Research Article

On Further Analogs of Hilbert's Inequality

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By introducing the function $|\ln x - \ln y|/(x + y + |x - y|)$, we establish new inequalities similar to Hilbert's type inequality for integrals. As applications, we give its equivalent form as well.

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1. Introduction

If f, g are real functions such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then we have (see [1])

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}, \quad (1.1)$$

where the constant factor π is the best possible. Inequality (1.1) is the well-known Hilbert's inequality. Inequality (1.1) had been generalized by Hardy-Riesz (see [2]) in 1925 as the following result.

If f, g are real functions such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty f^p(x)dx \right\}^{1/p} \left\{ \int_0^\infty g^q(x)dx \right\}^{1/q}, \quad (1.2)$$

where the constant factor $c = \pi/\sin(\pi/p)$ is the best possible. When $p = q = 2$, (1.2) reduces to (1.1). Inequality (1.2) is named Hardy-Hilbert's integral inequality, which is important in analysis and its applications (see [3]), it has been studied and generalized in many directions by a number of mathematicians (see [4–8]).

Under the same condition of (1.2), we have Hardy-Hilbert's type inequality (see [1, Theorems 341 and 342]):

$$\begin{aligned} \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x, y\}} dx dy &< 4 \left\{ \int_0^\infty f^2(x) dx \int_0^\infty g^2(x) dx \right\}^{1/2}, \\ \int_0^\infty \int_0^\infty \frac{\ln x - \ln y}{x - y} f(x)g(y) dx dy &< \pi^2 \left\{ \int_0^\infty f^2(x) dx \int_0^\infty g^2(x) dx \right\}^{1/2}, \end{aligned} \quad (1.3)$$

where the constant factors 4 and π^2 are both the best possible.

Recently, Li et al. [9] obtained the following result.

THEOREM 1.1. *If f, g are real functions such that $0 < \int_0^\infty f^2(x) dx < \infty$ and $0 < \int_0^\infty g^2(x) dx < \infty$, then one has*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x + y + \max\{x, y\}} dx dy < c \left(\int_0^\infty f^2(x) dx \right)^{1/2} \left(\int_0^\infty g^2(x) dx \right)^{1/2}, \quad (1.4)$$

where the constant factor $c = \sqrt{2}(\pi - 2 - \arctan \sqrt{2}) = 1.7408\dots$

In this paper, we give a further analogs of Hilbert's type inequality and its applications.

2. Main results and applications

THEOREM 2.1. *If $f(x), g(x) \geq 0$, $0 < \int_0^\infty f^2(x) dx < \infty$, $0 < \int_0^\infty g^2(x) dx < \infty$, then one has*

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x)g(y) dx dy < 4 \left(\int_0^\infty f^2(x) dx \right)^{1/2} \left(\int_0^\infty g^2(x) dx \right)^{1/2}, \quad (2.1)$$

where the constant factor 4 is the best possible.

Proof. Applying Hölder's inequality, we obtain

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x)g(y) dx dy \\ &= \int_0^\infty \int_0^\infty \left[\left(\frac{|\ln x - \ln y|}{x + y + |x - y|} \right)^{1/2} f(x) \left(\frac{x}{y} \right)^{1/4} \right] \\ &\quad \times \left[\left(\frac{|\ln x - \ln y|}{x + y + |x - y|} \right)^{1/2} g(y) \left(\frac{y}{x} \right)^{1/4} \right] dx dy \\ &\leq \left\{ \int_0^\infty \left(\int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} \left(\frac{x}{y} \right)^{1/2} dy \right) f^2(x) dx \right\} \\ &\quad \times \left\{ \int_0^\infty \left(\int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} \left(\frac{y}{x} \right)^{1/2} dx \right) g^2(y) dy \right\}. \end{aligned} \quad (2.2)$$

Define the weight function $\bar{\omega}(u)$ as

$$\omega(u) := \int_0^\infty \frac{|\ln u - \ln v|}{u + v + |u - v|} \left(\frac{u}{v} \right)^{1/2} dv. \quad (2.3)$$

For fixed u , letting $v = ut$, we have

$$\begin{aligned}
 \omega(u) &= \int_0^\infty \frac{|\ln u - \ln tu|}{u + tu + |u - tu|} \left(\frac{1}{t}\right)^{1/2} u dt = \int_0^\infty \frac{|\ln t|}{1+t+|1-t|} \left(\frac{1}{t}\right)^{1/2} dt \\
 &= -\int_0^1 \frac{\ln t}{2} \left(\frac{1}{t}\right)^{1/2} dt + \int_1^\infty \frac{\ln t}{2t} \left(\frac{1}{t}\right)^{1/2} dt \\
 &= -\int_0^1 (\ln t) \left(\frac{1}{t}\right)^{1/2} dt = -4 \int_0^1 \ln s ds (t^{1/2} = s) = 4.
 \end{aligned} \tag{2.4}$$

Thus

$$\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x)g(y) dx dy \leq 4 \left(\int_0^\infty f^2(x) dx \right)^{1/2} \left(\int_0^\infty g^2(x) dx \right)^{1/2}. \tag{2.5}$$

If (2.5) takes the form of the equality, then there exist constants c and d , such that they are not all zero and

$$c \frac{|\ln x - \ln y|}{x + y + |x - y|} f^2(x) \left(\frac{x}{y}\right)^{1/2} = d \frac{|\ln x - \ln y|}{x + y + |x - y|} g^2(y) \left(\frac{y}{x}\right)^{1/2}, \quad \text{a.e. on } (0, \infty) \times (0, \infty). \tag{2.6}$$

Then we have

$$cx f^2(x) = dy g^2(y), \quad \text{a.e. on } (0, \infty) \times (0, \infty). \tag{2.7}$$

Hence we have

$$cx f^2(x) = dy g^2(y) = \text{constant}, \quad \text{a.e. on } (0, \infty) \times (0, \infty). \tag{2.8}$$

Without losing the generality, suppose $c \neq 0$, then

$$\int_0^\infty f^2(x) dx = \int_0^\infty \frac{1}{x} \frac{\text{const}}{c} dx = \frac{\text{const}}{c} \int_0^\infty \frac{1}{x} dx, \tag{2.9}$$

which contradicts the facts that $0 < \int_0^\infty f^2(x) dx < \infty$. Hence (2.5) takes the form of strict inequality. So we have (2.1).

Assume that the constant factor 4 in (2.1) is not the best possible, then there exists a positive number K with $K < 4$ and $a > 0$; we have

$$\int_a^\infty \int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x)g(y) dx dy < K \left(\int_a^\infty f^2(x) dx \right)^{1/2} \left(\int_a^\infty g^2(x) dx \right)^{1/2}. \tag{2.10}$$

For $0 < \varepsilon < 1$, setting $f_\varepsilon(x) = x^{(-\varepsilon-1)/2}$, for $x \in [1, \infty)$; $f_\varepsilon(x) = 0$, for $x \in (0, 1)$, $g_\varepsilon(y) = y^{(-\varepsilon-1)/2}$, for $y \in [1, \infty)$; $g_\varepsilon(y) = 0$, for $y \in (0, 1)$.

Since

$$K \left(\int_a^\infty f^2(x) dx \right)^{1/2} \left(\int_a^\infty g^2(x) dx \right)^{1/2} = K \int_a^\infty x^{-1-\varepsilon} dx = K \cdot \frac{1}{\varepsilon a^\varepsilon}, \tag{2.11}$$

setting $y = ux$, we find

$$\begin{aligned}
 & \int_a^\infty \int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f_\varepsilon(x) g_\varepsilon(y) dx dy \\
 &= \int_a^\infty \int_b^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} x^{-(1+\varepsilon)/2} y^{-(1+\varepsilon)/2} dx dy \\
 &= \int_a^\infty \int_{b/x}^\infty \frac{|\ln x - \ln ux|}{x + ux + |1 - ux|} x \cdot x^{-(1+\varepsilon)} u^{-(1+\varepsilon)/2} dx du \\
 &= \int_a^\infty \int_{b/x}^\infty \frac{x^{-(1+\varepsilon)} u^{-(1+\varepsilon)/2} |\ln u|}{1 + u + |1 - u|} dx du.
 \end{aligned} \tag{2.12}$$

By (2.10) and for $b \rightarrow 0^+$, we have

$$\int_a^\infty \int_0^\infty \frac{x^{-(1+\varepsilon)} u^{-(1+\varepsilon)/2} |\ln u|}{1 + u + |1 - u|} dx dy \leq K \cdot \frac{1}{\varepsilon a^\varepsilon}, \tag{2.13}$$

or

$$\frac{1}{\varepsilon a^\varepsilon} \int_0^\infty \frac{u^{-(1+\varepsilon)/2} |\ln u|}{1 + u + |1 - u|} du \leq K \cdot \frac{1}{\varepsilon a^\varepsilon}, \tag{2.14}$$

that is

$$\int_0^\infty \frac{u^{-(1+\varepsilon)/2} |\ln u|}{1 + u + |1 - u|} du \leq K. \tag{2.15}$$

When $\varepsilon \rightarrow 0^+$, we have

$$\int_0^\infty \frac{u^{-(1+\varepsilon)/2} |\ln u|}{1 + u + |1 - u|} du = \int_0^\infty \frac{u^{-1/2} |\ln u|}{1 + u + |1 - u|} du + o(1) = 4 + o(1). \tag{2.16}$$

This contradicts the hypothesis. Hence the constant factor 4 in (2.1) is the best possible. $\square$

THEOREM 2.2. Suppose $f \geq 0$ and $0 < \int_0^\infty f^2(x) dx < \infty$. Then

$$\int_0^\infty \left[\int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x) \right]^2 dy < 16 \int_0^\infty f^2(x) dx, \tag{2.17}$$

where the constant factor 16 is the best possible. Inequality (2.17) is equivalent to (2.1).

Proof. Letting $g(y) = \int_0^\infty (|\ln x - \ln y|/(x + y + |x - y|))f(x)dx$, then by (2.5) we get

$$\begin{aligned}
 0 &< \int_0^\infty g^2(y)dy \\
 &= \int_0^\infty \left[\int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x)dx \right]^2 dy \\
 &= \int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x)g(y)dx dy \\
 &\leq 4 \left\{ \int_0^\infty f^2(x)dx \right\}^{1/2} \left\{ \int_0^\infty g^2(y)dy \right\}^{1/2}.
 \end{aligned} \tag{2.18}$$

Hence we obtain

$$0 < \int_0^\infty g^2(y)dy = 16 \int_0^\infty f^2(x)dx < \infty. \tag{2.19}$$

By (2.1), both (2.18) and (2.19) take the form of strict inequality, so we have (2.17). On the other hand, suppose that (2.17) is valid. By Hölder's inequality, we find

$$\begin{aligned}
 &\int_0^\infty \int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x)g(y)dx dy \\
 &= \int_0^\infty \left[\int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x)dx \right] g(y)dy \\
 &\leq \left\{ \int_0^\infty \left[\int_0^\infty \frac{|\ln x - \ln y|}{x + y + |x - y|} f(x)dx \right]^2 dy \right\}^{1/2} \left\{ \int_0^\infty g^2(x)dx \right\}^{1/2}.
 \end{aligned} \tag{2.20}$$

By (2.17) we have (2.1). Thus (2.1) and (2.17) are equivalent.

If the constant 16 in (2.17) is not the best possible, by (2.20), we may get a contradiction that the constant factor in (2.1) is not the best possible. This completes the proof. $\square$

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