F$, $\mathcal{O}_v$ the valuation ring of $F_v$ if $v$ is finite. Let $\mathbb{A}_v, \mathbb{A}_E$ be the rings of adeles of $F$ and $E$, respectively.

Let $\rho$ be the generator of $\text{Gal}(E/F)$. For $v \in v$, let $E_v = E \otimes F_v$. We may extend $\rho$ to $E_v$, denote it by $\rho_v$. Then $E_v$ is a quadratic extension of $F_v$, $\rho_v$ is an $E_v$-automorphism of $E_v$ of order 2. Corresponding to $v$ is split in $E$ or not, the couple $(E_v, \rho_v)$ belongs to one of the following two cases.

1. **Case NS:** $v$ remains prime in $E$. Hence $E_v$ is a quadratic field extension of $F_v, \rho_v \in \text{Gal}(E/F)$ is the nontrivial element.

2. **Case S:** $v$ splits in $E$. Then $E_v = F_v \oplus F_v$ and $(x, y)^{\rho_v} = (y, x)$ for $(x, y) \in E_v$.

Let $\gamma$ be a nontrivial Hecke character of $E$, that is, it is a continuous homomorphism
\[ \gamma : \mathbb{A}_E^\times \longrightarrow S^1 \tag{3.1} \]
such that $\gamma(E_v^\times) = 1$. For $v \in v$, let $\gamma_v$ be the restriction of $\gamma$ to $E_v^\times$, then $\gamma = \otimes_v \gamma_v$.

For an algebraic group $H$ defined over $F$, we let $H(F_v)$ be the set of $F_v$-points of $H$. Put
\[ H_a = \prod_{v \in a} H(F_v), \quad H_f = \prod_{v \in f} H(F_v), \tag{3.2} \]
where the prime indicates restricted product with respect to $H(\mathbb{C}_v)$. Then

$$H(\mathbb{A}) = H_a H_f. \quad (3.3)$$

Let $G = U(\eta_n)$ be the quasi-split even unitary group of rank $n$ defined over $F$. We have defined the standard Siegel parabolic subgroup $P = MN$ of $G$ in Section 2. Keep notations of last section. For $v \in \mathfrak{f}$, the localization of these algebraic groups are as follows.

1. Case NS: $v$ remains prime in $E$. In this case,

$$G(F_v) = U(\eta_n)(F_v),$$

$$M(F_v) = \{ m(a) \mid a \in \text{GL}_n(F_v) \}, \quad (3.4)$$

$$N(F_v) = \{ n(X) \mid X \in S(F_v) \}.$$  

2. Case S: $v$ splits in $E$. In this case,

$$G(F_v) = \text{GL}_{2n}(F_v),$$

$$M(F_v) = \left\{ m(A,B) \mid m(A,B) = \begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix}, \ A,B \in \text{GL}_n(F_v) \right\}, \quad (3.5)$$

$$N(F_v) = \left\{ n(X) \mid n(X) = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}, \ X \in \text{M}_{n \times n}(F_v) \right\}.$$  

If $v \in \mathfrak{f}$ is a finite place, let $K_{0,v} = G(\mathbb{C}_v)$ be a maximal open compact subgroup of $G(F_v)$. For $g \in G(F_v)$, we have Iwasawa decomposition

\begin{align*}
\text{(Case NS)} & \quad g = n(X)m(a)k, \\
\text{(Case S)} & \quad g = n(X)m(A,B)k \quad (3.6)
\end{align*}

for some $k \in K_{0,v}$, $n(X)m(a)$ or $n(X)m(A,B)$ belong to $P(F_v)$.

4. Local computation

Our result relies heavily on the $L$ function of unitary group in [5] derived by Li. So in this section, we review the doubling method of Gelbart et al. [6] briefly and the main theorem of [5].

Let $F$ be non-Archimedean local field with characteristic 0, $\mathcal{O}$ the valuation ring of $F$ with uniformizer $\omega$. Let $| \cdot |$ be the normalized absolute value of $F$. Let $(E, \rho)$ be a couple as in Section 1. If $E$ is a field extension of $F$, let $\mathcal{O}_E$ be the ring of integer of $E$ with uniformizer $\omega_E$, $| \cdot |_E$ the normalized absolute value of $E$.

Let $V$ be $2n$-dimensional space over $E$ with skew-Hermitian form $\varphi = \eta_{2n}$, $G = U(V)$. Then

$$G(F) = U(\eta_{2n}), \quad \text{Case NS};$$

$$G(F) = \text{GL}_{2n}, \quad \text{Case S}. \quad (4.1)$$

Let $-V$ be the space $V$ with Hermitian form $-\varphi$. Define

$$\mathbb{V} = V \oplus -V. \quad (4.2)$$
Then $\varphi \oplus (-\varphi)$ is a nonsingular skew-Hermitian form on $V$. Let $H = U(V)$ be the unitary group of $V$. Then $K = H(\mathbb{C})$ is a maximal open compact subgroup of $H(F)$. We embed $G \times G$ into $H$ as a closed subgroup.

Define two maximal isotropic subspaces of $V$ as follows:

$$X = \{(v, -v) \mid v \in V\}, \quad Y = \{(v, v) \mid v \in V\}. \quad (4.3)$$

Then $V = X \oplus Y$. Let $Q$ be the maximal parabolic subgroup of $H$ preserving $Y$. Following [5], we define a rational character $x$ of $Q$ by

$$x(p) = \det(p|_Y)^{-1}, \quad p \in Q. \quad (4.4)$$

Choose a basis of $V$ compatible with the decomposition (4.3), we can write $p$ as a matrix:

$$p = \begin{pmatrix} a & * \\ \hat{a} & \end{pmatrix}, \quad \text{with } a \in \mathrm{GL}_{2n}. \quad (4.5)$$

Then $x(p) = \det(a)^{\rho}$. Let $\gamma$ be an unramified character of $F^\times$. Then $p \mapsto \gamma(x(p))$ is a character of $Q(F)$. For $s \in \mathbb{C}$, let $I(s, \gamma)$ be the space of smooth functions $f : H(F) \to \mathbb{C}$ satisfying

$$f(pg) = \gamma(x(p))|x(p)|^{s+(4n+1)/2}f(g), \quad p \in Q(F), \ g \in G(F). \quad (4.6)$$

$H(F)$ acts on $I(s, \gamma)$ by right multiplication. Let $I(s, \gamma)^K$ be the subspace of $K$-invariant elements of $I(s, \gamma)$. Since $\gamma$ is unramified, by Frobenius reciprocity,

$$\dim \mathbb{C} I(s, \gamma)^K = 1. \quad (4.7)$$

Let $\Phi_{K, s}$ be the unique $K$-invariant function in $I(s, \gamma)$ such that

$$\Phi_{K, s}(1) = 1. \quad (4.8)$$

One important property of $\Phi_{K, s}$ is the following.

**Lemma 4.1** (see [5, Lemma 3.2]). Let $K_0 = G(\mathbb{C})$ be a maximal open compact subgroup of $G(F)$. Then for $k_1, k_2 \in K_0, \ g \in G(F)$,

$$\Phi_{K, s}(k_1 g k_2, 1) = \Phi_{K, s}(g, 1), \quad (4.9)$$

here $(g, 1) \in G \times G \mapsto H$.

**4.1. $L$ functions.** Let $(\pi, V)$ be an unramified irreducible representation of $G(F)$, $(\check{\pi}, \check{V})$ the contragredient of $\pi$. Let $\langle \cdot, \cdot \rangle_\pi$ be the canonical pairing between $V$ and $\check{V}$. For $v \in V$, $\check{v} \in \check{V}$, define a matrix coefficient of $\pi$ by

$$\omega_\pi(g; v, \check{v}) = \langle gv, \check{v} \rangle_\pi, \quad g \in G(F). \quad (4.10)$$
If \(v\) and \(\check{v}\) are \(K_0\)-fixed elements of \(\pi\) and \(\check{\pi}\), respectively, then \(\omega_\pi(g; v, \check{v})\) is a spherical function of \(\pi\). In addition, if \(\langle v, \check{v} \rangle_\pi = 1\), then \(\omega_\pi(1; v, \check{v}) = 1\), we get the zonal spherical function \(\omega_\pi\) of \(\pi\).

Let \(^LG\) be the dual group of \(G\). Then

\[
^LG = \text{GL}_{2n}(\mathbb{C}) \rtimes \text{Gal}(E/F), \quad \text{Case NS}
\]

\[
^LG = \text{GL}_{2n}(\mathbb{C}), \quad \text{Case S.}
\]

For Case NS, the action of \(\text{Gal}(E/F)\) on \(\text{GL}_{2n}\) is given by

\[
g^\rho = \Phi_{2n} \cdot g^{-1} \Phi_{2n}^{-1}, \quad g \in \text{GL}_{2n}(\mathbb{C}).
\]

Here

\[
\Phi_{2n} = \begin{pmatrix}
1 & -1 & \cdots & -1 \\
-1 & 1 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & 1
\end{pmatrix}
\]

(4.13)

Since \(\pi\) is an unramified irreducible representation of \(G(F)\), it determines a unique semisimple conjugacy class \((a_\pi, \rho)\) (Case NS) or \(a_\pi\) (Case S) in \(^LG\) [7]. We can take a representative of \(a_\pi\) as follows:

\[
a_\pi = \text{diag}(a_1, \ldots, a_n, 1, \ldots, 1), \quad \text{Case NS},
\]

\[
a_\pi = \text{diag}(a_1, \ldots, a_{2n}), \quad \text{Case S},
\]

with \(a_i \in \mathbb{C}^\times, i = 1, \ldots, 2n\) [7, Section 6.9].

Let \(r\) be the natural action of \(\text{GL}_{2n}(\mathbb{C})\) on \(\mathbb{C}^{2n}\), \(\sigma\) the induced representation

\[
\sigma = \text{Ind}_{\text{GL}_{2n}(\mathbb{C})}^{^LG} (r), \quad \text{Case NS},
\]

\[
\sigma = \text{Ind}_{\text{GL}_{2n}(\mathbb{C})}^{\text{GL}_{2n}(\mathbb{C}) \times \mathbb{Z}/2\mathbb{Z}} r, \quad \text{Case S},
\]

(4.15)

respectively. Associate a local \(L\) function \(L(s, \pi, \sigma)\) to \(\pi\) by

\[
\text{Case NS : } L(s, \pi, \sigma) = \det \left(1 - \sigma(a_\pi, \rho) q^{-s}\right)^{-1}
\]

\[
= \prod_{i \leq n} \left[ (1 - a_i q^{-2s})(1 - a_i^{-1} q^{-2s}) \right]^{-1},
\]

(4.16)

\[
\text{Case S : } L(s, \pi, \sigma) = \det \left(1 - \sigma(a_\pi) q^{-s}\right)^{-1}
\]

\[
= \prod_{i \leq 2n} \left[ (1 - a_i q^{-s})(1 - a_i^{-1} q^{-s}) \right]^{-1},
\]

where \(q\) is the cardinality of residue field of \(F\).
The relation between the functions $\Phi_{K,s}$, $\omega_n$, and $L(s,\pi,\sigma)$ is as follows.

**Theorem 4.2** (see [5, Theorem 3.1]). *Notations as above. For $s \in \mathbb{C}$,*

\[
\int_{G(F)} \Phi_{K,s}(g,1) \omega_n(g) = \frac{L(s+1/2,\pi,\sigma)}{d_H(s)}.
\]

(4.17)

*Here*

\[
(Case \ NS) \quad d_H(s) = \frac{L(2s+1,\epsilon_{E/F})}{L(2s+2n+1,\epsilon_{E/F})} \prod_{0 \leq j < n} \xi(2s+2n-2j)L(2s+2n-2j+1,\epsilon_{E/F}),
\]

\[
(Case \ S) \quad d_H(s) = \prod_{j=1}^{2n}(2s+j).
\]

(4.18)

$\xi(s)$ is the zeta function of $F$, $\epsilon_{E/F}$ is the character of order 2 associated to the extension $E/F$ by local class field theory, $L(s,\chi)$ is the local Hecke $L$ function for a character $\chi$ of $F^\times$.

We will derive a formula from (4.17) which is applicable for our computation later. For this purpose, for $g \in G(F)$, let

\[
(Case \ NS) \quad \delta(g) = \text{diag}(\omega_{E}^{l_1}, \ldots, \omega_{E}^{l_n}), \quad l_1 \geq \cdots \geq l_n \geq 0,
\]

\[
(Case \ S) \quad \delta(g) = \text{diag}(\omega_{E}^{l_1}, \ldots, \omega_{E}^{l_{2n}}), \quad l_1 \geq \cdots \geq l_{2n},
\]

(4.19)

such that $g \in K_0m(\delta(g))K_0$(Case NS) or $g \in K_0\delta(g)K_0$(Case S). Define a function $\Delta(g)$ on $G(F)$ by

\[
(Case \ NS) \quad \Delta(g) = |\det \delta(g)|_{E}^{-1},
\]

\[
(Case \ S) \quad \Delta(g) = |\det \delta(g)|^{-1}.
\]

(4.20)

By Lemma 4.1,

\[
(Case \ NS) \quad \Phi_{K,s}(g,1) = \Phi_{K,s}(m(\delta(g),1)),
\]

\[
(Case \ S) \quad \Phi_{K,s}(g,1) = \Phi_{K,s}(\delta(g),1).
\]

(4.21)

Furthermore, reasoning as in [5, page 197], one can show that

\[
\Phi_{K,s}(g,1) = \Delta(g)^{-(s+n)}.
\]

(4.22)

Hence Theorem 4.2 is equivalent to the following.

**Theorem 4.3.** *For $s \in \mathbb{C}$,*

\[
\int_{G(F)} \Delta(g)^{-(s+n)}(g) \omega_n(g) dg = \frac{L(s+1/2,\pi,\sigma)}{d_H(s)}.
\]

(4.23)

*Here $d_H(s)$ is the meromorphic functions in Theorem 4.2.*
Before we end this section, we record a formula for the value on $\Delta(g)$ for some special elements in $G(F)$. For $\beta \in M_{n \times n}(F)$, let $L(\beta)$ be the set of all minors of $\beta$.

**Lemma 4.4** (see [8, Proposition 3.9]). (1) (Case NS) Let

$$g = \begin{pmatrix} \hat{w} & \beta \\ w & 1 \end{pmatrix} \begin{pmatrix} v^* & v^{-1} \end{pmatrix} \in G(F)$$ (4.24)

with $v, w \in \text{GL}_n(E) \cap M_{n \times n}(\mathbb{O}_E)$. Then

$$\Delta(g) = | \det(vw) |_E^{-1} \max_{C \in L(\beta)} | \det C |_E. \quad (4.25)$$

(2) (Case S) Let

$$g = \begin{pmatrix} w^{-1} & \beta \\ v & 1 \end{pmatrix} \begin{pmatrix} v' & w' \end{pmatrix} \in G(F)$$ (4.26)

with $v, v', w, w' \in \text{GL}_n(F) \cap M_{n \times n}(\mathbb{O})$. Then

$$\Delta(g) = | \det(vv'ww') |^{-1} \left( \max_{C \in L(\beta)} | \det C | \right)^2. \quad (4.27)$$

5. Fourier coefficients

In this section, we will compute Fourier coefficients of $\Delta(g)$. Our method is similar to that of [4].

Notations are as in the last section. Let $\psi$ be a nontrivial additive character of $F$. Let $(\pi, V_0)$ be an unramified irreducible admissible representation of $G(F)$, $T$ a square matrix such that $T \in S(F)$(Case NS) or $T \in M_{n \times n}(F)$(Case S). Let $l_T$ be a linear functional on $V_0$ satisfying

$$l_T \left( \pi \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} v \right) = \psi(\text{tr}(XT)) l_T(v) \quad (5.1)$$

for all $v \in V_0, X \in S(F)$(Case NS) or $X \in M_{n \times n}(F)$(Case S).

**Example 5.1.** Let $F$ be a number field, $\pi$ an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ for a moment [9]. Then $\pi = \phi' \pi_v$ is a restricted product of irreducible admissible representations $\pi_v$ of $G(F_v)$, for almost all $v \in \mathfrak{v}$, $\pi_v$ is unramified irreducible admissible representation. Let $f$ be a cuspid form in $A(G(F) \setminus G(\mathbb{A}))_\pi$, the isotypic space of $\pi$. Let $v \in \mathfrak{f}$ such that $\pi_v$ is unramified irreducible admissible representation of $G(F_v)$. Let $T_v \in S(F_v)$(Case NS) or $T_v \in M_{n \times n}(F_v)$. Define a linear functional $l_{T_v}$ on $A(G(F) \setminus G(\mathbb{A}))_\pi$ by

$$l_{T_v}(f) = \int f \left( \begin{pmatrix} 1 & X_v \\ 0 & 1 \end{pmatrix} \right) \psi(\text{tr}(X_vT_v)) dX_v, \quad (5.2)$$
where the integral is taken on $S(F_v)(\text{Case NS})$ or $M_{n \times n}(F_v)(\text{Case S})$. We see that $l_T(f)$ is independent of $f|_{G(F_v)}$ for $w \in v, w \neq v$. But $\pi_v = \pi|_{G(F_v)}$, so $l_T$ is a linear functional on $\pi_v$ satisfying (5.1).

Back to the assumption that $F$ is non-Archimedean local field, $(\pi, V_0)$ is an unramified irreducible representation of $G(F)$. Define a subset $M(\mathbb{C})$ of $M_{2n}(E)(\text{Case NS})$ or of $M_{2n}(F)(\text{Case S})$ as follows:

$$
\begin{align*}
\text{(Case NS)} & \quad M(\mathbb{C}) = \left\{ m(a) = \begin{pmatrix} a & \lambda \\ \hat{a} & \lambda \end{pmatrix} \bigg| a \in M_{n \times n}(\mathbb{C}) \cap \text{GL}_n(E) \right\}; \\
\text{(Case S)} & \quad M(\mathbb{C}) = \left\{ m(A, B) = \begin{pmatrix} A & 0 \\ B^{-1} & 0 \end{pmatrix} \bigg| A, B \in M_{n \times n}(\mathbb{C}) \cap \text{GL}_n(F) \right\}.
\end{align*}
$$

Let $\gamma_0$ be a function on $M(\mathbb{C})$ defined by

$$
\begin{align*}
\text{(Case NS)} & \quad \gamma_0(m(a)) = |\det a|_E, \\
\text{(Case S)} & \quad \gamma_0(m(A, B)) = |\det A \det B|.
\end{align*}
$$

**Lemma 5.2.** Let $\psi$ be an unramified additive character of $F$. Let $T$ be a square matrix such that $T \in S(F)(\text{Case NS})$ or $T \in M_{n \times n}(F)(\text{Case S})$. Let $(\pi, V_0)$ be an unramified irreducible admissible representation of $G(F)$. Take $0 \neq f_0 \in V_0^{K_0}$, where $K_0 = G(\mathbb{C})$ is a maximal compact subgroup of $G(F)$. Let $l_T$ be a linear functional on $V_0$ satisfying (5.1). Then for $s \in \mathbb{C}$,

$$
\int_{G(F)} \Delta^{-(s+n)}(g) l_T(\pi(g) f_0) \, dg = l_T(f_0) \frac{L(s + 1/2, \pi, \sigma)}{d_H(s)}.
$$

**Proof.** As in [3], the convergence of left-hand side of the equation when $\text{Re} s$ is sufficiently large comes from the vanishing of $l_T(\pi(a) f_0)$ when $a$ is sufficiently large, here $a$ belongs to the maximal $F$-torus consisting of diagonal elements in $G(F)$.

Since both sides are meromorphic functions of $s$, we only need to show the equation for $\text{Re} s$ sufficiently large. We first claim that

$$
\int_{K_0} l_T(\pi(kg) f_0) \, dk = l_T(f_0) \omega_\pi(g), \quad g \in G(F).
$$

In fact, the left-hand side is a bi-$K_0$-invariant matrix coefficient of $\pi$, so there is some $\lambda \in \mathbb{C}$ such that

$$
\int_{K_0} l_T(\pi(kg) f_0) \, dk = \lambda \omega_\pi(g), \quad g \in G(F).
$$

Let $g = 1$, then $\lambda = l_T(f_0)$.

Back to the proof of the lemma. If $\text{Re} s$ is sufficiently large, the left-hand side of (5.5) converges absolutely. Hence

$$
\text{L.H.S of (5.5)} = \int_{G(F)} \int_{K_0} \Delta^{-(s+n)}(kg) l_T(\pi(g) f_0) \, dk \, dg
$$

$$
= \int_{G(F)} \int_{K_0} \Delta^{-(s+n)}(g) l_T(\pi(kg) f_0) \, dk \, dg
$$

(5.8)
we have computed the inside integral in (5.6), so

\begin{equation}
(5.8) = l_T(f_0) \int_{G(F)} \Delta^{-(s+n)}(g) \omega_f(g) dg
\end{equation}

\begin{equation}
= l_T(f_0) \frac{L(s + 1/2, \pi, \sigma)}{d_H(s)}, \text{ by Theorem 4.3.}
\end{equation}

\[\square\]

Apply Iwasawa decomposition (3.6) \( g = n(X)m(a)k \) in the integrand of (5.5). When \( \text{Re} \ s \) is sufficiently large,

\begin{equation}
\int_{G(F)} \Delta^{-(s+n)}(g) l_T(\pi(g)f_0) df = \int_{K_0 \times M(F) \times N(F)} \Delta^{-(s+n)}(n(X)m(a)k) l_T(\pi(n(X)m(a)k)f_0)
\end{equation}

\begin{equation}
\times \delta_P(m(a))^{-1}dn(X)dm(a)dk.
\end{equation}

(5.10)

Here \( \delta_P(m(a)) \) is the modular function of \( P(F) \), hence \( \delta_P(m(a)) = | \det A |_F^n \) (Case NS) or \( \delta_P(m(A,B)) = | \det A \det B |_n \) (Case S). Note that \( f_0 \) is \( K_0 \) invariant, \( \Delta \) is bi-\( K_0 \) invariant,

\begin{equation}
(5.10) = \int_{M(F) \times N(F)} \Delta^{-(s+n)}(n(X)m(a)) \overline{\psi(\text{tr}(XT))}
\end{equation}

\begin{equation}
\times l_T(\pi(m(a)f_0) \delta_P(m(a))^{-1}dn(X)dm(a).
\end{equation}

(5.11)

If we let

\begin{equation}
J_T(s,a) = \int_{N(F)} \Delta^{-(s+n)}(n(X)m(a)) \overline{\psi(\text{tr}(XT))}dn(X),
\end{equation}

(5.12)

for \( m(a) \in M(F) \), then

\begin{equation}
(5.11) = \int_{M(F)} J_T(s,a)l_T(\pi(m(a))f_0) \delta_P^{-1}(m(a))dm(a).
\end{equation}

(5.13)

Properties of \( J_T(s,a) \), such as convergent when \( s \) sufficiently large, having meromorphic continuation to \( \mathbb{C} \), is discussed by Shimura [10], for example, Proposition 3.3 there.

**Lemma 5.3.** Let \( \psi \) be an unramified character of \( F \). Let \( T \) be a square matrix such that \( T \in \text{GL}_{n \times n}(\mathbb{C}_E) \cap S(F) \) or \( T \in \text{GL}_n(\mathbb{C})(\text{Case S}) \). Then

\begin{equation}
J_T(s,a) = \begin{cases} \nu_0(m(a))^{s+n} f_T(s), & a \in M(\mathbb{C}), \\ 0, & \text{if else}. \end{cases}
\end{equation}

(5.14)
Let $\xi \in X$ for all $m$ with $m \geq \cdots \geq 1$. Let $(\text{Case NS})$.

\[ j_T(s) = \int_{S(F)} \Delta^{-(s+n)}(n(X)) \psi(\text{tr}(TX))dX = \prod_{r=0}^{n-1} L(2s + 2n - r, \epsilon'_{E/F}), \]

$(\text{Case S})$.

\[ j_T(s) = \int_{M_{\infty}(F)} \Delta^{-(s+n)}(n(X)) \psi(\text{tr}(TX))dX = \prod_{r=0}^{n-1} \zeta(2s + 2n - r). \]

**Proof.** Both sides of (5.14) are meromorphic functions for a given $m(a) \in M(F)$. We only need to prove this lemma for $\operatorname{Res}$ sufficiently large.

$(\text{Case NS})$. Let $a \in \text{GL}_n(E)$. By the principle of elementary divisors, $a = t w^{-1} v'$ with $v, w \in M_{n \times n}(\mathbb{C}_E)$, $v = k \delta_1, w = k' \delta_2$ with $k, k' \in \text{GL}_n(\mathbb{C}_E)$ and

\[ \delta_1 = \text{diag}(\omega_{E}^{m_1}, \ldots, \omega_{E}^{m_i}, 1, \ldots, 1), \quad \delta_2 = \text{diag}(1, \ldots, 1, \omega_{E}^{m_{i+1}}, \ldots, \omega_{E}^{m_n}) \]  

with $m_1 \geq \cdots \geq m_i \geq 0, m_{i+1} \geq \cdots \geq m_n \geq 0$ for some $0 \leq i \leq n$. Then

\[ J_T(s, a) = J_T(s, t w^{-1} v') = \int_{S(F)} \Delta^{-(s+n)}(n(X)m(t w^{-1} v')) \psi(\text{tr}(TX))dX = \int_{S(F)} \Delta^{-(s+n)}(m(t w^{-1} m(t w^{-1})^{-1} n(X)m(t w^{-1} v'))) \times \psi(\text{tr}(TX))dX \]

\[ = | \text{det}(w) |_E^{-n} \int_{S(F)} \Delta^{-(s+n)}(m(t w^{-1} n(X)m(t v'))) \times \psi(\text{tr}(Xw^{-\rho} T w^{-1}))dX. \]

Let $S(\mathbb{C})$ be the set of elements in $S(F)$ with entries in $\mathbb{C}_E$. Let $\mathcal{J}$ be a set of representative of $S(F)/S(\mathbb{C})$. Decompose the integral in (5.17) as a sum of integrals indexed by $\mathcal{J}$:

\[ (5.17) = | \text{det} w |_E^{-n} \sum_{\xi \in \mathcal{J}} \int_{S(\mathbb{C})} \Delta^{-(s+n)}(m(t w^{-1} n(X)m(t v')) \times \psi(\text{tr}(Xw^{-\rho} T w^{-1}))dX. \]

Let $\xi \in S(F)$. If $\xi \notin S(\mathbb{C})$, by Lemma 4.4,

\[ \Delta^{-(s+n)}(m(t w^{-1} n(X)m(t v')) = | \text{det} \psi w |_E^{-n} \Delta^{-(s+n)}(n(\xi)) \]  

for all $X \in S(\mathbb{C})$, since

\[ \max_{C \in L(\xi+X)} | \text{det} C | = \max_{C \in L(\xi)} | \text{det} C | \]  

(5.20)
for \( \xi \notin S(\mathbb{C}) \). If \( \xi \in S(\mathbb{C}) \), then \( \Delta(n(\xi)) = 1 \),
\[
\Delta^{-(s+n)}(m(\,\cdot\,w^{-1})n(\xi + X)m(\,\cdot\,)) = |\det(vw)|^{s+n}_E \Delta^{-(s+n)}(n(\xi)) = |\det(vw)|^{s+n}_E .
\]
(5.21)

Hence for all \( \xi \in S(F) \), \( X \in S(\mathbb{C}) \),
\[
\Delta^{-(s+n)}(m(\,\cdot\,w^{-1})n(\xi + X)m(\,\cdot\,)) = |\det(vw)|^{s+n}_E \Delta^{-(s+n)}(n(\xi)).
\]
(5.22)

Apply (5.22) to (5.18), we then get
\[
(5.18) = |\det w|_E^{s-n} |\det(vw)|^{s+n}_E \sum_{\xi \in \mathcal{Y}} \Delta^{-(s+n)}(n(\xi)) \times \psi(\tr(\xi w^{-\rho}T^i w^{-1})) \int_{S(\mathbb{C})} \psi(\tr(Xw^{-\rho}T^i w^{-1})) dX.
\]
(5.23)

If \( a \notin M_{n \times n}(\mathbb{C}_E) \), then \( |\det w|_E < 1 \) and \( w^{-\rho}T^i w^{-1} \in S(\mathbb{C}) \). Hence
\[
\int_{S(\mathbb{C})} \psi(\tr(Xw^{-\rho}T^i w^{-1})) dX = 0,
\]
(5.24)

and \( J_T(s,a) = 0 \). If \( a \in \GL_n(E) \cap M_{n \times n}(\mathbb{C}_E) \), we compute \( J_T(s,a) \) directly:
\[
J_T(s,a) = \int_{S(F)} \Delta^{-(s+n)}(n(X)m(a)) \overline{\psi(\tr(XT))} dX
= |\det a|_E^{s+n} \int_{S(F)} \Delta^{-(s+n)}(n(X)) \overline{\psi(\tr(XT))} dX, \quad \text{by Lemma 4.4}
\]
(5.25)

\[
J_T(s,a) = |\det a|_E^{s+n} j_T(s),
\]
here
\[
j_T(s) = \int_{S(F)} \Delta^{-(s+n)}(n(X)) \overline{\psi(\tr(TX))} dX
\]
(5.26)

\[
= \prod_{r=0}^{n-1} L(2s+2n-r, c_{E/F}^r),
\]

where the second equality comes from [10, Proposition 6.2] by Shimura.

The proof for Case S is similar, and we omit it here. \( \Box \)

**Theorem 5.4.** Let \( \psi \) be an unramified character of \( F \), \( (\pi, V_0) \) an unramified irreducible admissible representation of \( G(F) \). Let \( T \) be a square matrix such that \( T \in \GL_n(\mathbb{C}_E) \cap S(F)(\text{Case NS}) \) or \( T \in \GL_n(\mathbb{C})(\text{Case S}) \). Let \( l_T \) be a linear functional on \( V_0 \) satisfying (5.1). Then for \( 0 \neq f_0 \in V_{0}^{K_0} \),
\[
\int_{M(\mathbb{C})} \gamma_0(m(a)) l_T(\pi(m(a))f_0) dm(a) = l_T(f_0) \frac{L(s+1/2, \pi, \sigma)}{j_T(s)d_H(s)},
\]
(5.27)

where \( d_H(s) \) and \( j_T(s) \) are given in Theorem 4.2 and Lemma 5.3.
Proof. Lemma 5.2 and the paragraph after Lemma 5.2 have shown that

$$l_T(f_0) \frac{L(s + 1/2, \pi, \sigma)}{d_H(s)} = \int_{G(F)} \Delta^{-s}(g)l_T(\pi(g)f_0)dg$$

$$= \int_{M(F)} J_T(s,a)l_T(\pi(m(a))f_0)\delta_p^{-1}(m(a))dm(a).$$

(5.28)

By Lemma 5.3, $J_T(s,a)$ vanishes when $a \notin M(\mathbb{C})$. Substitute the formula of $J_T(s,a)$ for $a \in M(\mathbb{C})$ and $\delta_p^{-1}$, the conclusion follows. 

6. Global computation

Let $F$ be a number field, $E$ a quadratic field extension of $F$. As usual, let $v$ be the set of all places of $F, a, f$ the set of archimedean and non-archimedean places of $F$ respectively. Let $F_v$ be the localization of $F$ at the place $v$ of $v$, $E_v = E \otimes F_v$. If $v \in \mathfrak{v}$, let $\mathcal{O}_v$ be the ring of integers of $F_v$. If $v$ remains prime in $E$, then $E_v$ is a quadratic field extension of $F_v$, let $\mathcal{O}_{E_v}$ be the ring of integer of $E_v$. The ring of adeles of $F$ (resp., $E$) is denoted by $\mathbb{A}$ (resp., $\mathbb{A}_E$). Denote by $| \cdot |$ (resp., $| \cdot |_E$) the normalized absolute value of $\mathbb{A}$ (resp., $\mathbb{A}_E$). Let $\psi$ be a nontrivial continuous character of $\mathbb{A}$ trivial on $F$.

Let $V$ be a 2n-dimensional vector space over $E$ with an anti-Hermitian form $\eta_{2n}$ on it. Let $W$ be an n-dimensional vector space over $E$ with a nonsingular Hermitian form $T$. Let $G = U(\eta_{2n}), G' = U(T)$ be the corresponding unitary groups. Then $G \times G'$ is a dual pair in $\text{Sp}(\mathbb{W})$, where $\mathbb{W} = V \otimes W$ is symplectic space with symplectic form $\text{tr}_{E/F}(\eta_{2n} \otimes T)$.

Let $P = MN$ be the maximal parabolic subgroup of $G$ defined in Section 2. For $v \in \mathfrak{v}$, let $K_v$ be a maximal compact subgroup of $G(F_v)$ such that for almost all $v \in \mathfrak{v}, K_v = G(\mathcal{O}_v)$. Let $K_v = \prod_{v \in \mathfrak{v}} K_v$. Then $G(\mathbb{A}) = P(\mathbb{A})K_\mathbb{A}$. For $v \in \mathfrak{v}$, let $d_{K_v}$ be the Haar measure on $K_v$ such that $\int_{K_v} d_{K_v} = 1$. Then $d_{K_v}$ is an Haar measure on $K_v$ such that $\int_{K_v} d_{K_v} = 1$. Let $d_\mathfrak{p}(p_v)$ be a left Haar measure on $P(F_v)$ for $v \in \mathfrak{v}$. Then $d_\mathfrak{p} = \prod_{v \in \mathfrak{v}} d_\mathfrak{p}(p_v)$ is a left Haar measure on $P(\mathbb{A})$. Since $P(\mathbb{A}) = M(\mathbb{A})N(\mathbb{A}), d_\mathfrak{p} = | \text{det} a |_E^{-n} d^\times a dX$ if $p = m(a)n(X)$ for $a \in \text{GL}_n(\mathbb{A}_E), X \in S(\mathbb{A})$, where $d^\times a, dX$ are Haar measure on $\text{GL}_n(\mathbb{A}_E), S(\mathbb{A})$, respectively. We then let $dg = d_\mathfrak{p}d\mathfrak{d}d\mathfrak{k}$ be an Haar measure on $G(\mathbb{A})$.

Let $s \in \mathbb{C}$, let $\gamma$ be a Hecke character of $E$. Denote by $I(s, \gamma)$ the set of smooth functions $f : G(\mathbb{A}) \rightarrow \mathbb{C}$ satisfying

(i) $f(pg) = \gamma(x(p))|x(p)|_E^s f(g)$, for $p \in P(\mathbb{A}), g \in G(\mathbb{A})$,

(ii) $f$ is $K_v$-finite for all $v \in \mathfrak{a}$.

$G(\mathbb{A})$ acts on $I(s, \gamma)$ by right multiplication. Let $\Phi(g, s)$ be a smooth function in $I(s, \gamma)$ holomorphic at $s$. The Eisenstein series associated to $\Phi(g, s)$ is given by

$$E(g, s; \gamma, \Phi) = \sum_{\xi \in P(\mathbb{F}) \backslash G(\mathbb{F})} \Phi(\xi g, s).$$

(6.1)

In [9], it has been shown that (6.1) is convergent when $\text{Res} > n/2$ and has a meromorphic continuation to the whole complex plane.
Let \( \pi \) be a cuspidal automorphic representation of \( G(\mathbb{A}) \) (cf. [9]). Let \( f \) be cusp form in the isotypic space of \( \pi \). Let \( \beta \in S(F) \). The \( \beta \)-th Fourier coefficient of \( f \) is

\[
f_{\beta}(g) = \int_{S(F) \backslash S(\mathbb{A})} f(n(X)g) \psi(\text{tr}(X\beta))dX, \quad g \in G(\mathbb{A}). \tag{6.2}
\]

If \( \beta_1, \beta_2 \in S(F), \) \( \beta_1 = a^t \beta_2 a \) for some \( a \in \text{GL}_n(E) \), then

\[
f_{\beta_1}(g) = f_{\beta_2}(m(a)g), \quad g \in G(\mathbb{A}). \tag{6.3}
\]

Let \( \chi \) be a Hecke character of \( E \) satisfying \( \chi|_{\mathbb{A}^\times/F^\times} = \epsilon_{E/F}^s \), where \( \epsilon_{E/F} \) is the quadratic character of \( E/F \) by global class field theory. Associate with \( \psi \) a Weil representation \( \omega_{\psi} \) of \( G(\mathbb{A}) \) acting on \( \mathcal{S}(\mathbb{A}) \), the set of Schwartz-Bruhat functions on \( \mathbb{A} \). In fact, \( \omega_{\psi} \) is the restriction of Weil representation (associated with \( \psi \)) of \( \hat{\mathbb{S}}(\mathbb{A}) \) to \( G(\mathbb{A}) \) (see Section 2 for the definition of \( \mathbb{S}, \mathbb{W} \)). We will omit the subscript \( \psi \) when \( \psi \) is clear from the context. The explicit formula of \( \omega \) is given in [11], we cite here the formula on \( P(\mathbb{A}) \).

Let \( f \in \mathcal{S}(\mathbb{A}), \) \( a \in \text{GL}_n(\mathbb{A}_E), \) \( n(X) \in N(\mathbb{A}) \), then

\[
\omega(m(a))\phi(y) = \chi(\text{det}a)|\text{det}a|_E^{s/2}\phi(ya),
\]

\[
\omega(n(X))\phi(y) = \psi(\text{tr}(b\mu(y)))\phi(y), \quad y \in \mathbb{A}. \tag{6.4}
\]

Here \( \mu = \prod_{\nu} \mu_{\nu} : \mathbb{A} \rightarrow \mathcal{S}(\mathbb{A}), \) \( \mu_{\nu} \) is the moment map defined at Section 2 for local field \( F_{\nu} \).

The theta series \( \theta_\Phi \) for \( \phi \in \mathcal{S}(\mathbb{A}) \) is a smooth function on \( G(\mathbb{A}) \) of moderate growth

\[
\theta_\Phi(g) = \sum_{\xi \in S(F)} \omega(g)\phi(\xi), \quad g \in G(\mathbb{A}). \tag{6.5}
\]

### 6.1. Vanishing lemma.

Let \( \pi \) be a cuspidal automorphic representation of \( G(\mathbb{A}) \). We make the following assumption: There is some cusp form \( f \) in the isotypic space of \( \pi \) such that

\[
\int_{N(F) \backslash N(\mathbb{A})} f(n(X)g) \psi(\text{tr}(XT)) \neq 0. \tag{6.6}
\]

In [4], Piatetski-Shapiro and Rallis do not propose this assumption, because Li has shown in [12] that every cusp forms supports some nonsingular symmetric matrix.

For \( \phi \in \mathcal{S}(\mathbb{A}), \Phi(g,s) \in \mathcal{I}(s,y), \) \( f \in A(G(F) \setminus G(\mathbb{A}))_\pi \) the isotypic space of \( \pi \) in the space of automorphic forms on \( G(\mathbb{A}) \), define

\[
I(s, \phi, \Phi, f) = \int_{G(F) \backslash G(\mathbb{A})} f(g)E(g, s, \Phi)\theta_\Phi(g)dg. \tag{6.7}
\]

Although \( \theta_\Phi \) is slowly increasing function on \( G(\mathbb{A}), \) \( E(g, s, \Phi) \) is of moderate growth, but \( f \) is rapidly decreasing on \( G(\mathbb{A}), \) (6.7) is convergent at \( s \) where the Eisenstein series is holomorphic. We will show that when we choose appropriate \( \phi, \Phi, f, I(s, \phi, \Phi, f) \) is product of meromorphic function with partial \( L \) function of \( \pi \).
Substitute Eisenstein series (6.1), theta series (6.5) into (6.7), then

\[
(6.7) = \int_{P(F)\backslash G(A)} f(g)\Phi(g, s) \sum_{\xi \in \mathcal{Y}(F)} \omega(g)\phi(\xi)dg
\]

\[
= \int_{K_\mathcal{A}} \int_{P(F)\backslash P(A)} f(pk)\Phi(pk, s) \sum_{\xi \in \mathcal{Y}(F)} \omega(pk)\phi(\xi)dpdk. \quad (6.8)
\]

By the assumption that \(\Phi(g, s) \in I(s, \gamma)\), \(\Phi(pk, s) = \gamma(x(p))|x(p)|^{s+n/2}\Phi(k, s)\). Apply the formula of Weil representation (6.4) to (6.8), then

\[
(6.8) = \int_{K_\mathcal{A}} \int_{M(F)\backslash M(A)} \int_{N(F)\backslash N(A)} f(n(X)m(a)k)\Phi(k, s)
\]

\[
\times (\gamma|\cdot|_{E}^t)(\det a) \sum_{\xi \in \mathcal{Y}(F)} \psi(\text{tr}(b\mu(\xi)))\omega(k)\phi(\xi a)dXd^\infty adk. \quad (6.9)
\]

Recall that in Section 2, we let \(\mathcal{C} \subset S(F)\) be the image of moment map, which is invariant under the action of \(M(F)\). Let \(\mathcal{J}\) be a set of representatives of orbits \(\mathcal{C}/M(F)\) such that \(T \in \mathcal{J}\). We then write (6.9) as a sum of integrals indexed by \(\mathcal{J}\):

\[
(6.9) = \sum_{\beta \in \mathcal{J}} \int_{K_\mathcal{A}} \int_{M(F)\backslash M(A)} \sum_{\beta \in \mathcal{C}} \sum_{\xi \in \mu^{-1}(\beta)} f_\beta(m(a)k)\Phi(k, s)
\]

\[
\times (\gamma|\cdot|_{E}^t)(\det a) \omega(k)\phi(\xi a)d^\infty adk \quad (6.10)
\]

Here \(f_\beta\) is \(\beta\)th Fourier coefficient of \(f\), \(M_\beta\) is the stabilizer of \(\beta\) under the action of \(M\) (cf. Section 2). For \(\beta \in \mathcal{J}\), let

\[
I_\beta(s) = \int_{K_\mathcal{A}} \int_{M(F)\backslash M(A)} \sum_{\alpha' \in M_\beta(F)\backslash M(F)} \sum_{\xi \in \mu^{-1}(\beta)} f_\beta(m(a')m(a)k)\Phi(k, s)
\]

\[
\times (\gamma|\cdot|_{E}^t)(\det a) \omega(k)\phi(\xi a'a)d^\infty adk. \quad (6.11)
\]

Then

\[
I(s, \phi, \Phi, f) = \sum_{\beta \in \mathcal{J}} I_\beta(s). \quad (6.12)
\]

**Lemma 6.1.** \(I_\beta(s) = 0\) for all \(\beta \in \mathcal{J}\) with \(\det \beta = 0\).

**Proof.** If \(\beta = 0\), then for all \(g \in G(A)\),

\[
f_\beta(g) = \int_{N(F)\backslash N(A)} f(ng)dn = 0 \quad (6.13)
\]
since $f$ is a cusp form. Hence

$$
I_\beta(s) = \int_{K_\alpha} \int_{M(F) \setminus M(A)} \sum_{a' \in M_\beta(F) \setminus M(F)} \sum_{\xi \in \mu^{-1}(\beta)} f_\beta(m(a')m(a)k) \Phi(k,s) \times (\gamma \chi | \cdot |_F \mid \frac{E}{F}) (\det a) \omega(k) \phi(\xi a' a) d^\times a d k = 0.
$$

Let $0 \neq \beta \in \mathcal{B}$ with $\det \beta = 0$. Then

$$
I_\beta(s) = \int_{K_\alpha} \int_{M(F) \setminus M(A)} \sum_{a' \in M_\beta(F) \setminus M(F)} \sum_{\xi \in \mu^{-1}(\beta)} f_\beta(m(a')m(a)k) \Phi(k,s) \times (\gamma \chi | \cdot |_F \mid \frac{E}{F}) (\det a) \omega(k) \phi(\xi a' a) d^\times a d k
$$

$$
= \int_{K_\alpha} \int_{M_\beta(A) \setminus M(A)} \int_{M_\beta(F) \setminus M_\beta(A)} f_\beta(m_1 m k) \Phi(k,s) \times (\gamma \chi | \cdot |_F \mid \frac{E}{F}) (x(m_1 m)) \sum_{\xi \in \mu^{-1}(\beta)} \omega(k) \phi(\xi m_1 m) d m_1 \ d m \ d k.
$$

Let $x \in \mathcal{Y}$ such that $\beta = \mu(x) = \iota x^\rho T x$, $r = \text{rank}(\beta)$. Then $r < n$. Let $a \in \text{GL}_n(F)$ such that

$$
\iota A^\rho \beta A = \begin{pmatrix} 0 & 0 \\ 0 & T' \end{pmatrix},
$$

where $T'$ is a nondegenerate $r \times r$ Hermitian matrix. So without loss of generality, we assume that $\beta = \text{diag}(0_{n-r}, T')$. Then

$$
M_\beta = \left\{ m \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in M \mid D \in U(T'), \ i C^\rho T'' C = 0, \ i C^\rho T'' D = 0 \right\}.
$$

Define two subgroups $M_1$, $L$ of $M_\beta$:

$$
M_1 = \left\{ m \left( \begin{array}{cc} A & 0 \\ C & D \end{array} \right) \in M \mid D \in U(T'), \ i C^\rho T'' C = 0, \ i C^\rho T'' D = 0 \right\},
$$

$$
L = \left\{ m \left( \begin{array}{cc} 1_{n-r} & B \\ 0 & 1_r \end{array} \right) \in M \mid B \in M_{n-r \times n-r}(E) \right\}.
$$

Then $M_\beta = M_1 \cdot L$. We use this decomposition to compute the inner integral over $M_\beta(F) \setminus M_\beta(A)$ of (6.15),

$$
\int_{M_\beta(F) \setminus M(A)} f_\beta(m_1 m k) (\gamma \chi | \cdot |_F \mid \frac{E}{F}) (x(m_1 m)) \sum_{\xi \in \mu^{-1}(\beta)} \omega(k) \phi(\xi m_1 m) d m_1.
$$
(Here because $\Phi(k,s)$ is independent of $m_1$ so we remove it from the integral over $M_\beta(F) \setminus M(\mathbb{A})$.) The above integral equals to

$$
\int_{M_1(F) \setminus M_1(\mathbb{A})} \int_{L(F) \setminus L(\mathbb{A})} \int_{S(F) \setminus S(\mathbb{A})} f(n(X)\ell m_1 m k) \psi(\text{tr}(X\beta)) \\
\times (\gamma \chi | \cdot |_E) (x(\ell m_1 m)) \sum_{\xi \in \mu^{-1}(\beta)} \omega(k) \phi(\xi \ell m_1 m) dX d\ell dm_1.
$$

Let $U$ be the subgroup of $N$ consisting of elements of the following form:

$$
n\left( \begin{array}{cc} c & d \\ t \, d^\rho & 0 \end{array} \right) \quad \text{with } c \in M_{(n-r) \times (n-r)}. \tag{6.21}
$$

Then $LU$ is the unipotent radical of the maximal parabolic group $P'$ preserving the flag $0 \subset \otimes_{i=1}^{r-1} Ee_{r+i} \subset Y$ (see Section 2 for the choice of basis of $V$). On the other hand, let $\Delta_+$ be the set of positive roots of $G$ with respect to the Borel subgroup of $G$ consisting of element of following form:

$$
\begin{pmatrix} A & B \\ \hat{A} & \end{pmatrix} \quad \text{with } A \text{ be upper triangular matrix.} \tag{6.22}
$$

For $\alpha \in \Delta_+$, let $N_\alpha$ be the 1-parameter unipotent subgroup of $G$ corresponding to $\alpha$. Set $\Gamma = \{ \alpha \in \Delta_+ | N_\alpha \subset N \}$. Let $\alpha_0$ be the simple root corresponding to $P'$, $w = s_{\alpha_0}$ be the simple reflection of $\alpha_0$. Then $U = \prod_{\beta \in \Gamma, \omega_\beta \in \Gamma} N_\beta$. If we put $U_1 = \prod_{\beta \in \Gamma, \omega_\beta \in -\Gamma} N_\beta$, then $N = U \cdot U_1$. Hence we have decomposition

$$
N(F) \setminus N(\mathbb{A}) = U(F) \setminus U(\mathbb{A}) \cdot U_1(F) \setminus U_1(\mathbb{A}). \tag{6.23}
$$

Corresponding to the decomposition of $N$, we have a decomposition of $S(F)$:

$$
S_U(F) = \left\{ \begin{pmatrix} c & d \\ t \, d^\rho & 0 \end{pmatrix} \in S(F) \mid c \in M_{(n-r) \times (n-r)}(F) \right\},
$$

$$
S_{U_1}(F) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \in S(F) \mid d \in M_{r \times r}(F) \right\}. \tag{6.24}
$$

Then the isomorphism $n : S(F) \to N$ send $S_U$ and $S_{U_1}$ onto $U$ and $U_1$, respectively. Substitute the decomposition of $S(F)$ into (6.20), then

$$
(6.20) = \int_{M_1(F) \setminus M_1(\mathbb{A})} \int_{L(F) \setminus L(\mathbb{A})} \int_{S_U(F) \setminus S_U(\mathbb{A})} \int_{S_{U_1}(F) \setminus S_{U_1}(\mathbb{A})} \int_{S_U(F) \setminus S_U(\mathbb{A})} f(n(X_U + X_{U_1})\ell m_1 m k) \psi(\text{tr}((X_U + X_{U_1})\beta)) \\
\times (\gamma \chi | \cdot |_E) (x(\ell m_1 m)) \sum_{\xi \in \mu^{-1}(\beta)} \omega(k) \phi(\xi \ell m_1 m) dX_U dX_{U_1} d\ell dm_1 dm_1 dm. \tag{6.25}
$$
Direct computation shows that $L$ centralizes $U_1$. We can change the order of the above integration, then

$$(6.20) = \int_{M_1(F)\backslash M_1(A)} \int_{S_U(F)\backslash S_U(A)} \int_{L(F)\backslash L(A)} \int_{S_U(F)\backslash S_U(A)} F \psi(\text{tr}((X_U + X_{U_1})\beta))$$

$$\times f(n(X_U)\ell n(X_{U_1}) m_1 m k) \omega(k) \phi(\xi \ell m_1 m) dX_U d\ell dX_{U_1} \omega(d m_1 d m).$$

Let $X_U = (c \ d \ 0)$ be an element of $S_U(A)$. Then

$$\beta X_U = \begin{pmatrix} 0 & 0 \\ 0 & T' \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} c \ d \\ T' \end{pmatrix}.$$ (6.27)

So

$$\text{tr}((X_U + X_{U_1})) = \text{tr}(\beta X_U)$$ (6.28)

which is independent of $X_U$. Since $\chi(\ell) = 1$ for $\ell \in L(A)$, we see that

$$(\gamma \chi| \cdot |_{\xi}) (\ell) = 1, \quad \ell \in L(A).$$ (6.29)

If $\xi \in \mu^{-1}(\beta)$, then $\text{rank}(\xi) = r$. Let $a_1, \ldots, a_n$ be the column vectors of $\xi$. Recall that the right lower corner of $\xi$ is an $r \times r$ nonsingular matrix $T'$, the space generated by $a_{n-r+1}, \ldots, a_n$ is of rank $r$. Hence there is $a \in M_\beta$ (depends on $\xi$, but it does not affect our computation) such that

$$\xi' = \xi a^{-1} = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}$$ (6.30)

for some nonsingular $r \times r$ matrix $u$. If $\ell = m(1 \ x) \in L$, then

$$\xi' \ell = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \begin{pmatrix} 1 \\ x \\ 1 \end{pmatrix} = \xi'.$$ (6.31)

The integral for fixed $\xi \in \mu^{-1}(\beta)$ on $L(F)\backslash L(A) \times U(F)\backslash U(A)$ in (6.26) is

$$\int_{L(F)\backslash L(A)} \int_{U(F)\backslash U(A)} \int_{L(F)\backslash L(A)} \int_{S_U(F)\backslash S_U(A)} f(n(X_U)\ell n(X_{U_1}) m_1 m k) \psi(\text{tr}((X_U + X_{U_1})\beta))$$

$$\times (\gamma \chi| \cdot |_{\xi}) (\ell m_1 m) \omega(k) \phi(\xi \ell m_1 m) dX_U d\ell.$$ (6.32)

By (6.28), (6.29), and (6.31),

$$(6.32) = \int_{L(F)\backslash L(A)} \int_{U(F)\backslash U(A)} f(n(X_U)\ell n(X_{U_1}) m_1 m k) \psi(\text{tr}(X_U \beta))$$

$$\times (\gamma \chi| \cdot |_{\xi}) (m_1 m) \omega(k) \phi(\xi' m_1 m) dX_U d\ell,$$ (6.33)

which is 0, since $LU$ is the unipotent radical of $P'$. This finishes the proof of the lemma. □
By Lemma 6.1, $I_\beta(s) = 0$ if $\beta$ is singular. Recall that we choose $T$ to be the representative of the open orbit of $\mathcal{C}/\mathcal{M}$. The stabilizer $M_T$ is isomorphic to $G' = U(T)$ the unitary group of $W$. Then (6.12) reduces to

\[
I(s, \phi, \Phi, f) = \int_{K_\mathbb{A}} \int_{M(\mathbb{A})} \sum_{a' \in G'(\mathbb{F}) M(\mathbb{A})} f_T(m(a')m(a)k) \Phi(k,s) \times (\gamma \chi| \cdot |_E^k) (\det a) \sum_{\xi \in \mathcal{G}(\mathbb{F})} \omega(k)\phi(\xi a) d^\times a \, dk
\]

\[
= \int_{K_\mathbb{A}} \int_{M(\mathbb{A})} f_T(m(a)k) \Phi(k,s) \omega(k)\phi(\xi a) (\gamma \chi| \cdot |_E^k) a d^\times a \, dk. \tag{6.34}
\]

\[
6.2. \textbf{Main theorem.} \text{ Let } y_v = y|_{E_v}, \text{ then } y = \prod_v y_v. \text{ Similarly, } \chi = \prod_v \chi_v. \text{ Let } \Phi_v \text{ be a standard section of } I(y, s) \text{ of } G(F_v) \text{ for all } v \in \mathfrak{v}. \text{ Set } \Phi = \prod_v \Phi_v. \text{ Assume that } \phi = \prod_v \phi_v \text{ in } \mathcal{H}(\mathcal{V}). \text{ Let } f \text{ be a cusp form in the isotypic space of a cuspidal automorphic representation of } G(\mathbb{A}). \text{ Let } S \text{ be a finite subset of } \mathfrak{v} \text{ containing all archimedean places such that if } v \notin S, \chi_v, \gamma_v \text{ are unramified, } T_v \in \text{GL}_{n \times n}(\mathbb{O}_E) \cap S(F_v) \text{ and } \psi_v \text{ is unramified character of } F_v. \text{ Since } \pi = \otimes_v \pi_v \text{ for almost all } v \text{, } \pi_v \text{ is unramified for almost all places. Assume that } \pi_v \text{ is unramified if } v \notin S \text{ and } f \text{ is } K_v \text{ fixed. Moreover, } \phi_v = \text{char}(\mathcal{V}(\mathbb{O}_v)) \text{ if } v \notin S.

Let } \Omega \text{ be a finite subset of } \mathfrak{v} \text{ containing } S. \text{ Put }

\[
G_\Omega = \prod_{\mathfrak{v} \in \Omega} G, \quad K_\Omega = \prod_{\mathfrak{v} \in \Omega} K_v, \quad M_\Omega = \prod_{\mathfrak{v} \in \Omega} M_v. \tag{6.35}
\]

They embed naturally into $G(\mathbb{A}), K_\mathbb{A}, M(\mathbb{A})$, respectively. If $a \in M(\mathbb{A}), a = \prod_{\mathfrak{v} \in \Omega} a_v$, put

\[
a_\Omega = \prod_{\mathfrak{v} \in \Omega} a_v. \text{ Similarly, if } k \in K_{\Omega \cup \{v\}}, \text{ then } k = k_\Omega \cdot k_v, \text{ for } k_\Omega \in K_\Omega, k_v \in K_v. \text{ To compute } (6.34), \text{ we define }

\[
I_\Omega(s) = \int_{K_\Omega} \int_{M_\Omega} f_T(m(a)k) \Phi(k,s) \omega(k)\phi(a) (\gamma \chi| \cdot |_E^k) a d^\times a \, dk. \tag{6.36}
\]

\[
\textbf{Theorem 6.2. Notations as above. Then }

\[
I_{\Omega \cup \{v\}}(s) = \frac{L(s + 1/2, \pi_v, \gamma_v \chi_v, \sigma)}{j_T(s) d_H(s)} I_\Omega(s), \tag{6.37}
\]

where $j_T, d_H(s)$ are $j_T(s), d_H(s)$ in Theorem 5.4 for $T_v, H_v$, respectively,

\[
L \left( s + \frac{1}{2}, \pi_v, \gamma_v \chi_v, \sigma \right) = L \left( s + \frac{1}{2} + \lambda_v, \pi_v, \sigma \right), \tag{6.38}
\]

where $\lambda_v \in \mathbb{C}$ such that $(\gamma_v \chi_v)(a) = |a|_E^{\lambda_v}$ for all $a \in E_v^\times (\text{Case NS})$, or $(\gamma_v \chi_v)(a) = |a|_E^{\lambda_v}$ for all $a \in F_v^\times (\text{Case S})$ (See Section 3 for the definition of Case NS and Case S).
Proof. We will apply results in Section 5, $F_v$ will be $F$ there,

$$I_{\Omega \cup \{v\}}(s) = \int_{K_{0\Omega \cup \{v\}}} \int_{M_{0\Omega \cup \{v\}}} f_T(m(a)k)\Phi(k)s\omega(k)\phi(a)(\gamma\chi|\cdot|_{E}^s)(\det a) d^\times a dk$$

$$= \int_{K_{0\Omega}M_{0\Omega}} \int_{K_vM_{v}(F_v)} \Phi(K_{\Omega},s)\Phi_v(k_v,s) f'_T(m(a_v)m(a_\Omega)k_vk_\Omega)$$

$$\times \gamma\chi|\cdot|_{E}^s(\det a_va_\Omega)\omega(k_\Omega)\phi(a_\Omega)\phi(a_v)d^\times a_v dk_v da_\Omega dk_\Omega. \quad (6.39)$$

$\Phi_v$ is the standard section, then $\Phi_v(k_v,s) = 1$ for all $k_v \in K_v$. Moreover, $f$ is $K_v$-fixed, hence $f_T(m(a_va_\Omega)k_vk_\Omega) = f_T(m(a_va_\Omega)k_\Omega)$ for all $k_v \in K_v$. $\phi_v = \text{char}(\mathcal{Y}(\mathcal{C}_v))$ which is $K_v$ fixed element for the Weil representation, hence $\omega(k_v)\phi_v = \phi_v$,

$$= \int_{K_{0\Omega}M_{0\Omega}} \Phi(k_\Omega,s) f'_T(m(a_v)m(a_\Omega)k_\Omega)$$

$$\times \gamma\chi|\cdot|_{E}^s(\det a_va_\Omega)\omega(k_\Omega)\phi(a_\Omega)\phi(a_v)d^\times a_v dk_v da_\Omega dk_\Omega. \quad (6.40)$$

As $\phi_v = \text{char}(\mathcal{Y}(\mathcal{C}_v))$, $M_v \cap \mathcal{Y}(\mathcal{C}_v) = M(\mathcal{C}_v)$ (cf. Section 5),

$$\int_{M_{v}(F_v)} f_T(m(a_v)m(a_\Omega)k_\Omega)\phi(a_v)\gamma_\Omega^0(a_v)(\gamma\chi)(\det a_v)d^\times a_v$$

$$= \int_{M_{\mathcal{C}_v}} f_T(m(a_v)m(a_\Omega)k_\Omega)\gamma_\Omega^0(a_v)(\gamma\chi)(\det a_v)d^\times a_v \quad (6.41)$$

$$= \frac{L(s+1/2,\pi_v,\gamma_v\chi_v,\sigma)}{j_{T_v}(s)d_{H_v}(s)} f_T(m(a_\Omega)k_\Omega), \quad \text{by Theorem 5.4.}$$

Here we are viewing $f_T(m(a_v)m(a_\Omega)k_\Omega)$ as a functional $l_{T_v}$ on $\pi_v$ by Example 5.1 in Section 5. Hence

$$I_{\Omega \cup \{v\}} = \frac{L(s+1/2,\pi_v,\gamma_v\chi_v,\sigma)}{j_{T_v}(s)d_{H_v}(s)} I_\Omega(s). \quad (6.42)$$

To complete the computation of our global integral, let

$$j_{T_v}^\Omega(s) = \prod_{v \notin S} j_{T_v}(s), \quad d_{H_v}^\Omega(s) = \prod_{v \notin S} d_{H_v}(s). \quad (6.43)$$

Define partial $L$ function of $\pi$ as

$$L^\Omega\left(s + \frac{1}{2}, \pi, \gamma\chi, \sigma \right) = \prod_{v \in S} L\left(s + \frac{1}{2}, \pi_v, (\gamma_v\chi_v), \sigma \right). \quad (6.44)$$

Since $I(s) = \lim_\Omega I_\Omega(s)$, by Theorem 6.2, let $\Omega$ be a finite set of $v$ approaching to $v$ by adding one place each time, then the following holds.
Choose \( f, \phi, \Phi \) and \( S \subset \mathfrak{v} \) as in Section 6.1. Then for all \( s \in \mathbb{C} \),

\[
I(s, \phi, \Phi, f) = \frac{R(s)}{j^1_t(s)d^3_H(s)} L^S \left( s + \frac{1}{2}, \pi, \chi, \sigma \right),
\]

where \( R(s) = I_S(s) \) is a meromorphic function of \( s \).

**Proof.** Argue as [6, Theorem 6.1], the partial \( L \) function is a meromorphic function. Also by the analytic property of Eisenstein series, \( I(s, \phi, \Phi, f) \) itself is a meromorphic function, hence \( R(s) = I_S(s) \) is a meromorphic function of \( s \).

**Remark 6.4.** We remark here that following [4, pages 118-119], under our assumption one can show that by choosing appropriate \( \phi, \Phi, f \), we can let that \( R(s) \neq 0 \).

**Acknowledgments**

The author is grateful to J.-S. Li for his generosity and encouragement in these years. He also thanks the referee for the careful reading and for pointing out mistakes in an earlier version of this paper.

**References**


Yujun Qin: Department of Mathematics, East China Normal University, Shanghai 200062, China

Email address: yjqin@math.ecnu.edu.cn