Research Article

On Certain Multivalent Functions

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Let $\mathcal{S}^*(p, \alpha)$ be the class of functions $f(z)$ which are analytic and p -valently starlike of order α in the open unit disk $\mathbb{E}$. The object of the present paper is to derive an interesting condition for $f(z)$ to be in the class $\mathcal{S}^*(p, \alpha)$.

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1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}) \quad (1.1)$$

which are analytic in the open unit disk $\mathbb{E} = \{z : z \in \mathbb{C}, |z| < 1\}$. A function $f(z) \in \mathcal{A}(p)$ is said to be p -valently starlike of order α ($0 \leq \alpha < p$) in $\mathbb{E}$ if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \quad (z \in \mathbb{E}). \quad (1.2)$$

We denote by $\mathcal{S}^*(p, \alpha)$ the subclass of $\mathcal{A}(p)$ consisting of functions which are p -valently starlike of order α in $\mathbb{E}$. We only call a function $f(z) \in \mathcal{S}^*(p, 0)$ to be p -valently starlike in $\mathbb{E}$. Further, a function $f(z) \in \mathcal{A}(p)$ is said to be p -valently convex of order α ($0 \leq \alpha < p$) in $\mathbb{E}$ if and only if

$$1 + \operatorname{Re} \frac{zf''(z)}{f'(z)} > \alpha \quad (z \in \mathbb{E}). \quad (1.3)$$

We denote by $\mathcal{C}(p, \alpha)$ the subclass of $\mathcal{A}(p)$ consisting of all p -valently convex functions of order α in $\mathbb{E}$. We also call a function $f(z) \in \mathcal{C}(p, 0)$ p -valently convex function. From the definition, it is trivial that if $f(z)$ is a p -valently convex function, then $zf'(z)$ is p -valently starlike in $\mathbb{E}$.

2. Preliminaries

In this paper, we need the following lemmas.

LEMMA 2.1. If $M(z) = z^p + \sum_{n=p+k}^{\infty} a_n z^n$ ($1 \leq p$ and $1 \leq k$) and $N(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ are analytic in $\mathbb{E}$ and $N(z)$ satisfies $\operatorname{Re}(N(z)/zN'(z)) > \delta$ ($0 \leq \delta < 1/p$), then

$$\operatorname{Re}\left(\frac{M'(z)}{N'(z)}\right) > \beta \quad \text{implies} \quad \operatorname{Re}\left(\frac{M(z)}{N(z)}\right) > \frac{2\beta + k\delta}{2 + k\delta}. \quad (2.1)$$

Remark 2.2. This lemma holds to be true for $N(z)$ which is multivalently starlike in $\mathbb{E}$.

We owe the above lemma to Ponnusamy and Karunakaran [1].

LEMMA 2.3. Let $f(z) \in \mathcal{S}^*(p, 0)$. Then

$$\frac{F(z)}{p+1} = \int_0^z f(t)dt \in \mathcal{S}^*(p+1, 0) \quad (2.2)$$

or

$$\operatorname{Re} \frac{zF'(z)}{F(z)} > 0 \quad (z \in \mathbb{E}). \quad (2.3)$$

The proof of this result can be found in [2].

3. Main result

THEOREM 3.1. If $f(z) \in \mathcal{A}(p)$ satisfies the following condition:

$$\operatorname{Re} \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > \alpha \quad (z \in \mathbb{E}) \quad (3.1)$$

for $\alpha(0 \leq \alpha < 1)$, then $f(z) \in \mathcal{S}^*(p, \alpha + p - 1)$ or

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha + p - 1 \quad (z \in \mathbb{E}). \quad (3.2)$$

Proof. From assumption (3.1), $f^{(p-1)}(z)$ is univalently starlike of order α and it is trivial that

$$(zf^{(p-1)}(z) - f^{(p-2)}(z))' = zf^{(p)}(z). \quad (3.3)$$

Note that $f^{(p-2)}(z)$ is starlike in $\mathbb{E}$ by Lemma 2.3. Therefore, applying Lemma 2.1 and (3.3), we have

$$\operatorname{Re} \frac{zf^{(p-1)}(z) - f^{(p-2)}(z)}{f^{(p-2)}(z)} > \alpha \quad (z \in \mathbb{E}). \quad (3.4)$$

From Lemma 2.3, $f^{(p-2)}(z)$ is 2-valently starlike in $\mathbb{E}$. Now then, it is trivial that

$$(zf^{(p-2)}(z) - 2f^{(p-3)}(z))' = zf^{(p-1)}(z) - f^{(p-2)}(z). \quad (3.5)$$

Then, from Lemma 2.1, 2-valently starlikeness of $f^{(p-2)}(z)$, and (3.5), we have

$$\operatorname{Re} \frac{zf^{(p-2)}(z) - 2f^{(p-3)}(z)}{f^{(p-3)}(z)} > \alpha \quad (z \in \mathbb{E}). \quad (3.6)$$

Further, it is trivial that

$$(zf^{(p-3)}(z) - 3f^{(p-4)}(z))' = zf^{(p-2)}(z) - 2f^{(p-3)}(z), \quad (3.7)$$

and applying the same method and reason as above, we have

$$\operatorname{Re} \frac{zf^{(p-3)}(z) - 3f^{(p-4)}(z)}{f^{(p-4)}(z)} > \alpha \quad (z \in \mathbb{E}), \quad (3.8)$$

where $f^{(p-3)}(z)$ is 3-valently starlike in $\mathbb{E}$. Applying the mathematical induction, we have

$$\operatorname{Re} \frac{zf'(z) - (p-1)f(z)}{f(z)} > \alpha \quad (z \in \mathbb{E}), \quad (3.9)$$

where $f(z)$ is p -valently starlike in $\mathbb{E}$. This shows that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha + p - 1 \quad (z \in \mathbb{E}), \quad (3.10)$$

or $f(z) \in \mathcal{S}^*(p, \alpha + p - 1)$. □

Our main result shows the following.

COROLLARY 3.2. *Let $f(z) \in \mathcal{A}(p)$, $2 \leq p$, $0 \leq \alpha < 1$,*

$$\operatorname{Re} \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > \alpha \quad (z \in \mathbb{E}), \quad (3.11)$$

and put $f(z) = z^{p-1}f_1(z)$ where

$$f_1(z) = z + \sum_{n=p+1}^{\infty} a_n z^{n-p+1}. \quad (3.12)$$

Then $f_1(z)$ is univalently starlike of order α in $\mathbb{E}$.

Proof. From the definition of $f_1(z)$ and Theorem 3.1, it follows that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} = p - 1 + \operatorname{Re} \frac{zf_1'(z)}{f_1(z)} > \alpha + p - 1. \quad (3.13)$$

This completes the proof. □

COROLLARY 3.3. Let $f(z) \in \mathcal{A}(p)$, $2 \leq p$, $0 \leq \alpha < 1$, and

$$1 + \operatorname{Re} \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} > \alpha \quad (z \in \mathbb{E}). \quad (3.14)$$

Then one has

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \beta(\alpha) + p - 1 \quad (z \in \mathbb{E}), \quad (3.15)$$

where

$$\beta(\alpha) = \begin{cases} \frac{1-2\alpha}{2^{2-2\alpha}[1-2^{2\alpha-1}]} & \text{if } \alpha \neq \frac{1}{2}, \\ \frac{1}{2\log 2} & \text{if } \alpha = \frac{1}{2}. \end{cases} \quad (3.16)$$

Proof. Putting

$$g(z) = \frac{f^{(p-1)}(z)}{p!} = z + \sum_{n=2}^{\infty} b_n z^n, \quad (3.17)$$

then from assumption (3.14), $g(z)$ is univalently convex of order α , and therefore from Wilken-Feng result [3] and Theorem 3.1, we have

$$\operatorname{Re} \frac{zg'(z)}{g(z)} = \operatorname{Re} \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > \beta(\alpha) \quad (z \in \mathbb{E}), \quad (3.18)$$

and therefore it follows that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \beta(\alpha) + p - 1 \quad (z \in \mathbb{E}). \quad (3.19)$$

□

We will give here an open problem.

Problem 3.4. Let $f(z) \in \mathcal{S}^*(p, \alpha)$ and $0 \leq \alpha < p$.

Then

$$\frac{F(z)}{p+1} = \int_0^z f(t) dt \in \mathcal{S}^*(p+1, \beta(p, \alpha)). \quad (3.20)$$

What is the best $\beta(p, \alpha)$?

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