Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2007, Article ID 80451, 12 pages
doi:10.1155/2007/80451

Research Article

Near S^*-Compactness in L-Topological Spaces

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Received 16 October 2006; Accepted 7 February 2007

Recommended by Heinz Peter Gumm

The notion of near S^*-compactness is introduced in L-topological spaces based on S^*-compactness. Its properties are researched and the relations between it and other near compactness are obtained. Moreover many characterizations of near S^*-compactness are presented.

1. Introduction

The concept of compactness is one of the most important concepts in general topology. The concept of compactness in [0, 1]-fuzzy set theory was first introduced by Chang in terms of open covers [1]. Goguen pointed out a deficiency in Chang's compactness theory by showing that the Tychonoff theorem is false [2]. Since Chang's compactness has some limitations, Gantner et al. introduced α-compactness [3], Lowen introduced fuzzy compactness, strong fuzzy compactness, and ultrafuzzy compactness [4, 5], Liu introduced Q-compactness [6], Li introduced strong Q-compactness [7] which is equivalent to strong fuzzy compactness in [5], and Wang and Zhao introduced N-compactness [8, 9]. Recently Shi introduced S^*-compactness [10].

Near compactness is one of the good weak compactness in topology. It was generalized and studied by many authors in L-topological spaces. In [11], Eş introduced a definition of fuzzy near compactness in $[0,1]$-topological spaces by using the notion of Chang's compactness which is not a good extension of compactness. In [12], Kudri and Warner generalized the concept of near compactness to L-topological spaces by using the notion of Kudri's compactness which is equivalent to strong compactness in [13]. Meng also presented a definition of fuzzy near compactness in L-fuzzy topological spaces in [14] by using the notion of N-compactness. Moreover Bülbül and Warner introduced

Lo-fuzzy near compactness of $[0,1]$-topological spaces [15] based on Lowen's fuzzy compactness. Recently, Shi and Xu [16] gave a new definition of fuzzy near compactness in L-topological spaces by using an inequality, where L is a complete de Morgan algebra.

The aim of this paper is to study near S^*-compactness in L-topological spaces. We will discuss the properties of near S^*-compactness and give its characterizations. Moreover we will investigate the relations among different notions of near compactness in L-topological spaces.

2. Preliminaries

Throughout this paper, $(L,\bigvee,\bigwedge,')$ is a completely distributive de Morgan algebra, and X a nonempty set. L^X is the set of all L-fuzzy sets on X. The smallest element and the largest element in L^X are denoted by $\underline{0}$ and $\underline{1}$.

An element a in L is called a prime element if $a \geqslant b \wedge c$ implies $a \geqslant b$ or $a \geqslant c$. a in L is called a coprime element if a' is a prime element [17]. The set of nonunit prime elements in L is denoted by $P(L)$. The set of nonzero coprime elements in L is denoted by $M(L)$. The set of nonzero coprime elements in L^X is denoted by $M(L^X)$.

The binary relation $\prec$ in L is defined as follows: for $a,b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leqslant \sup D$ always implies the existence of $d \in D$ with $a \leqslant d$ [18]. In a completely distributive de Morgan algebra L, each element b is a sup of $\{a \in L \mid a \prec b\}$. In the sense of [13, 19], $\{a \in L \mid a \prec b\}$ is the greatest minimal family of b, in symbol $\beta(b)$. Moreover for $b \in L$, define $\alpha(b) = \{a \in L \mid a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we use the following notations in [10, 20]:

$$A_{[a]} = \{x \in X \mid A(x) \geqslant a\}, \qquad A_{(a)} = \{x \in X \mid a \in \beta(A(x))\},$$
$$A^{(a)} = \{x \in X \mid A(x) \nleqslant a\}. \tag{2.1}$$

An L-topological space (or L-space for short) is a pair $(X,\mathcal{T})$, where $\mathcal{T}$ is a subfamily of L^X which contains $\underline{0}$, $\underline{1}$ and is closed with respect to suprema and finite infima. $\mathcal{T}$ is called an L-topology on X. Each member of $\mathcal{T}$ is called an open L-set and its complement is called a closed L-set.

Definition 2.1 [13, 19]. For a topological space $(X,\mathcal{T})$, let $\omega_L(\mathcal{T})$ denote the family of all lower semicontinuous maps from $(X,\mathcal{T})$ to L, that is, $\omega_L(\mathcal{T}) = \{A \in L^X \mid A^{(a)} \in \mathcal{T},\ a \in L\}$. Then $\omega_L(\mathcal{T})$ is an L-topology on X, in this case, $(X,\omega_L(\mathcal{T}))$ is called topologically generated by $(X,\mathcal{T})$.

Definition 2.2 [13, 19]. An L-space $(X,\mathcal{T})$ is called weakly induced if for all $a \in L$, for all $A \in \mathcal{T}$, it follows that $A^{(a)} \in [\mathcal{T}]$, where $[\mathcal{T}]$ denotes the topology formed by all crisp sets in $\mathcal{T}$.

It is obvious that $(X,\omega_L(\mathcal{T}))$ is weakly induced.

Lemma 2.3 [10]. *Let $(X,\mathcal{T})$ be a weakly induced L-space, $a \in L$, $A \in \mathcal{T}$. Then $A_{(a)}$ is an open set in $[\mathcal{T}]$.*

Definition 2.4. $A \in L^X$ is called (1) semiopen [21] if $A \leqslant A^{\circ -}$, the complement of a semiopen L-set is called semiclosed; (2) regularly open [21] if $A^{-\circ} = A$, the complement of a regularly open L-set is called regularly closed; (3) α-open [22] if $A \leqslant A^{\circ -\circ}$, the complement of an α-open L-set is called α-closed.

Definition 2.5. Let $(X,\mathcal{T}_1)$ and $(Y,\mathcal{T}_2)$ be two L-spaces. A map $f:(X,\mathcal{T}_1) \to (Y,\mathcal{T}_2)$ is called

(1) almost continuous [21] if $f_L^{\leftarrow}(G) \in \mathcal{T}_1$ for each regularly open L-set G in $(Y,\mathcal{T}_2)$;
(2) completely continuous [23, 24] if $f_L^{\leftarrow}(G)$ is regularly open L-set in $(X,\mathcal{T}_1)$ for each $G \in \mathcal{T}_2$;
(3) R-irresolute if $f_L^{\leftarrow}(G)$ is regularly open in $(X,\mathcal{T}_1)$ for each regularly open L-set G in $(Y,\mathcal{T}_2)$.

Definition 2.6 [25]. A net S with directed index set D is also denoted by $\{S(n) \mid n \in D\}$ or $\{S(n)\}_{n\in D}$. For $G \in L^X$, a net S is said to quasicoincide with G if for all $n \in D$, $S(n) \nleqslant G'$.

Definition 2.7 [25]. Let $\alpha \in M(L)$. A net $\{s(n) \mid n \in D\}$ in L^X is called an α^--net if there exists $n_0 \in D$ such that for all $n \geqslant n_0$, $V(S(n)) \leqslant \alpha$, where $V(S(n))$ denotes the height of $S(n)$. A net $\{S(n)\}_{n\in D}$ in L^X is said to be a constant α-net if the height of each $S(n)$ is a constant value α.

Obviously, each constant α-net must be an α^--net.

Definition 2.8 [13]. Let $(X,\mathcal{T})$ be an L-space. $A \in \mathcal{T}'$ is called a closed remote neighborhood of a fuzzy point x_a if $x_a \nleqslant A$. Let $\eta^-(x_a)$ denote the set of all closed remote neighborhoods of x_a

Definition 2.9 [9]. Let $A \in L^X$, $a \in M(L)$. $\Phi \subseteq \mathcal{T}'$ is called an a-remote neighborhood family (briefly a-RF) of A, if for each $x_a \leqslant A$, there is $P \in \Phi$ such that $P \in \eta^-(x_a)$. Φ is called an a^--RF of A if there exists $b \in \beta^*(a)$ such that Φ is a b-RF of A.

Definition 2.10 [26, 27]. Let $A \in L^X$, $a \in L$, and $\Omega \subseteq L^X$. Ω is called

(1) an a-shading of A if for each $x \in X$, it follows that $(A' \vee \bigvee_{U\in\Omega} U)(x) \nleqslant a$;
(2) a strong a-shading of A if $\bigwedge_{x\in X}(A' \vee \bigvee_{U\in\Omega} U)(x) \nleqslant a$.

It is obvious that for all $a \in P(L)$, Ω is an a-shading (a strong a-shading) of A if and only if Ω is an a-cover (a^+-cover) of A in the sense of [14].

Definition 2.11 [10]. Let $(X,\mathcal{T})$ be an L-space, $a \in M(L)$, and $G \in L^X$. A subfamily $\mathcal{U}$ of L^X is called a β_a-cover of G if for any $x \in X$ with $a \notin \beta(G'(x))$, there exists an $A \in \mathcal{U}$ such that $a \in \beta(A(x))$. A β_a-cover $\mathcal{U}$ of G is called an open(regularly open, α-open, etc.) β_a-cover of G if each member of $\mathcal{U}$ is open(regularly open, α-open, etc.).

It is obvious that $\mathcal{U}$ is a β_a-cover of G if and only if for any $x \in X$ it follows that $a \in \beta(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x))$.

Definition 2.12 [10]. Let $(X,\mathcal{T})$ be an L-space, $a \in M(L)$, and $G \in L^X$. A subfamily $\mathcal{U}$ of L^X is called a Q_a-cover of G if for any $x \in X$ with $G(x) \nleqslant a'$, it follows that $\bigvee_{A\in\mathcal{U}} A(x) \geqslant a$. A Q_a-cover $\mathcal{U}$ of G is called an open(regularly open, α-open, etc.) Q_a-cover of G if each member of $\mathcal{U}$ is open(regularly open, α-open, etc.).

Definition 2.13 [10]. Let $(X,\mathcal{T})$ be an L-space and $G \in L^X$. G is called S^*-compact if for any $a \in M(L)$, each open β_a-cover of G has a finite subfamily $\mathcal{V}$ which is an open Q_a-cover of G. $(X,\mathcal{T})$ is said to be S^*-compact if $\underline{1}$ is S^*-compact.

Definition 2.14 [28]. An L-space $(X,\mathcal{T})$ is said to be regular if and only if each open L-set A is a union of open L-sets whose closures are less than A.

3. Definitions and properties of near S^*-compactness

Definition 3.1. Let $(X,\mathcal{T})$ be an L-space and $G \in L^X$. Then G is called nearly S^*-compact if for any $a \in M(L)$, each open β_a-cover of G has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^{-\circ} = \{A^{-\circ} \mid A \in \mathcal{V}\}$ is a Q_a-cover of G. $(X,\mathcal{T})$ is said to be nearly S^*-compact if $\underline{1}$ is nearly S^*-compact.

Obviously, we have the following theorems.

Theorem 3.2. *S^*-compactness implies near S^*-compactness.*

Theorem 3.3. *If G is nearly S^*-compact and H is regularly closed, then $G \wedge H$ is nearly S^*-compact.*

Theorem 3.4. *Let $(X,\mathcal{T})$ be an L-space and $G \in L^X$. Then G is nearly S^*-compact if and only if for any $a \in M(L)$, each regularly open β_a-cover of G has a finite subfamily which is a Q_a-cover of G.*

Theorem 3.5. *Let $(X,\mathcal{T})$ be a regular L-space and $G \in L^X$. Then G is nearly S^*-compact if and only if G is S^*-compact.*

Proof. The sufficiency is obvious, we need only prove the necessity. Let $\mathcal{A} = \{A_i\}_{i\in I}$ be an open β_a-cover of G. By regularity of $(X,\mathcal{T})$, we know that for each $i \in I$, there exists a family $\{B_{ij} \mid j \in J_i\}$ of open L-sets such that $A_i = \bigvee_{j\in J_i} B_{ij}$ and $B_{ij}^{-} \leqslant A_i$. Let $\mathcal{B} = \{B_{ij} \mid i \in I, j \in J_i\}$, then $\mathcal{B}$ is an open β_a-cover of G. By near S^*-compactness of G, we know that $\mathcal{B}$ has a finite subfamily $\mathcal{C}$ such that $\mathcal{C}^{-\circ} = \{C^{-\circ} \mid C \in \mathcal{C}\}$ is a Q_a-cover of G. Suppose $\mathcal{C} = \{B_{ij} \mid i \in I_0, j \in J_{i0}\}$, where I_0 and J_{i0} are finite subfamilies of I and J_i, respectively. Obviously, $\bigvee_{i\in I_0}\bigvee_{j\in J_{i0}} B_{ij}^{-\circ} \leqslant \bigvee_{i\in I_0}\bigvee_{j\in J_{i0}} B_{ij}^{-} \leqslant \bigvee_{i\in I_0} A_i$, hence $\{A_i \mid i \in I_0\}$ is a finite Q_a-cover of G. Therefore G is S^*-compact. □

Theorem 3.6. *If both G and H are nearly S^*-compact, then $G \vee H$ is nearly S^*-compact.*

Proof. For any $a \in M(L)$, suppose that $\mathcal{U}$ is an open β_a-cover of $G \vee H$, then by

$$(G \vee H)'(x) \vee \bigvee_{A\in\mathcal{U}} A(x) = \left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \wedge \left(H'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right), \tag{3.1}$$

we obtain that for any $x \in X$, $a \in \beta(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x))$, and $a \in \beta(H'(x) \vee \bigvee_{A\in\mathcal{U}} A(x))$. This shows that $\mathcal{U}$ is an open β_a-cover of G and H. By near S^*-compactness of G and H, we know that $\mathcal{U}$ has finite subfamilies $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mathcal{V}_1^{-\circ}$ is a Q_a-cover of G and $\mathcal{V}_2^{-\circ}$ is a Q_a-cover of H. Hence for any $x \in X$, $a \leqslant G'(x) \vee \bigvee_{A\in\mathcal{V}_1} A^{-\circ}(x)$ and $a \leqslant H'(x) \vee \bigvee_{A\in\mathcal{V}_2} A^{-\circ}(x)$. Take $\mathcal{W} = \mathcal{V}_1 \cup \mathcal{V}_2$, then $\mathcal{W}$ is a finite subfamily of $\mathcal{U}$ and it satisfies the condition $a \leqslant G'(x) \vee \bigvee_{A\in\mathcal{W}} A^{-\circ}(x)$ and $a \leqslant H'(x) \vee \bigvee_{A\in\mathcal{W}} A^{-\circ}(x)$, hence $a \leqslant (G \vee H)'(x) \vee \bigvee_{A\in\mathcal{W}} A^{-\circ}(x)$. Therefore $G \vee H$ is nearly S^*-compact. □

Theorem 3.7. *Let $f : (X,\mathcal{T}_1) \to (Y,\mathcal{T}_2)$ be almost continuous. If G is S^*-compact in $(X,\mathcal{T}_1)$, then $f_L^{\rightarrow}(G)$ is nearly S^*-compact in $(Y,\mathcal{T}_2)$.*

Proof. For any $a \in M(L)$, suppose that $\mathcal{U} \subseteq \mathcal{T}_2$ is an open β_a-cover of $f_L^{\rightarrow}(G)$. Then $\mathcal{U}^{-\circ} = \{A^{-\circ} \mid A \in \mathcal{U}\}$ is a regularly open β_a-cover of $f_L^{\rightarrow}(G)$. For any $y \in Y$, we have that $a \in \beta(f_L^{\rightarrow}(G)'(y) \vee \bigvee_{A\in\mathcal{U}} A^{-\circ}(y))$. Since f is almost continuous, by the following equation:

$$f_L^{\rightarrow}(G)'(y) \vee \bigvee_{A\in\mathcal{U}} A^{-\circ}(y) = \bigwedge_{x\in f^{-1}(y)} \left(G'(x) \vee \bigvee_{A\in\mathcal{U}} f_L^{\leftarrow}(A^{-\circ})(x) \right), \tag{3.2}$$

we know that $f_L^{\leftarrow}(\mathcal{U}^{-\circ}) = \{f_L^{\leftarrow}(A^{-\circ}) \mid A \in \mathcal{U}\}$ is an open β_a-cover of G. By S^*-compactness of G, $\mathcal{U}$ has a finite subfamily $\mathcal{V}$ such that $f_L^{\leftarrow}(\mathcal{V}^{-\circ})$ is an open Q_a-cover of G. Hence for any $y \in Y$, $a \leqslant f_L^{\rightarrow}(G)'(y) \vee \bigvee_{A\in\mathcal{V}} A^{-\circ}(y)$. This shows that $\mathcal{V}^{-\circ}$ is an open Q_a-cover of $f_L^{\rightarrow}(G)$. Therefore $f_L^{\rightarrow}(G)$ is nearly S^*-compact. □

Similarly, we can obtain the following theorems.

Theorem 3.8. *Let $f : (X,\mathcal{T}_1) \to (Y,\mathcal{T}_2)$ be completely continuous. If G is nearly S^*-compact in $(X,\mathcal{T}_1)$, then $f_L^{\rightarrow}(G)$ is S^*-compact in $(Y,\mathcal{T}_2)$.*

Theorem 3.9. *Let $f : (X,\mathcal{T}_1) \to (Y,\mathcal{T}_2)$ be R-irresolute. If G is nearly S^*-compact in $(X,\mathcal{T}_1)$, then so is $f_L^{\rightarrow}(G)$ in $(Y,\mathcal{T}_2)$.*

The following theorem shows that near S^*-compactness is a good extension of near compactness in general topology.

Theorem 3.10. *If $(X,\mathcal{T})$ is a weakly induced L-space, then $(X,\mathcal{T})$ is nearly S^*-compact if and only if $(X,[\mathcal{T}])$ is nearly compact.*

Proof. Let $(X,[\mathcal{T}])$ be nearly compact. For $a \in M(L)$, let $\mathcal{U}$ be an open β_a-cover of $\underline{1}$ in $(X,\mathcal{T})$. By Lemma 2.3, $\{A_{(a)} \mid A \in \mathcal{U}\}$ is an open cover of $(X,[\mathcal{T}])$. By near compactness of $(X,[\mathcal{T}])$, we know that there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $(\mathcal{V}_{(a)})^{-\circ} = \{(A_{(a)})^{-\circ} \mid A \in \mathcal{V}\}$ is a cover of $(X,[\mathcal{T}])$. For any $A \in \mathcal{V}$, by $(A_{(a)})^{-\circ} \subseteq (A_{[a]})^{-\circ} \subseteq (A^{-\circ})_{[a]}$ we know that $\mathcal{V}^{-\circ}$ is a Q_a-cover of $\underline{1}$ in $(X,\mathcal{T})$. This shows that $(X,\mathcal{T})$ is nearly S^*-compact.

Conversely, let $(X,\mathcal{T})$ be nearly S^*-compact and $\mathcal{W}$ be an open cover of $(X,[\mathcal{T}])$. Then for each $a \in \beta^*(1)$, $\{\chi_A \mid A \in \mathcal{W}\}$ is an open β_a-cover of $\underline{1}$ in $(X,\mathcal{T})$ since $(\chi_A)^\circ = \chi_{A^\circ} = \chi_A$ for any $A \in \mathcal{W}$. By near S^*-compactness of $(X,\mathcal{T})$, we know that there exists a finite subfamily $\mathcal{V}$ of $\mathcal{W}$ such that $\{(\chi_A)^{-\circ} \mid A \in \mathcal{V}\}$ is a Q_a-cover of $\underline{1}$ in $(X,\mathcal{T})$. Obviously,

$\mathcal{V}^{-\circ}$ is a cover of $(X,[\mathcal{T}])$ since $(\chi_A)^{-\circ} = \chi_{A^{-\circ}}$ for any $A \in \mathcal{V}$. This shows that $(X,[\mathcal{T}])$ is nearly compact. □

As is well known, if $(X,\omega(\tau))$ is generated topologically by the topological space (X,τ), then $(X,\omega(\tau))$ is an induced L-space; naturally, it also is a weakly induced L-space. Hence we obtain the following result.

Corollary 3.11. *Let (X,τ) be a topological space and $(X,\omega(\tau))$ be generated topologically by (X,τ). Then $(X,\omega(\tau))$ is nearly S^*-compact if and only if (X,τ) is nearly compact.*

4. Some other characterizations of near S^*-compactness

In this section, we will show that near S^*-compactness can be characterized by nets.

Definition 4.1. Let $(X,\mathcal{T})$ be an L-space, a regularly open L-set U is called a strong regularly open neighborhood of a fuzzy point x_λ, if $\lambda \in \beta(U(x))$.

Definition 4.2. Let $\{S(n) \mid n \in D\}$ be a net in $(X,\mathcal{T})$, $x_\lambda \in M(L^X)$, x_λ is called a weak O_R-cluster point of S, if for each strong regularly open neighborhood U of x_λ, S is frequently in U; x_λ is called a weak O_R-limit point of S, if for each strong regularly open neighborhood U of x_λ, S is eventually in U, in this case, we also say that S weakly O_R-converges to x_λ, denoted by $S \xrightarrow{WO_R} x_\lambda$.

From [10], we know that S weakly O-converges to x_λ implies that S weakly O_R-converges to x_λ, and x_λ is a weak O-cluster point of S implies that x_λ is a weak O_R-cluster point of S.

Theorem 4.3. *An L-set G is nearly S^*-compact in $(X,\mathcal{T})$ if and only if for all $a \in M(L)$, each constant a-net quasicoinciding with G has a weak O_R-cluster point $x_a \notin \beta(G')$.*

Proof. Suppose that G is nearly S^*-compact. For $a \in M(L)$, let $\{S(n) \mid n \in D\}$ be a constant a-net quasicoinciding with G. Suppose that S has no weak O_R-cluster point $x_a \notin \beta(G')$, then for each $x_a \notin \beta(G')$, there exists a strong regularly open neighborhood U_x of x_a and $n_x \in D$ such that for all $n \geqslant n_x$, $S(n) \nleqslant U_x$. Take $\Phi = \{U_x \mid x_a \notin \beta\,(G')\}$, then Φ is a regularly open β_a-cover of G. Since G is nearly S^*-compact, Φ has a finite subfamily $\Psi = \{U_{x^i} \mid i = 1,2,\ldots,k\}$ such that Ψ is an open Q_a-cover of G. Since D is a directed set, there exists $n_0 \in D$ such that $n_0 \geqslant n_{x^i}$ for each $i \leqslant k$. Thus we obtain that for all $n \geqslant n_0$, $S(n) \nleqslant \bigvee\{U_{x^i} \mid i = 1,2,\ldots,k\}$. This contradicts Ψ being an open Q_a-cover of G. Therefore S has a weak O_R-cluster point $x_a \notin \beta(G')$.

Conversely, suppose that for each $a \in M(L)$, each constant a-net quasicoinciding with G has a weak O_R-cluster point $x_a \notin \beta(G')$. We now prove that G is nearly S^*-compact. Let Φ be a regularly open β_a-cover of G. If each finite subfamily Ψ of Φ is not an open Q_a-cover of G, then for each finite subfamily Ψ of Φ, there exists $S(\Psi) \in M(L^X)$ with height a such that $S(\Psi) \nleqslant G'$ and $S(\Psi) \nleqslant \bigvee\Psi$. Take $S = \{S(\Psi) \mid \Psi$ is a finite subfamily of $\Phi\}$, then S is a constant a-net quasicoinciding with G. Suppose that S has a weak O_R-cluster point $x_a \notin \beta(G')$. Then for each finite subfamily Ψ of Φ, we have that $x_a \notin \beta(\bigvee\Psi)$ (because if $x_a \in \beta(\bigvee\Psi)$, so there exists an $A \in \Psi$ such that $x_a \in \beta(A)$, that is, A is a strong regularly open neighborhood of x_a. Hence there exists a finite subfamily Ψ_0 of Φ such that for

all $\Psi \subseteq \Psi_0$ it follows that $S(\Psi_0) \leqslant A \leqslant \bigvee \Psi \leqslant \bigvee \Psi_0$. This contradicts the definition of S), in particular $x_a \notin \beta(B)$ for each $B \in \Phi$. But since Φ is a regularly open β_a-cover of G, we know that there exists $B \in \Phi$ such that $x_a \in \beta(B)$. This yields a contradiction with $x_a \notin \beta(B)$. So G is nearly S^*-compact. □

Theorem 4.4. *An L-set G is nearly S^*-compact in $(X, \mathcal{T})$ if and only if for all $a \in M(L)$, each a^--net quasicoinciding with G has a weak O_R-cluster point $x_a \notin \beta(G')$.*

The proof is omitted.

Definition 4.5. Let $A \in L^X$. The α-closure of A is defined to be

$$\mathrm{cl}_\alpha(A) = \bigwedge \{B \mid A \leqslant B \text{ and } B \text{ is } \alpha\text{-closed}\}. \tag{4.1}$$

The α-interior of A is defined to be $\mathrm{cl}_\alpha(A')'$, written as $\mathrm{int}_\alpha(A)$.

Lemma 4.6. *If A is a semiopen L-set, then* $\mathrm{cl}_\alpha(A) = A^-$. *If A is a semiclosed L-set, then* $\mathrm{int}_\alpha(A) = A^\circ$.

Proof. Obviously, $\mathrm{cl}_\alpha(A) \leqslant A^-$. In order to prove that $A^- \leqslant \mathrm{cl}_\alpha(A)$, suppose that $x_a \nleqslant \mathrm{cl}_\alpha(A)$. There exists an α-closed set B such that $A \leqslant B$ and $x_a \nleqslant B$. Hence $A^- \leqslant A^{\circ -} \leqslant B^{\circ -} \leqslant B^{-\circ -} \leqslant B$ since A is semiopen and B is α-closed. And so $x_a \nleqslant A^-$, which implies that $A^- \leqslant \mathrm{cl}_\alpha(A)$. Therefore $\mathrm{cl}_\alpha(A) = A^-$. Similarly, we can prove the other result. □

Theorem 4.7. *An L-set G is nearly S^*-compact in $(X, \mathcal{T})$ if and only if for all $a \in M(L)$, each α-open β_a-cover $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ such that* $\mathrm{int}_\alpha(\mathrm{cl}_\alpha(\mathcal{V}))$ *is a Q_a-cover of G.*

Proof. ($\Rightarrow$) Suppose that G is nearly S^*-compact. For any $a \in M(L)$, let $\mathcal{U}$ be an α-open β_a-cover of G. Let $\mathcal{W} = \{A^{\circ -\circ} \mid A \in \mathcal{U}\}$, then $\mathcal{W}$ is an open β_a-cover of G. By near S^*-compactness of G, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\{A^{\circ -\circ -\circ} \mid A \in \mathcal{V}\}$ is a Q_a-cover of G. Since $A^{\circ -\circ -\circ} = A^{-\circ} = \mathrm{int}_\alpha(\mathrm{cl}_\alpha(A))$, $\mathrm{int}_\alpha(\mathrm{cl}_\alpha(\mathcal{V})) = \{\mathrm{int}_\alpha(\mathrm{cl}_\alpha(A)) \mid A \in \mathcal{V}\}$ is also a Q_a-cover of G.

($\Leftarrow$) For any $a \in M(L)$, let $\mathcal{U}$ be an open β_a-cover of G. Then $\mathcal{U}$ is also an α-open β_a-cover of G. By the hypothesis, there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{int}_\alpha(\mathrm{cl}_\alpha(\mathcal{V}))$ is a Q_a-cover of G. Since $\mathrm{int}_\alpha(\mathrm{cl}_\alpha(A)) = A^{-\circ}$ for any $A \in \mathcal{V}$, G is nearly S^*-compact. □

5. The relationships between different notions of near compactness

In this section, we will investigate some relationships between different notions of near compactness. Firstly, we recall some other notions of near compactness.

Definition 5.1 [16]. Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is called nearly compact if for every family $\mathcal{U} \subseteq \mathcal{T}$, it follows that

$$\bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \right) \leqslant \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X} \left(G'(x) \vee \bigvee_{A \in \mathcal{V}} A^{-\circ}(x) \right). \tag{5.1}$$

Lemma 5.2 [16]. *Let $(X,\mathcal{T})$ be an L-space and $G \in L^X$. Then G is nearly compact if and only if for any $a \in M(L)$ and any $b \in \beta^*(a)$, each open Q_a-cover of G has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^{-\circ}$ is a Q_b-cover of G.*

Definition 5.3 [29]. Let $(X,\mathcal{T})$ be an L-space and $G \in L^X$. Then G is called nearly N-compact if for any $a \in M(L)$, each a-RF Φ of G has a finite subfamily Ψ such that $\Psi^{\circ-} = \{P^{\circ-} \mid P \in \Psi\}$ is an a^--RF of G. $(X,\mathcal{T})$ is said to be nearly N-compact if $\underline{1}$ is nearly N-compact.

Definition 5.4 [12]. Let $(X,\mathcal{T})$ be an L-space and $G \in L^X$. Then G is called nearly strongly compact if for each $a \in P(L)$, each open a-shading $\mathcal{U}$ of G has a finite subfamily $\mathcal{V}$ such that $\mathcal{V}^{-\circ}$ is an a-shading of G. $(X,\mathcal{T})$ is said to be nearly strongly compact if $\underline{1}$ is nearly strongly compact.

Theorem 5.5. *Near S^*-compactness implies near compactness.*

Proof. Let G be nearly S^*-compact. For each $a \in M(L)$, suppose that Φ is an open Q_a-cover of G. Then $a \leqslant G'(x) \vee \bigvee_{A\in\Phi} A(x)$ for any $x \in X$. Thus for all $b \in \beta^*(a)$, Φ is an open β_b-cover of G. By near S^*-compactness of G we know that Φ has a finite subfamily Ψ such that $\Psi^{-\circ}$ is Q_b-cover of G. Therefore G is nearly compact by Lemma 5.2. □

But near compactness need not imply near S^*-compactness in general. This can be seen in the following example.

Example 5.6. Let $L = [0,1]$, $X = \{2,3,4,\dots\}$, and let $\mathcal{T}$ be an L-topology generated by $\Phi = \{A_n, B_n \mid n \in X\}$, where

$$A_n(x) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{n}, & x = n, \\ 0, & x \neq n, \end{cases} \qquad B_n(x) = \begin{cases} \dfrac{1}{2} - \dfrac{1}{n}, & x = n, \\ 0, & x \neq n. \end{cases} \tag{5.2}$$

By

$$A'_n(x) = 1 - A_n(x) = \begin{cases} \dfrac{1}{2} - \dfrac{1}{n}, & x = n, \\ 1, & x \neq n, \end{cases} \qquad B'_n(x) = 1 - B_n(x) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{n}, & x = n, \\ 1, & x \neq n, \end{cases} \tag{5.3}$$

we obtain

$$A_n^{-\circ}(x) = \begin{cases} \dfrac{1}{2} + \dfrac{1}{n}, & x = n, \\ \dfrac{1}{2} - \dfrac{1}{x}, & x \neq n, \end{cases} \qquad B_n^{-\circ}(x) = \frac{1}{2} - \frac{1}{x}. \tag{5.4}$$

Obviously, for any $a \in (0.5,1]$, none of all subfamilies of Φ is an open Q_a-cover of $\underline{1}$. Thus we only need to consider $a \in (0,0.5]$. Suppose that $\mathcal{U}$ is an open Q_a-cover of $\underline{1}$. For each $b \in (0,a)$, we can take $A_m \leqslant U \in \mathcal{U}$ or $B_n \leqslant U \in \mathcal{U}$. Then $b \leqslant A_m^{-\circ}(x) \leqslant U^{-\circ}(x)$ or $b \leqslant B_n^{-\circ}(x) \leqslant U^{-\circ}(x)$ when $x \geqslant l = 1/(0.5 - b)$ and $x \in X$. Let $I = \{x \mid x \in X \text{ and } x < l\}$, then I is finite. For each $x \in I$, there exists $U_x \in \mathcal{U}$ such that $b < U_x(x)$. Let $\mathcal{C} = \{U_x, x \in I\} \bigcup \{U\}$. Then $\mathcal{C}$ is a finite subfamily of $\mathcal{U}$ and $\mathcal{C}^{-\circ}$ is a Q_b-cover of $\underline{1}$. Therefore $(X,\mathcal{T})$ is nearly compact.

At the same time, obviously $\mathcal{U} = \{A_n\}_{n\in X}$ is an open $\beta_{0.5}$-cover of $\underline{1}$, but $\mathcal{U}$ has no finite subfamily $\mathcal{V}$ such that $\mathcal{V}^{-\circ}$ is a $Q_{0.5}$-cover of $\underline{1}$. Hence $(X,\mathcal{T})$ is not nearly S^*-compact.

The following lemma is obvious.

Lemma 5.7. *Let $(X,\mathcal{T})$ be an L-space and $G \in L^X$, $\Omega \subseteq \mathcal{T}'$. Then*
(1) *Ω is a-RF of G if and only if $a \nleqslant G(x) \wedge \bigwedge_{A\in\Omega} A(x)$ for any $x \in X$;*
(2) *Ω is a^--RF of G if and only if $a \nleqslant \bigvee_{x\in X}(G(x) \wedge \bigwedge_{A\in\Omega} A(x))$.*

Theorem 5.8. *Near N-compactness implies near strong compactness.*

Proof. Suppose that G is near N-compact. For any $r \in P(L)$, let $\mathcal{U}$ be an open a-shading of G. Then $\mathcal{U}'$ is an r'-RF of G. By near N-compactness of G, we know that there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $r' \nleqslant \bigvee_{x\in X}(G(x) \wedge \bigwedge_{A\in\mathcal{V}} A'^{\circ-}(x))$. Since

$$\begin{aligned} r' \nleqslant \bigvee_{x\in X}\left(G(x) \wedge \bigwedge_{A\in\mathcal{V}} A'^{\circ-}(x)\right) &\Longleftrightarrow \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A'^{\circ-\prime}(x)\right) \nleqslant r \\ &\Longleftrightarrow \bigwedge_{x\in X}\left(G'(x) \vee \bigvee_{A\in\mathcal{V}} A^{-\circ}(x)\right) \nleqslant r, \end{aligned} \tag{5.5}$$

for any $x \in X$, $G'(x) \vee \bigvee_{A\in\mathcal{V}} A^{-\circ}(x) \nleqslant r$, that is, $\mathcal{V}^{-\circ}$ is an a-shading of G. Therefore G is nearly strongly compact. □

But near strong compactness need not imply near N-compactness. This can be seen from the following example.

Example 5.9. Let $X = (0,1)$, $\mathcal{T}$ an L-topology generated by A, B, and all constant L-sets, where $A(x) = x$, $B(x) = 1 - x$. It is obvious that $A^{-\circ} = A$, $B^{-\circ} = B$.

For $a \in [0,1)$, suppose that $\mathcal{U}$ is an open a-shading of $\underline{1}$.

(1) If $a \geqslant 0.5$, take $x = 0.5$, then $A(x) = B(x) = 0.5$. In this case, there exists $U \in \mathcal{U}$ such that $U(x) > a \geqslant 0.5$. This implies that there exists a constant fuzzy set $\underline{s} \leqslant U$ such that $s > a$. Therefore $\{U^{-\circ}\}$ is an a-shading of $\underline{1}$.

(2) If $a < 0.5$, then from the structure of $\mathcal{T}$, we know that there exists a subfamily $\mathcal{B}$ of $\{\underline{r}, \underline{r} \wedge A, \underline{r} \wedge B, \underline{r} \wedge A \wedge B \mid r \in [0,1]\}$ such that $\mathcal{B}$ is a refinement of $\mathcal{U}$ and $\mathcal{B}$ is a-shading of $\underline{1}$. Obviously, $\mathcal{B}$ has a finite subfamily $\mathcal{D}$ which is an a-shading of $\underline{1}$, hence $\mathcal{U}$ has a finite subfamily which is an a-shading of $\underline{1}$.

This shows that $(X,\mathcal{T})$ is nearly strongly compact.

Take $\mathcal{U} = \{A\}$. Then $\mathcal{U}$ is a 1-RF of $\underline{1}$. But there is no $t < 1$ such that $t \nleqslant A(x) = A^{\circ-}(x)$ for all $x \in X$. So $(X,\mathcal{T})$ is not nearly N-compact.

Theorem 5.10. *When $L = [0,1]$, near strong compactness implies near S^*-compactness.*

Proof. Suppose that G is nearly strongly compact and $\mathcal{U}$ is an open β_a-cover of G. Then $\mathcal{U}$ is an a-shading of G since

$$a \in \beta\left(G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x)\right) \Longleftrightarrow a < G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x) \Longleftrightarrow G'(x) \vee \bigvee_{A\in\mathcal{U}} A(x) \nleqslant a. \tag{5.6}$$

By near strong compactness of G, we know that there exists a finite subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}^{-\circ} = \{A^{-\circ} \mid A \in \mathcal{V}\}$ is an a-shading of G. Obviously $\mathcal{V}^{-\circ}$ is a Q_a-cover of G. Therefore G is nearly S^*-compact. □

Remark 5.11. When $L \neq [0,1]$, does near strong compactness imply near S^*-compactness? We leave it as an open question.

In general, near S^*-compactness need not imply near strong compactness. This can be seen from the following example.

Example 5.12. Let $L = [0,1]$, $X = \{2,3,4,\dots\}$ and $\mathcal{T}$ be an L-topology generated by $\{A_n, B_n, C_n \mid n \in X\}$, where

$$A_n(x) = \begin{cases} \frac{1}{2} - \frac{1}{n}, & x = n, \\ 0, & x \neq n, \end{cases} \qquad B_n(x) = \begin{cases} \frac{1}{2} + \frac{1}{n}, & x = n, \\ \frac{1}{2}, & x \neq n, \end{cases} \qquad C_n(x) = \begin{cases} \frac{1}{2}, & x = n, \\ 0, & x \neq n. \end{cases} \tag{5.7}$$

It is obvious that when $m \neq n$, we have that

$$\begin{gathered} A_n \wedge A_m = C_n \wedge C_m = A_n \wedge C_m = \underline{0}, \qquad B_n \wedge B_m = \underline{\frac{1}{2}}, \qquad A_n \wedge B_m = A_n, \\ C_n \wedge B_m = C_n, \qquad A_n \wedge \underline{\frac{1}{2}} = A_n, \qquad B_n \wedge \underline{\frac{1}{2}} = \underline{\frac{1}{2}}, \qquad C_n \wedge \underline{\frac{1}{2}} = C_n. \end{gathered} \tag{5.8}$$

Thus $\{A_n, B_n, C_n \mid n = 2,3,4,\dots\} \cup \{1/\underline{2}\}$ is a base of $(X,\mathcal{T})$. By

$$A'_n(x) = \begin{cases} \frac{1}{2} + \frac{1}{n}, & x = n, \\ 1, & x \neq n, \end{cases} \qquad B'_n(x) = \begin{cases} \frac{1}{2} - \frac{1}{n}, & x = n, \\ \frac{1}{2}, & x \neq n, \end{cases} \qquad C'_n(x) = \begin{cases} \frac{1}{2}, & x = n, \\ 1, & x \neq n, \end{cases} \tag{5.9}$$

we have that

$$A_n^{-\circ}(x) = \frac{1}{2} - \frac{1}{x}, \qquad B_n^{-\circ}(x) = B_n(x), \qquad \left(\underline{\frac{1}{2}}\right)^{-\circ} = \underline{\frac{1}{2}}, \qquad C_n^{-\circ}(x) = \begin{cases} \frac{1}{2}, & x = n, \\ \frac{1}{2} - \frac{1}{x}, & x \neq n. \end{cases} \tag{5.10}$$

Obviously, for any $a \in (0.5,1]$, none of all subfamily of Φ is an open β_a-cover of $\underline{1}$. Thus we only need to consider $a \in (0,0.5]$. Suppose that $\mathcal{U}$ is an open β_a-cover of $\underline{1}$. If we can take $B_k \leqslant U \in \mathcal{U}$ or $1/\underline{2} \leqslant U \in \mathcal{U}$, then $\{U^{-\circ}\}$ is an open Q_a-cover of $\underline{1}$. Otherwise $a < 0.5$. We can take $A_m \leqslant U \in \mathcal{U}$ or $C_n \leqslant U \in \mathcal{U}$. Then $a \leqslant A_m^{-\circ}(x) \leqslant U^{-\circ}(x)$ or $a \leqslant C_n^{-\circ}(x) \leqslant U^{-\circ}(x)$ when $x \geqslant l = 1/(0.5 - a)$ and $x \in X$. Let $I = \{x \mid x \in X \text{ and } x < l\}$, then I is finite. For each $x \in I$, there exists $U_x \in \mathcal{U}$ such that $a < U_x(x)$. Let $\mathcal{C} = \{U_x, x \in I\} \bigcup \{U\}$, then $\mathcal{C}$ is a finite subfamily of $\mathcal{U}$ and $\mathcal{C}^{-\circ}$ is a Q_a-cover of $\underline{1}$. Hence $(X,\mathcal{T})$ is nearly S^*-compact.

Take $\mathcal{U} = \{B_n\}_{n\in X}$, a 0.5-shading of $\underline{1}$. For any finite subfamily $\mathcal{V}$ of $\mathcal{U}$, there exists $x \in X$ such that $\bigvee_{A\in\mathcal{V}} A^{-\circ}(x) = 0.5$. So $(X, \mathcal{T})$ is not nearly strongly compact.

Corollary 5.13. *When $L = [0,1]$, near N-compactness implies near S^*-compactness.*

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