Research Article

Best Simultaneous Approximation in Orlicz Spaces

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Let X be a Banach space and let $L^\Phi(I, X)$ denote the space of Orlicz X -valued integrable functions on the unit interval I equipped with the Luxemburg norm. In this paper, we present a distance formula $\text{dist}_\Phi(f_1, f_2, L^\Phi(I, G))$, where G is a closed subspace of X , and $f_1, f_2 \in L^\Phi(I, X)$. Moreover, some related results concerning best simultaneous approximation in $L^\Phi(I, X)$ are presented.

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1. Introduction

A function $\Phi : (-\infty, \infty) \rightarrow [0, \infty)$ is called an Orlicz function if it satisfies the following conditions:

- (1) Φ is even, continuous, convex, and $\Phi(0) = 0$;
- (2) $\Phi(x) > 0$ for all $x \neq 0$;
- (3) $\lim_{x \rightarrow 0} \Phi(x)/x = 0$ and $\lim_{x \rightarrow \infty} \Phi(x)/x = \infty$.

We say that a function Φ satisfies the Δ_2 condition if there are constants $k > 1$ and $x_0 > 0$ such that $\Phi(2x) \leq k\Phi(x)$ for $x > x_0$. Examples of Orlicz functions that satisfy the Δ_2 conditions are widely available such as $\Phi(x) = |x|^p$, $1 \leq p < \infty$, and $\Phi(x) = (1 + |x|)\log(1 + |x|) - |x|$. In fact, Orlicz functions are considered to be a subclass of Young functions defined in [1].

Let X be a Banach space and let (I, μ) be a measure space. For an Orlicz function Φ , let $L^\Phi(I, X)$ be the Orlicz-Bochner function space that consists of strongly measurable functions $f : I \rightarrow X$ with $\int_I \Phi(\alpha \|f\|) d\mu(t) < \infty$ for some $\alpha > 0$. It is known that $L^\Phi(I, X)$ is a Banach space under the Luxemburg norm

$$\|f\|_{\Phi} = \inf \left\{ k > 0, \int_I \Phi \left(\frac{1}{k} \|f\| \right) d\mu(t) \leq 1 \right\}. \quad (1.1)$$

It should be remarked that if $\Phi(x) = |x|^p$, $1 \leq p < \infty$, the space $L^{\Phi}(I, X)$ is simply the p -Lebesgue Bochner function space $L^p(I, X)$ with

$$\|f\|_{\Phi} = \Phi^{-1} \int_I \Phi(\|f\|) d\mu(t) = \left(\int_I \|f\|^p d\mu(t) \right)^{1/p} = \|f\|_p. \quad (1.2)$$

On the other hand, if $\Phi(x) = (1 + |x|) \log(1 + |x|) - |x|$, then the space $L^{\Phi}(I, X)$ is the well-known Zygmund space, $L \log L^+$. For excellent monographs on $L^{\Phi}(I, X)$, we refer the readers to [1–3].

For a function $F = (f_1, f_2) \in (L^{\Phi}(I, X))^2$, we define $\|F\|$ by

$$\|F\| = \| \|f_1(\cdot)\| + \|f_2(\cdot)\| \|_{\Phi}. \quad (1.3)$$

In this paper, for a given closed subspace G of X and $F = (f_1, f_2) \in (L^{\Phi}(I, X))^2$, we show the existence of a pair $G_0 = (g_0, g_0) \in (L^{\Phi}(I, G))^2$ such that

$$\|F - G_0\| = \inf_{g \in G} \|F - (g, g)\|. \quad (1.4)$$

If such a function g exists, it is called a best simultaneous approximation of $F = (f_1, f_2)$. The problem of best simultaneous approximation can be viewed as a special case of vector-valued approximation. Recent results in this area are due to Pinkus [4], where he considered the problem when a finite-dimensional subspace is a unicity space. Characterization results for linear problems were given in [5] based on the derivation of an expression for the directional derivative, and these results generalize the earlier results presented in [6]. Results on best simultaneous approximation in general Banach spaces may be found in [7, 8]. Related results on $L^p(I, X)$, $1 \leq p < \infty$, are given in [9]. In [9], it is shown that if G is a reflexive subspace of a Banach space X , then $L^p(I, G)$ is simultaneously proximal in $L^p(I, X)$. If $L^{\Phi}(I, X) = L^1(I, X)$, Abu-Sarhan and Khalil [10] proved that if G is a reflexive subspace of the Banach space X or G is a 1-summand subspace of X , then $L^1(I, G)$ is simultaneously proximal in $L^1(I, X)$.

It is the aim of this work to prove a distance formula $\text{dist}_{\Phi}(f_1, f_2, L^{\Phi}(I, G))$, where $f_1, f_2 \in L^{\Phi}(I, X)$, similar to that of best approximation. This will allow us to generalize some recent results on $L^1(I, X)$ to $L^{\Phi}(I, X)$.

Throughout this paper, X is a Banach space, Φ is an Orlicz function, and $L^{\Phi}(I, X)$ is the Orlicz-Bochner function space equipped with the Luxemburg norm.

2. Distance formula

Let G be a closed subspace of X . For $x, y \in X$, define

$$\text{dist}(x, y, G) = \inf_{z \in G} \|x - z\| + \|y - z\|. \quad (2.1)$$

For $f_1, f_2 \in L^\Phi(I, X)$, we define $\text{dist}_\Phi(f_1, f_2, L^\Phi(I, G))$ by

$$\begin{aligned} \text{dist}_\Phi(f_1, f_2, L^\Phi(I, G)) &= \inf_{g \in L^\Phi(I, G)} \|(f_1, f_2) - (g, g)\| \\ &= \inf_{g \in L^\Phi(I, G)} \|\|f_1(\cdot) - g(\cdot)\| + \|f_2(\cdot) - g(\cdot)\|\|_\Phi. \end{aligned} \quad (2.2)$$

Our main result is the following.

THEOREM 2.1. *Let G be a subspace of the Banach space X and let Φ be an Orlicz function that satisfies the Δ_2 condition. If $f_1, f_2 \in L^\Phi(I, X)$, then the function $\text{dist}(f_1(\cdot), f_2(\cdot), G)$ belongs to $L^\Phi(I)$ and*

$$\|\text{dist}(f_1(\cdot), f_2(\cdot), G)\|_\Phi = \text{dist}_\Phi(f_1, f_2, L^\Phi(I, G)). \quad (2.3)$$

Proof. Let $f_1, f_2 \in L^\Phi(I, X)$. Then there exist two sequences $(f_{n,1}), (f_{n,2})$ of simple functions in $L^\Phi(I, X)$ such that

$$\|f_{n,1}(t) - f_1(t)\| \rightarrow 0, \quad \|f_{n,2}(t) - f_2(t)\| \rightarrow 0, \quad \text{as } n \rightarrow \infty \quad (2.4)$$

for almost all t in I . The continuity of $\text{dist}(x, y, G)$ implies that

$$|\text{dist}(f_{n,1}(t), f_{n,2}(t), G) - \text{dist}(f_1(t), f_2(t), G)| \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (2.5)$$

Set $H_n(t) = \text{dist}(f_{n,1}(t), f_{n,2}(t), G)$. Then each H_n is a measurable function. Thus $\text{dist}(f_1(\cdot), f_2(\cdot), G)$ is measurable and

$$\text{dist}(f_1(t), f_2(t), G) \leq \|f_1(t) - z\| + \|f_2(t) - z\| \quad (2.6)$$

for all z in G . Therefore,

$$\text{dist}(f_1(t), f_2(t), G) \leq \|f_1(t) - g(t)\| + \|f_2(t) - g(t)\| \quad (2.7)$$

for all $g \in L^\Phi(I, G)$. Thus

$$\|\text{dist}(f_1(\cdot), f_2(\cdot), G)\|_\Phi \leq \|\|f_1(t) - g(t)\| + \|f_2(t) - g(t)\|\|_\Phi \quad (2.8)$$

for all $g \in L^\Phi(I, G)$. Hence $\text{dist}(f_1(\cdot), f_2(\cdot), G) \in L^\Phi(I)$ and

$$\|\text{dist}(f_1(\cdot), f_2(\cdot), G)\|_\Phi \leq \text{dist}_\Phi(f_1, f_2, L^\Phi(I, G)). \quad (2.9)$$

Fix $\epsilon > 0$. Since the set of simple functions are dense in $L^\Phi(I, X)$, there exist simple functions f_i^* in $L^\Phi(I, X)$ such that $\|f_i - f_i^*\|_\Phi \leq \epsilon/6$ for $i = 1, 2$. Assume that $f_i^*(t) = \sum_{k=1}^n x_k^i \chi_{A_k}(t)$ with A_k 's are measurable sets, $x_k^i \in X$, $k = 1, 2, \dots, n$, $i = 1, 2$, $A_k \cap A_j = \emptyset$, $k \neq j$, and $\bigcup_{k=1}^n A_k = I$. We can assume that $\mu(A_k) > 0$ and $\Phi(1) \leq 1$. For each $k = 1, 2, \dots, n$, let $y_k \in G$ be such that

$$\|x_k^1 - y_k\| + \|x_k^2 - y_k\| \leq \text{dist}(x_k^1, x_k^2, G) + \frac{\epsilon}{3}. \quad (2.10)$$

Set $g(t) = \sum_{k=1}^n y_k \chi_{A_k}(t)$ and

$$F(t) = \text{dist}(f_1(t), f_2(t), G) + \|f_1(t) - f_1^*(t)\| + \|f_2(t) - f_2^*(t)\| + \frac{\epsilon}{3}. \quad (2.11)$$

Then

$$\begin{aligned} & \int_I \Phi \left(\frac{\|f_1^*(t) - g(t)\| + \|f_2^*(t) - g(t)\|}{\|F\|_\Phi} \right) d\mu(t) \\ &= \sum_{k=1}^n \int_{A_k} \Phi \left(\frac{\|f_1^*(t) - g(t)\| + \|f_2^*(t) - g(t)\|}{\|F\|_\Phi} \right) d\mu(t) \\ &= \sum_{k=1}^n \int_{A_k} \Phi \left(\frac{\|x_k^1 - y_k\| + \|x_k^2 - y_k\|}{\|F\|_\Phi} \right) d\mu(t) \\ &< \sum_{k=1}^n \int_{A_k} \Phi \left(\frac{\text{dist}(x_k^1, x_k^2, G) + \epsilon/3}{\|F\|_\Phi} \right) d\mu(t) \\ &= \int_I \Phi \left(\frac{\text{dist}(f_1^*(t), f_2^*(t), G) + \epsilon/3}{\|F\|_\Phi} \right) d\mu(t) \\ &\leq \int_I \Phi \left(\frac{\|f_1(t) - f_1^*(t)\| + \|f_2(t) - f_2^*(t)\| + \text{dist}(f_1(t), f_2(t), G) + \epsilon/3}{\|F\|_\Phi} \right) d\mu(t) \\ &= \int_I \Phi \left(\frac{F(t)}{\|F\|_\Phi} \right) d\mu(t) \leq 1. \end{aligned} \quad (2.12)$$

Consequently,

$$\| \|f_1^*(\cdot) - g(\cdot)\| + \|f_2^*(\cdot) - g(\cdot)\| \|_\Phi \leq \left\| \begin{aligned} & \|f_1(\cdot) - f_1^*(\cdot)\| + \|f_2(\cdot) - f_2^*(\cdot)\| \\ & + \text{dist}(f_1(\cdot), f_2(\cdot), G) + \frac{\epsilon}{3} \end{aligned} \right\|_\Phi. \quad (2.13)$$

Notice that

$$\begin{aligned} \text{dist}_\Phi(f_1, f_2, L^\Phi(I, G)) &\leq \text{dist}_\Phi(f_1^*, f_2^*, L^\Phi(I, G)) + \|f_1 - f_1^*\|_\Phi + \|f_2 - f_2^*\|_\Phi \\ &< \frac{\epsilon}{3} + \| \|f_1^*(\cdot) - g(\cdot)\| + \|f_2^*(\cdot) - g(\cdot)\| \|_\Phi \\ &\leq \frac{\epsilon}{3} + \left\| \begin{aligned} & \text{dist}(f_1(\cdot), f_2(\cdot), G) + \|f_1(\cdot) - f_1^*(\cdot)\| \\ & + \|f_2(\cdot) - f_2^*(\cdot)\| + \frac{\epsilon}{3} \end{aligned} \right\|_\Phi \\ &\leq \frac{2\epsilon}{3} + \| \text{dist}(f_1(\cdot), f_2(\cdot), G) \|_\Phi \\ &\quad + \|f_1(\cdot) - f_1^*(\cdot)\|_\Phi + \|f_2(\cdot) - f_2^*(\cdot)\|_\Phi \\ &\leq \epsilon + \| \text{dist}(f_1(\cdot), f_2(\cdot), G) \|_\Phi, \end{aligned} \quad (2.14)$$

which (since ϵ is arbitrary) implies that

$$\text{dist}_\Phi(f_1, f_2, L^\Phi(I, G)) \leq \|\text{dist}(f_1(\cdot), f_2(\cdot), G)\|_\Phi. \quad (2.15)$$

Hence by (2.9) and (2.15) the proof is complete. $\square$

A direct consequence of Theorem 2.1 is the following result.

THEOREM 2.2. *Let G be a closed subspace of the Banach space X and let Φ be an Orlicz function that satisfies the Δ_2 condition. For $g \in L^\Phi(I, G)$ to be a best simultaneous approximation of a pair of elements (f_1, f_2) in $L^\Phi(I, G)$, it is necessary and sufficient that $g(t)$ is a best simultaneous approximation of $(f_1(t), f_2(t))$ in G for almost all $t \in I$.*

3. Proximality of $L^\Phi(I, G)$ in $L^\Phi(I, X)$

A closed subspace G of X is called 1-summand in X if there exists a closed subspace Y such that $X = G \oplus_1 Y$, that is, any element $x \in X$ can be written as $x = g + y$, $g \in G$, $y \in Y$, and $\|x\| = \|g\| + \|y\|$. It is known that a 1-summand subspace G of X is proximal in X , and $L^1(I, G)$ is proximal in $L^1(I, X)$, [11].

Our first result in this section is the following.

THEOREM 3.1. *If G is simultaneously proximal in X , then every pair of simple functions admits a best simultaneous approximation in $L^\Phi(I, G)$.*

Proof. Let f_1, f_2 be two simple functions in $L^\Phi(I, X)$. Then f_1, f_2 can be written as $f_1(s) = \sum_{k=1}^n u_k^1 \chi_{I_k}(s)$, $f_2(s) = \sum_{k=1}^n u_k^2 \chi_{I_k}(s)$, where I_k 's are disjoint measurable subsets of I satisfying $\bigcup_{k=1}^n I_k = I$, and χ_{I_k} is the characteristic function of I_k . Since f_1 and f_2 represent classes of functions, we may assume that $\mu(I_k) > 0$ for each $1 \leq k \leq n$. By assumption, we know that for each $1 \leq k \leq n$ there exists a best simultaneous approximation w_k in G of the pair of elements $(u_k^1, u_k^2) \in X^2$ such that

$$\text{dist}(u_k^1, u_k^2, G) = \|u_k^1 - w_k\| + \|u_k^2 - w_k\|. \quad (3.1)$$

Set $g = \sum_{k=1}^n w_k \chi_{I_k}(s)$. Then, for any $\alpha > 0$ and $h \in L^\Phi(I, G)$, we obtain that

$$\begin{aligned} \int_I \Phi\left(\frac{\|f_1(t) - h(t)\| + \|f_2(t) - h(t)\|}{\alpha}\right) d\mu(t) &= \sum_{k=1}^n \int_{I_k} \Phi\left(\frac{\|u_k^1 - h(t)\| + \|u_k^2 - h(t)\|}{\alpha}\right) d\mu(t) \\ &\geq \sum_{k=1}^n \int_{I_k} \Phi\left(\frac{\|u_k^1 - w_k\| + \|u_k^2 - w_k\|}{\alpha}\right) d\mu(t) \\ &= \int_I \Phi\left(\frac{\|f_1(t) - g(t)\| + \|f_2(t) - g(t)\|}{\alpha}\right) d\mu(t). \end{aligned} \quad (3.2)$$

Taking the infimum over all such α 's, we have that

$$\| \|f_1(\cdot) - h(\cdot)\| + \|f_2(\cdot) - h(\cdot)\| \|_\Phi \geq \| \|f_1(\cdot) - g(\cdot)\| + \|f_2(\cdot) - g(\cdot)\| \|_\Phi \quad (3.3)$$

for all $h \in L^\Phi(I, G)$. Hence

$$\begin{aligned} \text{dist}_\Phi(f_1, f_2, L^\Phi(I, G)) &= \| \|f_1(\cdot) - g(\cdot)\| + \|f_2(t) - g(\cdot)\| \|_\Phi \\ &\geq \| \|f_1(\cdot) - h(\cdot)\| + \|f_2(\cdot) - h(\cdot)\| \|_\Phi. \end{aligned} \quad (3.4)$$

□

Now we prove the following 2-dimensional analogous of [12, Theorem 4].

THEOREM 3.2. *Let G be a closed subspace of the Banach space X and let Φ be an Orlicz function that satisfies the Δ_2 condition. If $L^1(I, G)$ is simultaneously proximal in $L^1(I, X)$, then $L^\Phi(I, G)$ is simultaneously proximal in $L^\Phi(I, X)$.*

Proof. Let $f_1, f_2 \in L^\Phi(I, X)$. Then $f_1, f_2 \in L^1(I, X)$; see [13]. By assumption, there exists $g \in L^1(I, G)$ such that

$$\| \|f_1(\cdot) - g(\cdot)\| + \|f_2(\cdot) - g(\cdot)\| \|_1 \leq \| \|f_1(\cdot) - h(\cdot)\| + \|f_2(\cdot) - h(\cdot)\| \|_1 \quad (3.5)$$

for every $h \in L^1(I, G)$. By Theorem 2.2 [10],

$$\|f_1(t) - g(t)\| + \|f_2(t) - g(t)\| \leq \|f_1(t) - h(t)\| + \|f_2(t) - h(t)\| \quad (3.6)$$

for almost all t in I . But $0 \in G$. Thus

$$\|f_1(t) - g(t)\| + \|f_2(t) - g(t)\| \leq \|f_1(t)\| + \|f_2(t)\| \quad (3.7)$$

or

$$\|g(t)\| \leq \|f_1(t)\| + \|f_2(t)\|. \quad (3.8)$$

Hence $g \in L^\Phi(I, G)$ and

$$\| \|f_1(\cdot) - g(\cdot)\| + \|f_2(\cdot) - g(\cdot)\| \|_\Phi \leq \| \|f_1(\cdot) - h(\cdot)\| + \|f_2(\cdot) - h(\cdot)\| \|_\Phi \quad (3.9)$$

for all $h \in L^1(I, G)$. □

THEOREM 3.3. *Let G be a 1-summand subspace of the Banach space X . Then $L^\Phi(I, G)$ is simultaneously proximal in $L^\Phi(I, X)$.*

The proof follows from Theorem 3.2 and [10, Theorem 2.4].

THEOREM 3.4. *Let G be a reflexive subspace of the Banach space X . Then $L^\Phi(I, G)$ is simultaneously proximal in $L^\Phi(I, X)$.*

The proof follows from Theorem 3.2 and [10, Theorem 3.2].

References

- [1] J. Musielak, *Orlicz Spaces and Modular Spaces*, vol. 1034 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1983.
- [2] M. A. Krasnosel'skiĭ and Y. B. Rutickii, *Convex Functions and Orlicz Spaces*, P. Noordhoff, Groningen, The Netherlands, 1961.

- [3] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, vol. 146 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1991.
- [4] A. Pinkus, "Uniqueness in vector-valued approximation," *Journal of Approximation Theory*, vol. 73, no. 1, pp. 17–92, 1993.
- [5] G. A. Watson, "A characterization of best simultaneous approximations," *Journal of Approximation Theory*, vol. 75, no. 2, pp. 175–182, 1993.
- [6] S. Tanimoto, "On best simultaneous approximation," *Mathematica Japonica*, vol. 48, no. 2, pp. 275–279, 1998.
- [7] P. D. Milman, "On best simultaneous approximation in normed linear spaces," *Journal of Approximation Theory*, vol. 20, no. 2, pp. 223–238, 1977.
- [8] B. N. Sahney and S. P. Singh, "On best simultaneous approximation in Banach spaces," *Journal of Approximation Theory*, vol. 35, no. 3, pp. 222–224, 1982.
- [9] F. B. Saidi, D. Hussein, and R. Khalil, "Best simultaneous approximation in $L^p(I, E)$," *Journal of Approximation Theory*, vol. 116, no. 2, pp. 369–379, 2002.
- [10] I. Abu-Sarhan and R. Khalil, "Best simultaneous approximation in function spaces," to appear in *Journal of Nonlinear Functional Analysis and Differential Equations*.
- [11] R. Khalil, "Best approximation in $L^p(I, X)$," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 94, no. 2, pp. 277–279, 1983.
- [12] M. Khandaqji, R. Khalil, and D. Hussein, "Proximality in Orlicz-Bochner function spaces," *Tamkang Journal of Mathematics*, vol. 34, no. 1, pp. 71–75, 2003.
- [13] M. Khandaqji, *Certain properties of Orlicz Bochner function spaces*, Ph.D. thesis, Jordan University, Amman, Jordan, 2000.

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