Research Article

Uniqueness of Transcendental Meromorphic Functions with Their Nonlinear Differential Polynomials Sharing the Small Function

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We deal with some uniqueness theorems of two transcendental meromorphic functions with their nonlinear differential polynomials sharing a small function. These results in this paper improve those given by C.-Y. Fang and M.-L. Fang (2002), by Lahiri and Pal (2006), and by Lin and Yi (2004).

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1. Introduction and main results

In this paper, we use the standard notations and terms in the value distribution theory [4]. For any nonconstant meromorphic function $f(z)$ on the complex plane $\mathbb{C}$, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ except possibly for a set of r of finite linear measures. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r, a) = S(r, f)$. Let $S(f)$ be the set of meromorphic functions in the complex plane $\mathbb{C}$ which are small functions with respect to f . Set $E(a(z), f) = \{z \mid f(z) - a(z) = 0\}$, $a(z) \in S(f)$, where a zero point with multiplicity m is counted m times in the set. If these zero points are only counted once, then we denote the set by $\bar{E}(a(z), f)$. Let k be a positive integer. Set $E_k(a(z), f) = \{z : f(z) - a(z) = 0, \exists i, 1 \leq i \leq k, \text{ such that } f^{(i)}(z) - a^{(i)}(z) \neq 0\}$, where a zero point with multiplicity m is counted m times in the set.

Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, $a(z) \in S(f) \cap S(g)$. If $E(a(z), f) = E(a(z), g)$, then we say that $f(z)$ and $g(z)$ share the function $a(z)$ CM, especially, we say that $f(z)$ and $g(z)$ have the same fixed points when $a(z) = z$. If $\bar{E}(a(z), f) = \bar{E}(a(z), g)$, then we say that $f(z)$ and $g(z)$ share the function $a(z)$ IM. If $E_k(a(z), f) = E_k(a(z), g)$, we say that $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the multiplicities $\leq k$.

In addition, we also use the following notations.

We denote by $N_k(r, f)$ the counting function for poles of $f(z)$ with multiplicity $\leq k$, and by $\overline{N}_k(r, f)$ the corresponding one for which multiplicity is not counted. Let $\overline{N}_{(k)}(r, f)$ be the counting function for poles of $f(z)$ with multiplicity $\geq k$, and let $\overline{N}_{(k)}(r, f)$ be the corresponding one for which multiplicity is not counted. Set $N_k(r, f) = \overline{N}(r, f) + \overline{N}_{(2)}(r, f) + \cdots + \overline{N}_{(k)}(r, f)$.

Similarly, we have the notations

$$N_k\left(r, \frac{1}{f}\right), \overline{N}_k\left(r, \frac{1}{f}\right), N_{(k)}\left(r, \frac{1}{f}\right), \overline{N}_{(k)}\left(r, \frac{1}{f}\right), N_k\left(r, \frac{1}{f}\right). \quad (1.1)$$

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and $\overline{E}(1, f) = \overline{E}(1, g)$. We denote by $\overline{N}_L(r, 1/(f - 1))$ the counting function for 1-points of both $f(z)$ and $g(z)$ about which $f(z)$ has larger multiplicity than $g(z)$, with multiplicity not being counted, and denote by $N_{11}(r, 1/(f - 1))$ the counting function for common simple 1-points of both $f(z)$ and $g(z)$ where multiplicity is not counted. Similarly, we have the notation $\overline{N}_L(r, 1/(g - 1))$.

In 1929, Nevanlinna proved the following well-known result, which is the so-called Nevanlinna four-value theorem.

THEOREM 1.1 [5]. *Let f and g be two nonconstant meromorphic functions. If f and g share four distinct values CM, then f is a Möbius transformation of g .*

In 1979, G. G. Gundersen proved the following result, which is an improvement of Theorem 1.1.

THEOREM 1.2 [6]. *Let f and g be two nonconstant meromorphic functions. If f and g share three distinct values CM and a fourth value IM, then f is a Möbius transformation of g .*

In 1997, Li and Yang proved the following two results, which generalize Theorems 1.1 and 1.2 to small functions.

THEOREM 1.3 [7]. *Let f and g be two nonconstant meromorphic functions, and let a_j ($j = 1, \dots, 4$) be distinct small functions of f and g . If f and g share a_j ($j = 1, \dots, 4$) CM^* , then f is a quasi-Möbius transformation of g .*

THEOREM 1.4 [7]. *Let f and g be two nonconstant meromorphic functions, and let a_j ($j = 1, \dots, 4$) be distinct small functions of f and g . If f and g share a_j ($j = 1, \dots, 3$) CM^* and $a_4(z)$ IM, then f is a quasi-Möbius transformation of g .*

Recently, some papers studied the uniqueness of meromorphic functions and differential polynomials, and obtained some results as follows.

In 2002, C.-Y Fang and M.-L. Fang [1] proved the following result.

THEOREM 1.5 [1]. *Let f and g be two nonconstant meromorphic functions and let n (≥ 13) be an integer. If $f^n(f - 1)^2 f' = g^n(g - 1)^2 g'$ share the value 1 CM, then $f \equiv g$.*

In 2006, Lahiri and Pal [2] proved the following results, the first of which improves Theorem 1.5.

THEOREM 1.6 [2]. Let f and g be two nonconstant meromorphic functions and let $n(\geq 13)$ be an integer. If $E_3(1, f^n(f-1)^2 f') = E_3(1, g^n(g-1)^2 g')$, then $f \equiv g$.

Fang and Qiu [8] proved the following results.

THEOREM 1.7 [8]. Let f and g be two nonconstant meromorphic (entire) functions, $n \geq 11$ ($n \geq 6$) is a positive integer. If $f^n f'$ and $g^n g'$ share $z\text{CM}$, then either $f = c_1 e^{cz^2}$, $g = c_2 e^{-cz^2}$, where c_1, c_2 , and c are three constants satisfying $4(c_1 c_2)^{n+1} c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.

Lin and Yi [3] proved the following results.

THEOREM 1.8 [3]. Let f and g be two transcendental meromorphic functions, $n \geq 13$ is an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share $z\text{CM}$, then $f(z) \equiv g(z)$.

Question 1.9. Is it possible that the value 1 can be replaced by a small function $a(z)$ in Theorems 1.5 and 1.6?

Question 1.10. Is it possible to relax the nature of sharing z in Theorem 1.8 and if possible, how far?

The purpose of this paper is to answer the above questions, and we get the following results.

THEOREM 1.11. Let f and g be two transcendental meromorphic functions and let $n \geq 13$, $k \geq 3$ be two positive integers. If $E_k(z, f^n(f-1)^2 f') = E_k(z, g^n(g-1)^2 g')$, then $f \equiv g$.

THEOREM 1.12. Let f and g be two transcendental meromorphic functions and let $n \geq 15$ be a positive integer. If $E_2(z, f^n(f-1)^2 f') = E_2(z, g^n(g-1)^2 g')$, then $f \equiv g$.

THEOREM 1.13. Let f and g be two transcendental meromorphic functions and let $n \geq 23$ be a positive integer. If $E_1(z, f^n(f-1)^2 f') = E_1(z, g^n(g-1)^2 g')$, then $f \equiv g$.

THEOREM 1.14. Let f and g be two transcendental meromorphic functions and $n \geq 28$ be a positive integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share $z\text{IM}$, then $f \equiv g$.

2. Some lemmas

In order to prove our results, we need the following lemmas.

LEMMA 2.1 [9]. Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \dots, a_n$ are constants and $a_n \neq 0$. Then

$$T(r, P(f)) = nT(r, f) + S(r, f). \quad (2.1)$$

LEMMA 2.2 [10]. Let f and g be two meromorphic functions, and let k be a positive integer, then

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f). \quad (2.2)$$

LEMMA 2.3 [11]. Let

$$Q(w) = (n-1)^2(w^n - 1)(w^{n-2} - 1) - n(n-2)(w^{n-1} - 1)^2, \quad (2.3)$$

then

$$Q(w) = (w-1)^4(w-\beta_1)(w-\beta_2)\cdots(w-\beta_{2n-6}), \quad (2.4)$$

where $\beta_j \in C \setminus \{0, 1\}$ ($j = 1, 2, \dots, 2n-6$), which are distinct, respectively.

LEMMA 2.4. Let f and g be two transcendental meromorphic functions. Then $f^n(f-1)^2f'g^n(g-1)^2g' \not\equiv z^2$, where $n \geq 8$ is a positive integer.

Proof. If possible, let $f^n(f-1)^2f'g^n(g-1)^2g' \equiv z^2$. Let $z_0 (\neq 0, \infty)$ be a 1-point of f with multiplicity $p (\geq 1)$. Then z_0 is a pole of g with multiplicity $q (\geq 1)$ such that $2p + p - 1 = (n+2)q + q + 1$, and so $p \geq (n+5)/3$.

Let $z_1 (\neq 0, \infty)$ be a zero of f with multiplicity $p (\geq 1)$ and let it be a pole of g with multiplicity $q (\geq 1)$. Then $np + p - 1 = (n+3)q + 1$, that is, $2q = (n+1)(p-q) - 2 \geq n-1$, that is, $q \geq (n-1)/2$. So $(n+1)p = (n+3)q + 2$, that is, $p \geq (n+1)/2$.

Since a pole of f is either a zero of $g(g-1)$ or a zero of g' , we get

$$\begin{aligned} \overline{N}(r, f) &\leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) + \overline{N}_0\left(r, \frac{1}{g'}\right) \\ &\leq \frac{2}{n+1}N\left(r, \frac{1}{g}\right) + \frac{3}{n+5}N\left(r, \frac{1}{g-1}\right) + \overline{N}_0\left(r, \frac{1}{g'}\right) \\ &\leq \left(\frac{2}{n+1} + \frac{3}{n+5}\right)T(r, g) + \overline{N}_0\left(r, \frac{1}{g'}\right), \end{aligned} \quad (2.5)$$

where $\overline{N}_0(r, 1/g')$ is the reduced counting function of those zeros of g' which are not the zeros of $g(g-1)$.

By the second fundamental theorem, we obtain

$$\begin{aligned} T(r, f) &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f-1}\right) - \overline{N}_0\left(r, \frac{1}{f'}\right) + S(r, f) \\ &\leq \frac{2}{n+1}N\left(r, \frac{1}{f}\right) + \frac{3}{n+5}N\left(r, \frac{1}{f-1}\right) + \left(\frac{2}{n+1} + \frac{3}{n+5}\right)T(r, g) \\ &\quad + \overline{N}_0\left(r, \frac{1}{g'}\right) - \overline{N}_0\left(r, \frac{1}{f'}\right) + 2\log r + S(r, f). \end{aligned} \quad (2.6)$$

So

$$\begin{aligned} & \left(1 - \frac{2}{n+1} - \frac{3}{n+5}\right) T(r, f) \\ & \leq \left(\frac{2}{n+1} + \frac{3}{n+5}\right) T(r, g) + \overline{N}_0\left(r, \frac{1}{g'}\right) - \overline{N}_0\left(r, \frac{1}{f'}\right) + 2\log r + S(r, f). \end{aligned} \quad (2.7)$$

Similarly, we get

$$\begin{aligned} & \left(1 - \frac{2}{n+1} - \frac{3}{n+5}\right) T(r, g) \\ & \leq \left(\frac{2}{n+1} + \frac{3}{n+5}\right) T(r, f) + \overline{N}_0\left(r, \frac{1}{f'}\right) - \overline{N}_0\left(r, \frac{1}{g'}\right) + 2\log r + S(r, g). \end{aligned} \quad (2.8)$$

Adding (2.7) and (2.8) we get

$$\left(1 - \frac{4}{n+1} - \frac{6}{n+5}\right) \{T(r, f) + T(r, g)\} \leq 4\log r + S(r, f) + S(r, g), \quad (2.9)$$

which is a contradiction. This proves this lemma. $\square$

LEMMA 2.5. *Let f and g be two transcendental meromorphic functions, $F = f^n(f-1)^2 f'/z$, and $G = g^n(g-1)^2 g'/z$, where $n(\geq 5)$ is a positive integer. If $F \equiv G$, then $f \equiv g$.*

Proof. If $F \equiv G$, that is,

$$F^* \equiv G^* + c, \quad (2.10)$$

where c is a constant,

$$\begin{aligned} F^* &= \frac{1}{n+3} f^{n+3} - \frac{2}{n+2} f^{n+2} + \frac{1}{n+1} f^{n+1}, \\ G^* &= \frac{1}{n+3} g^{n+3} - \frac{2}{n+2} g^{n+2} + \frac{1}{n+1} g^{n+1}. \end{aligned} \quad (2.11)$$

It follows that

$$T(r, f) = T(r, g) + S(r, f). \quad (2.12)$$

Suppose that $c \neq 0$. By the second fundamental theorem, from (2.10) and (2.12) we have

$$\begin{aligned} (n+3)T(r, g) &= T(r, G^*) < \overline{N}\left(r, \frac{1}{G^*}\right) + \overline{N}\left(r, \frac{1}{G^* + c}\right) + \overline{N}(r, G^*) + S(r, g) \\ &\leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g - \alpha_1}\right) + \overline{N}\left(r, \frac{1}{g - \alpha_2}\right) + \overline{N}(r, g) \\ &\quad + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f - \alpha_1}\right) + \overline{N}\left(r, \frac{1}{f - \alpha_2}\right) + S(r, f), \end{aligned} \quad (2.13)$$

where α_1, α_2 are distinct roots of the algebraic equation

$$\frac{1}{n+3}z^2 - \frac{2}{n+2}z + \frac{1}{n+1} = 0. \quad (2.14)$$

Then we can get

$$(n+3)T(r, g) < 7T(r, f) + S(r, f). \quad (2.15)$$

Since $n \geq 5$, we can get a contradiction. Therefore $F^* \equiv G^*$, that is,

$$f^{n+1} \left(\frac{1}{n+3}f^2 - \frac{2}{n+2}f + \frac{1}{n+1} \right) = g^{n+1} \left(\frac{1}{n+3}g^2 - \frac{2}{n+2}g + \frac{1}{n+1} \right). \quad (2.16)$$

Let $h = f/g$, we substitute $f = hg$ in (2.16), and it follows that

$$(n+2)(n+1)g^2(h^{n+3} - 1) - 2(n+3)(n+1)g(h^{n+2} - 1) + (n+2)(n+3)(h^{n+1} - 1) = 0. \quad (2.17)$$

If h is not constant, using Lemma 2.3 and (2.17), we can conclude that

$$\{(n+1)(n+2)(h^{n+3} - 1)g - (n+1)(n+3)(h^{n+2} - 1)\}^2 = -(n+3)(n+1)Q(h), \quad (2.18)$$

where $Q(h) = (h-1)^4(h-\beta_1)(h-\beta_2) \cdots (h-\beta_{2n})$, $\beta_j \in \setminus \{0, 1\}$ ($j = 1, 2, \dots, 2n$), which are pairwise distinct.

This implies that every zero of $h - \beta_j$ ($j = 1, 2, \dots, 2n$) has a multiplicity of at least 2. By the second fundamental theorem, we obtain that $n \leq 2$, which is again a contradiction. Therefore, h is a constant. We have from (2.17) that $h^{n+1} - 1 = 0$ and $h^{n+2} - 1 = 0$, which imply $h = 1$, and hence $f \equiv g$, so the lemma is proved. $\square$

LEMMA 2.6 [1]. Let f and g be two meromorphic functions, then and let k be a positive integer. If $E_k(1, f) = E_k(1, g)$, one of the following cases must occur:

(i)

$$\begin{aligned} T(r, f) + T(r, g) &\leq \overline{N}_2(r, f) + \overline{N}_2\left(r, \frac{1}{f-1}\right) + \overline{N}_2(r, g) + \overline{N}_2\left(r, \frac{1}{g}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{f-1}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) \\ &\quad - N_{11}\left(r, \frac{1}{f-1}\right) + \overline{N}_{(k+1)}\left(r, \frac{1}{f-1}\right) \\ &\quad + \overline{N}_{(k+1)}\left(r, \frac{1}{g-1}\right) + S(r, f) + S(r, g); \end{aligned} \quad (2.19)$$

(ii) $f = ((b+1)g + (a-b-1))/(bg + (a-b))$, where $a (\neq 0)$, b are two constants.

LEMMA 2.7 [12]. Let f and g be two meromorphic functions. If f and g share 1 IM, then one of the following cases must occur:

(i)

$$\begin{aligned} T(r, f) + T(r, g) &\leq 2 \left[\overline{N}_2(r, f) + \overline{N}_2\left(r, \frac{1}{f}\right) + \overline{N}_2(r, g) + \overline{N}_2\left(r, \frac{1}{g}\right) \right] \\ &\quad + 3\overline{N}_L\left(r, \frac{1}{f-1}\right) + 3\overline{N}_L\left(r, \frac{1}{g-1}\right) \\ &\quad + S(r, f) + S(r, g); \end{aligned} \quad (2.20)$$

(ii) $f = ((b+1)g + (a-b-1))/(bg + (a-b))$, where $a(\neq 0)$, b are two constants.

LEMMA 2.8. Let f and g be two transcendental meromorphic functions, let $n \geq 8$ be a positive integer, and let $F = f^n(f-1)^2 f'/z$ and $G = g^n(g-1)^2 g'/z$. If

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)}, \quad (2.21)$$

where $a(\neq 0)$, b are two constants, then $f \equiv g$.

Proof. By Lemma 2.1, we know

$$\begin{aligned} T(r, F) &= T\left(r, \frac{f^n(f-1)^2 f'}{z}\right) \\ &\leq T(r, f^n(f-1)^2) + T(r, f') + \log r \\ &\leq (n+2)T(r, f) + 2T(r, f') + \log r + S(r, f) \\ &= (n+4)T(r, f) + \log r + S(r, f), \\ (n+2)T(r, f) &= T(r, f^n(f-1)^2) + S(r, f) \\ &= N(r, f^n(f-1)^2) + m(r, f^n(f-1)^2) + S(r, f) \\ &\leq N\left(r, \frac{f^n(f-1)^2 f'}{z}\right) - N(r, f') + m\left(r, \frac{f^n(f-1)^2 f'}{z}\right) \\ &\quad + m\left(r, \frac{1}{f'}\right) + \log r + S(r, f) \\ &\leq T\left(r, \frac{f^n(f-1)^2 f'}{z}\right) + T(r, f') - N(r, f') - N\left(r, \frac{1}{f'}\right) \\ &\quad + \log r + S(r, f) \\ &\leq T(r, F) + T(r, f) - N(r, f) - N\left(r, \frac{1}{f'}\right) \\ &\quad + \log r + S(r, f). \end{aligned} \quad (2.22)$$

So

$$T(r, F) \geq (n+1)T(r, f) + N(r, f) + N\left(r, \frac{1}{f'}\right) + \log r + S(r, f). \quad (2.23)$$

Thus, by (2.22), (2.23) and $n \geq 8$, we get $S(r, F) = S(r, f)$. Similarly, we get

$$T(r, G) \geq (n+1)T(r, g) + N(r, g) + N\left(r, \frac{1}{g'}\right) + \log r + S(r, g). \quad (2.24)$$

Without loss of generality, we suppose that $T(r, f) \leq T(r, g)$, $r \in I$, where I is a set with infinite measures. Next, we consider three cases.

Case 1 $b \neq 0, -1$. If $a - b - 1 \neq 0$, then by (2.21) we know

$$\bar{N}\left(r, \frac{1}{G + (a - b - 1)/(b + 1)}\right) = \bar{N}\left(r, \frac{1}{F}\right). \quad (2.25)$$

By the Nevanlinna second fundamental theorem and Lemma 2.2, we have

$$\begin{aligned} T(r, G) &\leq \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{G + (a - b - 1)/(b + 1)}\right) + S(r, G) \\ &= \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{F}\right) + S(r, g) \\ &\leq \bar{N}(r, g) + \bar{N}\left(r, \frac{1}{g}\right) + T(r, g) + \bar{N}\left(r, \frac{1}{g'}\right) + \log r \\ &\quad + \bar{N}\left(r, \frac{1}{f}\right) + T(r, f) + N\left(r, \frac{1}{f}\right) + \bar{N}(r, f) + \log r + S(r, g) \\ &\leq 2T(r, g) + \bar{N}(r, g) + \bar{N}\left(r, \frac{1}{g'}\right) + \log r + 2N\left(r, \frac{1}{f}\right) \\ &\quad + T(r, f) + \bar{N}(r, f) + \log r + S(r, g) \\ &\leq 6T(r, g) + \bar{N}(r, g) + \bar{N}\left(r, \frac{1}{g'}\right) + 2\log r + S(r, g). \end{aligned} \quad (2.26)$$

Hence, by $n \geq 8$ and (2.24), we know $T(r, g) \leq S(r, g)$, $r \in I$, this is impossible.

If $a - b - 1 = 0$, then by (2.21) we know $F = ((b + 1)G)/(bG + 1)$. Obviously,

$$\bar{N}\left(r, \frac{1}{G + 1/b}\right) = \bar{N}(r, F). \quad (2.27)$$

By the Nevanlinna second fundamental theorem and Lemma 2.2, we have

$$\begin{aligned}
T(r, G) &\leq \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}\left(r, \frac{1}{G+1/b}\right) + S(r, G) \\
&= \bar{N}(r, G) + \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}(r, F) + S(r, g) \\
&\leq \bar{N}(r, g) + \bar{N}\left(r, \frac{1}{g}\right) + T(r, g) + \bar{N}\left(r, \frac{1}{g'}\right) + \log r + \bar{N}(r, f) \\
&\quad + \log r + S(r, g) \\
&\leq 2T(r, g) + \bar{N}(r, g) + \bar{N}\left(r, \frac{1}{g'}\right) + T(r, f) + 2\log r + S(r, g) \\
&\leq 3T(r, g) + \bar{N}(r, g) + \bar{N}\left(r, \frac{1}{g'}\right) + 2\log r + S(r, g).
\end{aligned} \tag{2.28}$$

Then by $n \geq 8$ and (2.24), we know $T(r, g) \leq S(r, g)$, $r \in I$, a contradiction.

Case 2 $b = -1$. Then (2.21) becomes $F = a/(a+1-G)$.

If $a+1 \neq 0$, then $\bar{N}(r, 1/(G-a-1)) = \bar{N}(r, F)$. Similarly, we can deduce a contradiction as in Case 1.

If $a+1 = 0$, then $FG \equiv 1$, that is,

$$f^n(f-1)^2 f' g^n(g-1)^2 g' \equiv z^2. \tag{2.29}$$

Since $n \geq 8$, by Lemma 2.4, a contradiction.

Case 3 $b = 0$. Then (2.21) becomes $F = (G+a-1)/a$.

If $a-1 \neq 0$, then $\bar{N}(r, 1/(G+a-1)) = \bar{N}(r, 1/F)$. Similarly, we can again deduce a contradiction as in Case 1.

If $a-1 = 0$, then $F \equiv G$, that is,

$$f^n(f-1)^2 f' \equiv g^n(g-1)^2 g'. \tag{2.30}$$

By Lemma 2.5, we obtain $f \equiv g$.

This completes the proof of this lemma. $\square$

3. Proof of theorems

Let F and G be defined as in Lemma 2.8.

Proof of Theorem 1.11. Since $k \geq 3$, we have

$$\begin{aligned}
&\bar{N}\left(r, \frac{1}{F-1}\right) + \bar{N}\left(r, \frac{1}{G-1}\right) - N_{11}\left(r, \frac{1}{F-1}\right) + \bar{N}_{(k+1)}\left(r, \frac{1}{F-1}\right) + \bar{N}_{(k+1)}\left(r, \frac{1}{G-1}\right) \\
&\leq \frac{1}{2}N\left(r, \frac{1}{F-1}\right) + \frac{1}{2}N\left(r, \frac{1}{G-1}\right) \leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + S(r, f) + S(r, g).
\end{aligned} \tag{3.1}$$

Then (i) in Lemma 2.6 becomes

$$T(r, F) + T(r, G) \leq 2 \left\{ N_2 \left(r, \frac{1}{F} \right) + N_2(r, F) + N_2 \left(r, \frac{1}{G} \right) + N_2(r, G) \right\} + S(r, f) + S(r, g). \quad (3.2)$$

Since

$$\begin{aligned} N_2 \left(r, \frac{1}{F} \right) + N_2(r, F) &= N_2 \left(r, \frac{z}{f^n(f-1)^2 f'} \right) + N_2 \left(r, \frac{f^n(f-1)^2 f'}{z} \right) \\ &\leq 2\bar{N} \left(r, \frac{1}{f} \right) + 2\bar{N} \left(r, \frac{1}{f-1} \right) + N \left(r, \frac{1}{f'} \right) + 2\bar{N}(r, f) + 2\log r. \end{aligned} \quad (3.3)$$

Similarly, we obtain

$$N_2 \left(r, \frac{1}{G} \right) + N_2(r, G) \leq 2\bar{N} \left(r, \frac{1}{g} \right) + 2\bar{N} \left(r, \frac{1}{g-1} \right) + N \left(r, \frac{1}{g'} \right) + 2\bar{N}(r, g) + 2\log r. \quad (3.4)$$

Suppose that

$$T(r, F) + T(r, G) \leq 2 \left\{ N_2 \left(r, \frac{1}{F} \right) + N_2(r, F) + N_2 \left(r, \frac{1}{G} \right) + N_2(r, G) \right\} + S(r, f) + S(r, g). \quad (3.5)$$

By Lemma 2.2 and (2.23), (2.24), and (3.3), we get

$$\begin{aligned} T(r, F) + T(r, G) &\leq 4\bar{N} \left(r, \frac{1}{f} \right) + 4\bar{N} \left(r, \frac{1}{f-1} \right) + 2N \left(r, \frac{1}{f'} \right) + 4\bar{N}(r, f) \\ &\quad + 4\bar{N} \left(r, \frac{1}{g} \right) + 4\bar{N} \left(r, \frac{1}{g-1} \right) + 2N \left(r, \frac{1}{g'} \right) + 4\bar{N}(r, g) \\ &\quad + 8\log r + S(r, f) + S(r, g) \\ &\leq 5N \left(r, \frac{1}{f} \right) + 4\bar{N} \left(r, \frac{1}{f-1} \right) + N \left(r, \frac{1}{f'} \right) + 5\bar{N}(r, f) \\ &\quad + 5N \left(r, \frac{1}{g} \right) + 4\bar{N} \left(r, \frac{1}{g-1} \right) + N \left(r, \frac{1}{g'} \right) + 5\bar{N}(r, g) \\ &\quad + 8\log r + S(r, f) + S(r, g) \\ &\leq 13T(r, f) + \bar{N}(r, f) + N \left(r, \frac{1}{f'} \right) + S(r, f) + 13T(r, g) \\ &\quad + \bar{N}(r, g) + N \left(r, \frac{1}{g'} \right) + 8\log r + S(r, g). \end{aligned} \quad (3.6)$$

By $n \geq 13$ and (2.23), (2.24), we can obtain a contradiction.

Thus, by Lemma 2.6, $F = ((b+1)G + (a-b-1))/(bG + (a-b))$, where $a(\neq 0), b$ are two constants. By Lemma 2.8, we get $f \equiv g$. This completes the proof of Theorem 1.11. $\square$

Proof of Theorem 1.12. Obviously, we have

$$\begin{aligned} & \overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) - N_{11}\left(r, \frac{1}{F-1}\right) + \frac{1}{2}\overline{N}_{(3)}\left(r, \frac{1}{F-1}\right) + \frac{1}{2}\overline{N}_{(3)}\left(r, \frac{1}{G-1}\right) \\ & \leq \frac{1}{2}N\left(r, \frac{1}{F-1}\right) + \frac{1}{2}N\left(r, \frac{1}{G-1}\right) \leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + S(r, f) + S(r, g). \end{aligned} \quad (3.7)$$

Then (i) in Lemma 2.6 becomes

$$\begin{aligned} T(r, F) + T(r, G) & \leq 2\left\{N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G)\right\} \\ & \quad + \overline{N}_{(3)}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(3)}\left(r, \frac{1}{G-1}\right) + S(r, f) + S(r, g). \end{aligned} \quad (3.8)$$

Consider

$$\begin{aligned} \overline{N}_{(3)}\left(r, \frac{1}{F-1}\right) & \leq \frac{1}{2}N\left(r, \frac{F}{F'}\right) = \frac{1}{2}N\left(r, \frac{F'}{F}\right) + S(r, f) \\ & \leq \frac{1}{2}\overline{N}(r, F) + \frac{1}{2}\overline{N}\left(r, \frac{1}{F}\right) + S(r, f) \\ & \leq \frac{1}{2}\left[\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f-1}\right) + N\left(r, \frac{1}{f'}\right) + \overline{N}(r, f)\right] + \log r + S(r, f) \\ & \leq \frac{5}{2}T(r, f) + \log r + S(r, f). \end{aligned} \quad (3.9)$$

Similarly, we get

$$\overline{N}_{(3)}\left(r, \frac{1}{G-1}\right) \leq \frac{5}{2}T(r, g) + \log r + S(r, g). \quad (3.10)$$

Suppose that

$$\begin{aligned} T(r, F) + T(r, G) & \leq 2\left\{N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G)\right\} \\ & \quad + \overline{N}_{(3)}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(3)}\left(r, \frac{1}{G-1}\right) + S(r, f) + S(r, g). \end{aligned} \quad (3.11)$$

Combining (3.3), (3.5) and (3.9)–(3.11), we can get

$$\begin{aligned} T(r, F) + T(r, G) & \leq \frac{31}{2}T(r, f) + \overline{N}(r, f) + N\left(r, \frac{1}{f'}\right) + S(r, f) + \frac{31}{2}T(r, g) \\ & \quad + \overline{N}(r, g) + N\left(r, \frac{1}{g'}\right) + 10\log r + S(r, g). \end{aligned} \quad (3.12)$$

From $n \geq 15$ and (2.23), (2.24), we can get a contradiction.

By Lemma 2.6, we obtain $F = ((b+1)G + (a-b-1))/(bG + (a-b))$, where $a (\neq 0)$, b are two constants. Then by Lemma 2.8, we can prove Theorem 1.12. $\square$

Proof of Theorem 1.13. Similarly, we get

$$\begin{aligned}\overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) - N_{11}\left(r, \frac{1}{F-1}\right) &\leq \frac{1}{2}N\left(r, \frac{1}{F-1}\right) + \frac{1}{2}N\left(r, \frac{1}{G-1}\right) \\ &\leq \frac{1}{2}T(r, F) + \frac{1}{2}T(r, G) + S(r, f) + S(r, g).\end{aligned}\quad (3.13)$$

Then (i) in Lemma 2.6 becomes

$$\begin{aligned}T(r, F) + T(r, G) &\leq 2\left\{N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G)\right. \\ &\quad \left.+ \overline{N}_{(2)}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G-1}\right)\right\} + S(r, f) + S(r, g).\end{aligned}\quad (3.14)$$

Consider

$$\begin{aligned}\overline{N}_{(2)}\left(r, \frac{1}{F-1}\right) &\leq N\left(r, \frac{F}{F'}\right) = N\left(r, \frac{F'}{F}\right) + S(r, f) \\ &\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, f) \\ &\leq 5T(r, f) + 2\log r + S(r, f).\end{aligned}\quad (3.15)$$

Similarly, we have

$$\overline{N}_{(2)}\left(r, \frac{1}{G-1}\right) \leq 5T(r, g) + 2\log r + S(r, g). \quad (3.16)$$

Suppose that

$$\begin{aligned}T(r, F) + T(r, G) &\leq 2\left\{N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G)\right. \\ &\quad \left.+ \overline{N}_{(2)}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(2)}\left(r, \frac{1}{G-1}\right)\right\} + S(r, f) + S(r, g).\end{aligned}\quad (3.17)$$

Considering (3.3), (3.4), (3.6), and (3.15)–(3.17), we know

$$\begin{aligned}T(r, F) + T(r, G) &\leq 23T(r, f) + \overline{N}(r, f) + N\left(r, \frac{1}{f'}\right) + S(r, f) + 23T(r, g) \\ &\quad + \overline{N}(r, g) + N\left(r, \frac{1}{g'}\right) + 12\log r + S(r, g).\end{aligned}\quad (3.18)$$

By $n \geq 23$ and (2.23), (2.24), we get a contradiction.

Applying Lemma 2.6, we know $F = ((b+1)G + (a-b-1))/(bG + (a-b))$, where $a(\neq 0), b$ are two constants. Then by Lemma 2.8, we can prove Theorem 1.13. $\square$

Proof of Theorem 1.14. Since

$$\begin{aligned}\overline{N}_L\left(r, \frac{1}{F-1}\right) &\leq N\left(r, \frac{F}{F'}\right) = N\left(r, \frac{F'}{F}\right) + S(r, f) \\ &\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, f) \\ &\leq 5T(r, f) + 2\log r + S(r, f).\end{aligned}\tag{3.19}$$

Similarly, we have

$$\overline{N}_L\left(r, \frac{1}{G-1}\right) \leq 5T(r, g) + 2\log r + S(r, g).\tag{3.20}$$

Suppose that F and G satisfied (i) in Lemma 2.7, then we get

$$\begin{aligned}T(r, F) + T(r, G) &\leq 2\left\{N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G)\right\} \\ &\quad + 3\overline{N}_L\left(r, \frac{1}{F-1}\right) + 3\overline{N}_L\left(r, \frac{1}{G-1}\right) + S(r, f) + S(r, g).\end{aligned}\tag{3.21}$$

Considering (3.3), (3.4), (3.6), and (3.19)–(3.21), we have

$$\begin{aligned}T(r, F) + T(r, G) &\leq 28T(r, f) + \overline{N}(r, f) + N\left(r, \frac{1}{f'}\right) + S(r, f) + 28T(r, g) \\ &\quad + \overline{N}(r, g) + N\left(r, \frac{1}{g'}\right) + 20\log r + S(r, g).\end{aligned}\tag{3.22}$$

From $n \geq 28$ and (2.23), (2.24), we get a contradiction.

Applying Lemma 2.7, we know $F = ((b+1)G + (a-b-1))/(bG + (a-b))$, where $a(\neq 0)$, b are two constants. Then by Lemma 2.8, we can prove Theorem 1.14. $\square$

4. Remarks

It follows from the proof of Theorems 1.11–1.14 that if “ z ” is replaced by “ $a(z)$ ” in Theorems 1.11–1.14, where $a(z)$ is a meromorphic function such that $a \neq 0, \infty$ and $T(r, a) = o\{T(r, f), T(r, g)\}$, then the conclusions of Theorems 1.11–1.14 still hold. So we obtain the following results.

THEOREM 4.1. *Let f and g be two transcendental meromorphic functions and let $n \geq 13$, $k \geq 3$ be two positive integers. If $E_k(a(z), f^n(f-1)^2 f') = E_k(a(z), g^n(g-1)^2 g')$, then $f \equiv g$.*

THEOREM 4.2. *Let f and g be two transcendental meromorphic functions and let $n(\geq 15)$ be a positive integer. If $E_2(a(z), f^n(f-1)^2 f') = E_2(a(z), g^n(g-1)^2 g')$, then the conclusion of Theorem 4.1 still holds.*

THEOREM 4.3. *Let f and g be two transcendental meromorphic functions and let $n(\geq 23)$ be a positive integer. If $E_1(a(z), f^n(f-1)^2 f') = E_1(a(z), g^n(g-1)^2 g')$, then the conclusion of Theorem 4.1 still holds.*

THEOREM 4.4. *Let f and g be two transcendental meromorphic functions and let $n(\geq 28)$ be a positive integer. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share $a(z)$ IM, then the conclusion of Theorem 4.1 still holds.*

Obviously, we can use the analog method of Theorems 1.11–1.14 to prove Theorems 4.1–4.4 easily. Here, we omit them.

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References

- [1] C.-Y. Fang and M.-L. Fang, "Uniqueness of meromorphic functions and differential polynomials," *Computers & Mathematics with Applications*, vol. 44, no. 5-6, pp. 607–617, 2002.
- [2] I. Lahiri and R. Pal, "Non-linear differential polynomials sharing 1-points," *Bulletin of the Korean Mathematical Society*, vol. 43, no. 1, pp. 161–168, 2006.
- [3] W. Lin and H. X. Yi, "Uniqueness theorems for meromorphic functions concerning fixed-points," *Complex Variables. Theory and Application*, vol. 49, no. 11, pp. 793–806, 2004.
- [4] L. Yang, *Value Distribution Theory*, Springer, Berlin, Germany, 1993.
- [5] R. Nevanlinna, *Le Théorème de Picard-Borel et la Théorie des Fonctions Méromorphes*, Gauthier-Villars, Paris, France, 1929.
- [6] G. G. Gundersen, "Meromorphic functions that share three or four values," *The Journal of the London Mathematical Society*, vol. 20, no. 3, pp. 457–466, 1979.
- [7] P. Li and C. C. Yang, "On two meromorphic functions that share pairs of small functions," *Complex Variables. Theory and Application*, vol. 32, no. 2, pp. 177–190, 1997.
- [8] M. Fang and H. Qiu, "Meromorphic functions that share fixed-points," *Journal of Mathematical Analysis and Applications*, vol. 268, no. 2, pp. 426–439, 2002.
- [9] C. C. Yang, "On deficiencies of differential polynomials—II," *Mathematische Zeitschrift*, vol. 125, no. 2, pp. 107–112, 1972.
- [10] H. X. Yi and C. C. Yang, *Uniqueness Theory of Meromorphic Functions*, Science Press, Beijing, China, 1995.
- [11] G. Frank and M. Reinders, "A unique range set for meromorphic functions with 11 elements," *Complex Variables. Theory and Application*, vol. 37, no. 1–4, pp. 185–193, 1998.
- [12] M. Fang and H. Guo, "On unique range sets for meromorphic or entire functions," *Acta Mathematica Sinica*, vol. 14, no. 4, pp. 569–576, 1998.

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