Research Article

About Some Linear Operators

Ovidiu T. Pop

Received 22 March 2007; Revised 24 May 2007; Accepted 15 July 2007

Recommended by Brigitte Forster-Heinlein

Using the method of Jakimovski and Leviatan from their work in 1969, we construct a general class of linear positive operators. We study the convergence, the evaluation for the rate of convergence in terms of the first modulus of smoothness and we give a Voronovskaja-type theorem for these operators.

Copyright © 2007 Ovidiu T. Pop. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

The aim of this paper is to construct a class of linear operators in more general conditions. The method was inspired by Jakimovski and Leviatan (see [1]). We do not study the convergence of these operators with the well-known theorem of Bohman-Korovkin. The evaluation theorems for the rate of convergence are different from the well-known theorem of Shisha-Mond. We prove the Voronovskaja-type theorem for these operators. In the end, we give particularizations of these operators.

We recall some notions and results which we will use in this paper.

Let $\mathbb{N}$ be the set of positive integer numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a given interval I , we will use the following function sets: $B(I) = \{f \mid f : I \rightarrow \mathbb{R}, f \text{ bounded on } I\}$, $C(I) = \{f \mid f : I \rightarrow \mathbb{R}, f \text{ continuous on } I\}$, and $C_B(I) = B(I) \cap C(I)$.

For any $x \in I$, consider the functions $\psi_x : I \rightarrow \mathbb{R}$ defined by $\psi_x(t) = t - x$ and $e_i : I \rightarrow \mathbb{R}$, $e_i(t) = t^i$ for any $t \in I$, $i \in \{0, 1, 2, 3, 4\}$.

For $f \in C_B(I)$, by the first-order modulus of smoothness of f is meant the function $\omega(f; \cdot) : [0, \infty) \rightarrow \mathbb{R}$ defined for any $\delta \geq 0$ by

$$\omega(f; \delta) = \sup \{ |f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta \}. \quad (1.1)$$

In the following, we take into account the properties of the first-order modulus of smoothness and the properties of the linear positive functional.

LEMMA 1.1. *Let $f \in C_B(I)$. Then, $\omega(f; \cdot)$ has the following properties:*

- (a) $\omega(f; 0) = 0$,
- (b) $\omega(f; \cdot)$ is an increasing function,
- (c) $\omega(f; \cdot)$ is a uniform continuous function on I ,
- (d) for any $\delta > 0$, $x, t \in I$, one has $|f(t) - f(x)| \leq [1 + \delta^{-2}\psi_x^2(t)]\omega(f; \delta)$.

LEMMA 1.2. *Let $A : E(I) \rightarrow \mathbb{R}$ be a linear positive functional. Then,*

- (a) for $f, g \in E(I)$ with $f(x) \leq g(x)$ for any $x \in I$, one has

$$A(f) \leq A(g); \quad (1.2)$$

- (b) $|A(f)| \leq A(|f|)$ for any $f \in E(I)$, where $E(I)$ is a subset of the set of real functions defined on I .

In [2] we have demonstrated the following theorem.

THEOREM 1.3. *Let I be an interval $x \in I$, and let the function $f : I \rightarrow \mathbb{R}$ be s times differentiable in x . According to the Taylor Expansion Theorem, one has*

$$f(t) = \sum_{i=0}^s \frac{(t-x)^i}{i!} f^{(i)}(x) + (t-x)^s \mu(t-x), \quad (1.3)$$

where μ is a bounded function and $\lim_{t \rightarrow x} \mu(t-x) = 0$. If $f^{(s)}$ is a continuous function on I , then for any $\delta > 0$ and $x \in I$ one has

$$|(\mu(t-x))| \leq \frac{1}{s!} [1 + \delta^{-2}\psi_x^2(t)] \omega(f^{(s)}; \delta). \quad (1.4)$$

2. Preliminaries

In this section, we construct a general class of linear and positive operators and we demonstrate for these operators an approximation theorem and a Voronovskaja-type theorem.

Let I, J be intervals and $I \cap J$ is a nonempty interval. For any $m \in \mathbb{N}$ and $k \in \mathbb{N}_0$, consider the function $\varphi_{m,k} : J \rightarrow \mathbb{R}$ with the property $\varphi_{m,k}(x) \geq 0$ for any $x \in J$ and the linear and positive functional $A_{m,k} : E(I) \rightarrow \mathbb{R}$.

In the following, let $E(I)$ and $F(J)$ be subsets of the set of real functions defined on I, J respectively, such that the series $\sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(f)$ is convergent for any $f \in E(I)$ and any $x \in J$. We suppose that $\psi_x^i \in E(I)$ for any $x \in I \cap J$ and any $i \in \{0, 1, \dots, s+2\}$.

In what follows $s \in \mathbb{N}_0$, s is even.

Definition 2.1. For $m \in \mathbb{N}$, define the operator $L_m : E(I) \rightarrow F(J)$ by

$$(L_m f)(x) = \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(f) \quad (2.1)$$

for any $f \in E(I)$ and $x \in J$.

PROPOSITION 2.2. *The operators $(L_m)_{m \geq 1}$ are linear and positive on $E(I \cap J)$.*

Proof. The proof follows immediately. □

Definition 2.3. For $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$, define T_i by

$$(T_i L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^i) \quad (2.2)$$

for any $x \in I \cap J$.

THEOREM 2.4. *If $f \in E(I)$ is an s -times differentiable function in $x \in I \cap J$, with $f^{(s)}$ continuous in x , and if there exist $\alpha_s, \alpha_{s+2} \in [0, \infty)$ and $m(s) \in \mathbb{N}$ such that*

$$\alpha_{s+2} < \alpha_s + 2 \quad (2.3)$$

and $(T_s L_m)(x)/m^{\alpha_s}, (T_{s+2} L_m)(x)/m^{\alpha_{s+2}}$ are bounded for any $m \in \mathbb{N}, m \geq (s)$, then

$$\lim_{m \rightarrow \infty} m^{s-\alpha_s} \left[(L_m f)(x) - \sum_{i=0}^s \frac{1}{i! m^i} (T_i L_m)(x) f^{(i)}(x) \right] = 0. \quad (2.4)$$

Assume that f is an s times differentiable function on I with $f^{(s)}$ continuous on I and an interval $K \subset I \cap J$ exists such that there exist $m(s) \in \mathbb{N}$ and the constants $k_j(K) \in \mathbb{R}$ depending on K , so that for any $m \in \mathbb{N}, m \geq m(s)$ and $x \in K$, one has

$$\frac{(T_j L_m)(x)}{m^{\alpha_j}} \leq k_j(K), \quad (2.5)$$

where $j \in \{s, s+2\}$. Then, the convergence given in (2.4) is uniform on K and

$$\begin{aligned} m^{s-\alpha_s} \left| (L_m f)(x) - \sum_{i=0}^s \frac{1}{i! m^i} (T_i L_m)(x) f^{(i)}(x) \right| \\ \leq \frac{1}{s!} (k_s(K) + k_{s+2}(K)) \omega \left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}} \right) \end{aligned} \quad (2.6)$$

for any $x \in K$ and $m \geq m(s)$.

Proof. According to Taylor's Theorem, we have

$$f(t) = \sum_{i=0}^s \frac{(t-x)^i}{i!} f^{(i)}(x) + (t-x)^s \mu(t-x), \quad (2.7)$$

where μ is a bounded function and $\lim_{t \rightarrow x} \mu(t-x) = 0$.

Hence, from (2.7), we have

$$A_{m,k}(f) = \sum_{i=0}^s \frac{f^{(i)}(x)}{i!} A_{m,k}(\psi_x^i) + A_{m,k}(\psi_x^s \mu_x), \quad (2.8)$$

where $\mu_x : I \rightarrow \mathbb{R}$, $\mu_x(t) = \mu(t-x)$, for any $t \in I \cap J$.

Multiplying by $\varphi_{m,k}(x)$ and summing over $k \in \mathbb{N}_0$, we obtain

$$(L_m f)(x) = \sum_{i=0}^s \frac{f^{(i)}(x)}{i!} (L_m \psi_x^i)(x) + \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^s \mu_x). \quad (2.9)$$

Thus,

$$m^{s-\alpha_s} \left[(L_m f)(x) - \sum_{i=0}^s \frac{f^{(i)}(x)}{i! m^i} (T_i L_m)(x) \right] = (R_m f)(x), \quad (2.10)$$

where

$$(R_m f)(x) = m^{s-\alpha_s} \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^s \mu_x). \quad (2.11)$$

Then,

$$|(R_m f)(x)| \leq m^{s-\alpha_s} \sum_{k=0}^{\infty} \varphi_{m,k}(x) |A_{m,k}(\psi_x^s \mu_x)| \quad (2.12)$$

and taking Lemma 1.2 into account, we obtain

$$|(R_m f)(x)| \leq m^{s-\alpha_s} \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^s |\mu_x|). \quad (2.13)$$

According to the relation (1.4), for any $\delta > 0$ and $t \in I \cap J$, we have

$$|\mu_x(t)| = |\mu(t-x)| \leq \frac{1}{s!} [1 + \delta^{-2} \psi_x^2(t)] \omega(f^{(s)}; \delta), \quad (2.14)$$

and so

$$(\psi_x^s |\mu_x|)(t) \leq \frac{1}{s!} [\psi_x^s(t) + \delta^{-2} \psi_x^{s+2}(t)] \omega(f^{(s)}; \delta). \quad (2.15)$$

From (2.13) and (2.15), it results that

$$|(R_m f)(x)| \leq \frac{1}{s!} m^{s-\alpha_s} \left[\sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^s) + \delta^{-2} \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^{s+2}) \right] \omega(f^{(s)}; \delta). \quad (2.16)$$

Thus,

$$|(R_m f)(x)| \leq \frac{1}{s!} \left[\frac{(T_s L_m)(x)}{m^{\alpha_s}} + \delta^{-2} \frac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}} m^{-2-\alpha_s+\alpha_{s+2}} \right] \omega(f^{(s)}; \delta). \quad (2.17)$$

Considering $\delta = 1/\sqrt{m^{2+\alpha_2-\alpha_{s+2}}}$, the inequality above becomes

$$|(R_m f)(x)| \leq \frac{1}{s!} \left[\frac{(T_s L_m)(x)}{m^{\alpha_s}} + \frac{(T_{s+2} L_m)(x)}{m^{\alpha_{s+2}}} \right] \omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right). \quad (2.18)$$

Taking into account that $(T_s L_m)(x)/m^{\alpha_s}$ and $(T_{s+2} L_m)(x)/m^{\alpha_{s+2}}$ are bounded for any $m \in \mathbb{N}$, $m \geq m(s)$, and considering the fact that

$$\lim_{m \rightarrow \infty} \omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right) = \omega(f^{(s)}; 0) = 0, \quad (2.19)$$

we have that

$$\lim_{m \rightarrow \infty} (R_m f)(x) = 0. \quad (2.20)$$

From (2.10) and (2.20), (2.4) follows.

If in addition (2.5) takes place then, (2.18) becomes

$$|(R_m f)(x)| \leq \frac{1}{s!} (k_s(K) + k_{s+2}(K)) \omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right), \quad (2.21)$$

for $m \geq m(s)$ and $x \in K$. Therefore, the convergence from (2.4) is uniform on K . Now, (2.10) and (2.21) yield (2.6). $\square$

In the following, we suppose that for any $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$, we have

$$A_{m,k}(e_0) = 1, \quad (2.22)$$

and for any $x \in I \cap J$ and $m \in \mathbb{N}$

$$\sum_{k=0}^{\infty} \varphi_{m,k}(x) = 1. \quad (2.23)$$

Remark 2.5. Taking (2.22) and (2.23) into account, it results that

$$(T_0 L_m)(x) = 1 \quad (2.24)$$

for any $x \in I \cap J$ and $m \in \mathbb{N}$.

Remark 2.6. In Theorem 2.4, we choose the smallest α_s and α_{s+2} , if they exist.

Remark 2.7. Taking (2.24) into account, we choose $\alpha_0 = 0$.

Remark 2.8. For $s = 0, s = 2$, respectively, we state two corollaries which we will use in the section Main results.

COROLLARY 2.9. *If $f \in E(I)$ is a continuous function in $x \in I \cap J$, and if there exist α_2 and $m(0) \in \mathbb{N}$ such that*

$$0 \leq \alpha_2 < 2 \quad (2.25)$$

and $(T_2L_m)(x)/m^{\alpha_2}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(0)$, then

$$\lim_{m \rightarrow \infty} (L_m f)(x) = f(x). \quad (2.26)$$

Assume that f is continuous on I and an interval $K \subset I \cap J$ exists, such that there exist $m(0) \in \mathbb{N}$ and $k_2(K)$ so that for any $m \in \mathbb{N}$, $m \geq m(0)$, and $x \in K$, one has

$$\frac{(T_2L_m)(x)}{m^{\alpha_2}} \leq k_2(K). \quad (2.27)$$

Then, the convergence given in (2.26) is uniform on K and

$$|(L_m f)(x) - f(x)| \leq (1 + k_2(K)) \omega\left(f; \frac{1}{\sqrt{m^{2-\alpha_2}}}\right) \quad (2.28)$$

for any $x \in K$ and $m \geq m(0)$.

COROLLARY 2.10. *If $f \in E(I)$ is a two-times differentiable function in $x \in I \cap J$, with $f^{(2)}$ continuous in x , and if there exist α_2, α_4 and $m(2) \in \mathbb{N}$ such that*

$$\begin{aligned} 0 &\leq \alpha_2 < 2, \\ 0 &\leq \alpha_4 < \alpha_2 + 2, \end{aligned} \quad (2.29)$$

$(T_2L_m)(x)/m^{\alpha_2}$ and $((T_4L_m)(x))/m^{\alpha_4}$ are bounded for any $m \in \mathbb{N}$, $m \geq m(2)$, then

$$\lim_{m \rightarrow \infty} m^{2-\alpha_2} \left[(L_m f)(x) - f(x) - \frac{1}{m} (T_1L_m)(x) f^{(1)}(x) - \frac{1}{2m^2} (T_2L_m)(x) f^{(2)}(x) \right] = 0. \quad (2.30)$$

Assume that f is a two-times differentiable function on I with $f^{(2)}$ continuous on I and an interval $K \subset I \cap J$ exists, such that there exist $m(2) \in \mathbb{N}$ and $k_j(K)$, so that for any $m \geq m(2)$ and $x \in K$, one has

$$\frac{(T_jL_m)(x)}{m^{\alpha_j}} \leq k_j(K), \quad (2.31)$$

where $j \in \{2, 4\}$. Then, the convergence given in (2.30) is uniform on K .

Remark 2.11. Theorem 2.4, Corollary 2.9, and 2.10 are Voronovskaja-type theorems.

3. Main results

In this section, we construct a general class of linear positive operators. Let $\mathbb{R}_0 = [0, \infty)$ and J be an interval with $J \subset \mathbb{R}_0$. Let the sequence $(a_m)_{m \geq 1}$ so that $a_m x \in J$ for any $m \in \mathbb{N}$ and $x \in J$. The indefinitely differentiable functions $a, b : J \rightarrow \mathbb{R}$ have the property:

$$b(x) > 0 \quad (3.1)$$

for any $x \in \mathbb{R}_0$,

$$a(1) \neq 0 \quad (3.2)$$

and for any compact $K \subset J$ the constants $M_1(K)$, $M_2(K)$ depending on K exist, such that

$$\begin{aligned} |a^{(k)}(x)| &\leq M_1(K), \\ |b^{(k)}(x)| &\leq M_2(K) \end{aligned} \quad (3.3)$$

for any $x \in K$ and $k \in \mathbb{N}_0$.

Then, it is known that

$$\begin{aligned} a(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} a^{(n)}(0) x^n, \\ b(x) &= \sum_{p=0}^{\infty} \frac{1}{p!} b^{(p)}(0) x^p \end{aligned} \quad (3.4)$$

for any $x \in J$.

If $u, x, ux \in J$, we calculate

$$a(u)b(ux) = \left(\sum_{n=0}^{\infty} \frac{1}{n!} a^{(n)}(0) u^n \right) \left(\sum_{p=0}^{\infty} \frac{1}{p!} b^{(p)}(0) (ux)^p \right) \quad (3.5)$$

and we take it to the form

$$a(u)b(ux) = \sum_{k=0}^{\infty} p_k(x) u^k, \quad (3.6)$$

where

$$p_k(x) = \sum_{i=0}^k \frac{1}{i!(k-i)!} a^{(i)}(0) b^{(k-i)}(0) x^{k-i}. \quad (3.7)$$

Remark 3.1. If $u = 1$, then from (3.6), we obtain

$$a(1)b(a_mx) = \sum_{k=0}^{\infty} p_k(a_mx) \quad (3.8)$$

for any $m \in \mathbb{N}$ and $x \in J$.

Remark 3.2. We consider that the conditions $a^{(i)}(0)b^{(k-i)}(0)/a(1) \geq 0$, $i \in \{0, 1, \dots, k\}$ and $k \in \mathbb{N}_0$, hold and then it results that $a(1)p_k(x) \geq 0$ for any $x \in J$ and any $k \in \mathbb{N}_0$.

In the following, let a fixed function $w : \mathbb{R}_0 \rightarrow (0, \infty)$, called the weight function, and the set functions

$$E(w) = \{f \mid f : \mathbb{R}_0 \rightarrow \mathbb{R} \text{ such that } wf \text{ is bounded on } [0, \infty)\}. \quad (3.9)$$

For $f \in E(w)$, there exists a positive constant M such that $w(x)|f(x)| \leq M$ for any $x \in \mathbb{R}_0$. For $m \in \mathbb{N}$ and $x \in J$, and taking in the end (3.8) into account, we have

$$\begin{aligned} \left| \frac{1}{a(1)b(a_mx)} \sum_{k=0}^{\infty} p_k(a_mx) f\left(\frac{k}{m}\right) \right| &\leq \frac{1}{a(1)b(a_mx)} \sum_{k=0}^{\infty} p_k(a_mx) \left| f\left(\frac{k}{m}\right) \right| \\ &\leq \frac{M}{w(x)} \frac{1}{a(1)b(a_mx)} \sum_{k=0}^{\infty} p_k(a_mx) = \frac{M}{w(x)}, \end{aligned} \quad (3.10)$$

from where it results that the series $(1/a(1)b(a_mx)) \sum_{k=0}^{\infty} p_k(a_mx) f(k/m)$ is convergent.

Definition 3.3. For $m \in \mathbb{N}$, define the operator $L_m : E(w) \rightarrow F(J)$ by

$$(L_m f)(x) = \frac{1}{a(1)b(a_mx)} \sum_{k=0}^{\infty} p_k(a_mx) f\left(\frac{k}{m}\right) \quad (3.11)$$

for any $f \in E(w)$ and $x \in J$, where $F(J)$ is a subset of the set of real functions defined on J .

Remark 3.4. The operators $(L_m)_{m \geq 1}$ are linear and positive on $E(w)$.

In the following, we consider that for any $x \in J$, we have $\psi_x^i \in E(w)$, $i \in \{1, 2, 3, 4\}$.

Definition 3.5. For $m \in \mathbb{N}$ and $i \in \{1, 2, 3, 4\}$, define T_i by

$$(T_i L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \frac{1}{a(1)b(ax)} \sum_{k=0}^{\infty} p_k(ax) \left(\frac{k}{m} - x\right)^i \quad (3.12)$$

for any $x \in J$.

LEMMA 3.6. One has

$$(L_m e_0)(x) = 1, \quad (3.13)$$

$$(L_m e_1)(x) = \frac{a_m}{m} \frac{b^{(1)}(ax)}{b(ax)} x + \frac{1}{m} \frac{a^{(1)}(1)}{a(1)},$$

$$(L_m e_2)(x) = \left(\frac{a_m}{m}\right)^2 \frac{b^{(2)}(ax)}{b(ax)} x^2 + \frac{1}{m} \frac{a_m}{m} \frac{a(1) + 2a^{(1)}(1)}{a(1)} \frac{b^{(1)}(ax)}{b(ax)} x + \frac{1}{m^2} \frac{a^{(1)}(1) + a^{(2)}(1)}{a(1)},$$

$$(L_m e_3)(x) = \left(\frac{a_m}{m}\right)^3 \frac{b^{(3)}(ax)}{b(ax)} x^3 + \frac{1}{m} \left(\frac{a_m}{m}\right)^2 \frac{3a(1) + 3a^{(1)}(1)}{a(1)} \frac{b^{(2)}(ax)}{b(ax)} x^2$$

$$+ \frac{1}{m^2} \frac{a_m}{m} \frac{a(1) + 6a^{(1)}(1) + 3a^{(2)}(1)}{a(1)} \frac{b^{(1)}(ax)}{b(ax)} x + \frac{1}{m^3} \frac{a^{(1)} + 3a^{(2)}(1) + a^{(3)}(1)}{a(1)},$$

$$(L_m e_4)(x) = \left(\frac{a_m}{m}\right)^4 \frac{b^{(4)}(ax)}{b(ax)} x^4 + \frac{1}{m} \left(\frac{a_m}{m}\right)^3 \frac{6a(1) + 4a^{(1)}(1)}{a(1)} \frac{b^{(3)}(ax)}{b(ax)} x^3$$

$$+ \frac{1}{m^2} \left(\frac{a_m}{m}\right)^2 \frac{7a(1) + 18a^{(1)}(1) + 6a^{(2)}(1)}{a(1)} \frac{b^{(2)}(ax)}{b(ax)} x^2$$

$$+ \frac{1}{m^3} \frac{a_m}{m} \frac{a(1) + 14a^{(1)}(1) + 18a^{(2)}(1) + 4a^{(3)}(1)}{a(1)} \frac{b^{(1)}(ax)}{b(ax)} x$$

$$+ \frac{1}{m^4} \frac{a^{(1)}(1) + 7a^{(2)}(1) + 6a^{(3)}(1) + a^{(4)}(1)}{a(1)} \quad (3.14)$$

for any $x \in J$ and $m \in \mathbb{N}$.

Proof. The relation (3.13) results from (3.8). The proof of relations (3.14) follows immediately by differentiating (3.6) with respect to u , and after that take 1 for u and ax for x . $\square$

LEMMA 3.7. For $x \in J$ and $m \in \mathbb{N}$, the following hold

$$(T_0 L_m)(x) = 1, \quad (3.15)$$

$$(T_1 L_m)(x) = -m \left(1 - \frac{a_m}{m} \frac{b^{(1)}(a_m x)}{b(a_m x)} \right) x + \frac{a^{(1)}(1)}{a(1)},$$

$$(T_2 L_m)(x) = -m^2 \left[1 - \left(\frac{a_m}{m} \right)^2 \frac{b^{(2)}(a_m x)}{b(a_m x)} \right] x^2$$

$$+ m^2 \left(1 - \frac{a_m}{m} \frac{b^{(1)}(a_m x)}{b(a_m x)} \right) \left(2x^2 - \frac{1}{m} \frac{a(1) + 2a^{(2)}(1)}{a(1)} x \right)$$

$$+ mx + \frac{a^{(1)}(1) + a^{(2)}(1)}{a(1)}, \quad (3.16)$$

$$(T_4 L_m)(x) = -m^4 \left[1 - \left(\frac{a_m}{m} \right)^4 \frac{b^{(4)}(a_m x)}{b(a_m x)} \right] x^4$$

$$+ m^4 \left[1 - \left(\frac{a_m}{m} \right)^3 \frac{b^{(3)}(a_m x)}{b(a_m x)} \right] \left(4x^4 - \frac{1}{m} \frac{6a(1) + 4a^{(1)}(1)}{a(1)} x^3 \right)$$

$$+ m^4 \left[1 - \left(\frac{a_m}{m} \right)^2 \frac{b^{(2)}(a_m x)}{b(a_m x)} \right] \left(-6x^4 + 4 \frac{1}{m} \frac{3a(1) + 3a^{(1)}(1)}{a(1)} x^3 \right.$$

$$\left. - \frac{1}{m^2} \frac{7a(1) + 18a^{(1)}(1) + 6a^{(2)}(1)}{a(1)} x^2 \right)$$

$$+ m^4 \left(1 - \frac{a_m}{m} \frac{b^{(1)}(a_m x)}{b(a_m x)} \right) \left(4x^4 - 6 \frac{1}{m} \frac{a(1) + 2a^{(1)}(1)}{a(1)} x^3 \right.$$

$$+ 4 \frac{1}{m^2} \frac{a(1) + 6a^{(1)} + 3a^{(2)}(1)}{a(1)} x^2$$

$$\left. - \frac{1}{m^3} \frac{a(1) + 14a^{(1)} + 18a^{(2)}(1) + 4a^{(3)}(1)}{a(1)} x \right)$$

$$+ 3m^2 x^2 + \frac{a(1) + 10a^{(1)}(1) + 6a^{(2)}(1)}{a(1)} mx$$

$$+ \frac{a^{(1)}(1) + 7a^{(2)}(1) + 6a^{(3)}(1) + a^{(4)}(1)}{a(1)}. \quad (3.17)$$

Proof. The proof follows immediately from (3.12) and Lemma 3.6. $\square$

THEOREM 3.8. Let $f : \mathbb{R}_0 \rightarrow \mathbb{R}$ be a function, $f \in E(w)$. If $x \in \mathbb{R}_0$, f is continuous in x , α_2 and $m(0) \in \mathbb{N}$ exist such that

$$1 \leq \alpha_2 < 2 \quad (3.18)$$

and $m^{2-\alpha_2} |1 - (a_m/m)^i (b^{(i)}(a_m x)/b(a_m x))|$ is bounded for any $m \in \mathbb{N}$, $m \geq m(0)$, where $i \in \{1, 2\}$, then

$$\lim_{m \rightarrow \infty} (L_m f)(x) = f(x). \quad (3.19)$$

Assume that f is continuous on $\mathbb{R}_0$ and a compact interval $K \subset \mathbb{R}_0$ exists, such that there exist $m(0) \in \mathbb{N}$ and $l_i(K)$ so that for any $m \in \mathbb{N}$, $m \geq m(0)$, and $x \in K$, one has

$$m^{2-\alpha_2} \left| 1 - \left(\frac{a_m}{m} \right)^i \frac{b^{(i)}(a_mx)}{b(a_mx)} \right| \leq l_i(K), \quad (3.20)$$

where $i \in \{1, 2\}$.

Then, the convergence given in (3.19) is uniform in K and

$$|(L_m f)(x) - f(x)| \leq M(K) \omega \left(f; \frac{1}{\sqrt{m^{2-\alpha_2}}} \right) \quad (3.21)$$

for any $x \in K$ and any $m \geq m(0)$, where $M(K)$ is a constant depending on K .

Proof. Because $m^{2-\alpha_2} |1 - (a_m/m)^i (b^{(i)}(a_mx)/b(a_mx))|$ is bounded for any $m \in \mathbb{N}$, $m \geq m(0)$, it results that $(T_2 L_m)(x)/m^{\alpha_2}$ is bounded for any $m \in \mathbb{N}$, $m \geq m(0)$. Taking relation (3.16) into account, we apply now the Corollary 2.9. The proof is similar on a compact interval K . $\square$

THEOREM 3.9. Let $f : \mathbb{R}_0 \rightarrow \mathbb{R}$ be a function, $f \in E(w)$. If $x \in \mathbb{R}_0$, f is a two times differentiable function in x with $f^{(2)}$ continuous in x , α_2, α_4 and $m(2) \in \mathbb{N}$ exist such that

$$1 \leq \alpha_2 < 2, \quad (3.22)$$

$$2 \leq \alpha_4 < \alpha_2 + 2, \quad (3.23)$$

$m^{4-\alpha_4} |1 - (a_m/m)^i (b^{(i)}(a_mx)/b(a_mx))|$ is bounded for any $m \in \mathbb{N}$, $m \geq m(2)$, where $i \in \{1, 2, 3, 4\}$, then

$$\lim_{m \rightarrow \infty} m^{2-\alpha_2} \left[(L_m f)(x) - f(x) - \frac{1}{m} (T_1 L_m)(x) f^{(1)}(x) - \frac{1}{2m^2} (T_2 L_m)(x) f^{(2)}(x) \right] = 0. \quad (3.24)$$

In addition, if the limit $\lim_{m \rightarrow \infty} ((T_2 L_m)(x)/m^{\alpha_2})$ exists and

$$\lim_{m \rightarrow \infty} \frac{(T_2 L_m)(x)}{m^{\alpha_2}} = B_2(x) \in \mathbb{R}, \quad (3.25)$$

then

$$\lim_{m \rightarrow \infty} m^{2-\alpha_2} \left[(L_m f)(x) - f(x) - \frac{1}{m} (T_1 L_m)(x) f^{(1)}(x) \right] = \frac{1}{2} B_2(x) f^{(2)}(x). \quad (3.26)$$

Assume that f is a two-times differentiable function on $\mathbb{R}_0$ with $f^{(2)}$ continuous on $\mathbb{R}_0$ and a compact interval $K \subset \mathbb{R}_0$ exists, such that there exist $m(2) \in \mathbb{N}$ and $l_i(K)$ so that for any $m \geq m(2)$ and $x \in K$, one has

$$m^{4-\alpha_4} \left| 1 - \left(\frac{a_m}{m} \right)^i \frac{b^{(i)}(a_mx)}{b(a_mx)} \right| \leq l_i(K), \quad (3.27)$$

where $i \in \{1, 2, 3, 4\}$. Then, the convergence given in (3.24) is uniform on K .

Proof. From (3.23), it results that $4 - \alpha_4 > 2 - \alpha_2$, and then we have that $m^{2-\alpha_2} |1 - (a_m/m)^i (b^{(i)}(a_m x)/b(a_m x))|$, $i \in \{1, 2\}$ are bounded for any $m \geq m(2)$. So $(T_2 L_m)(x)/m^{\alpha_2}$ is bounded for any $m \geq m(2)$. Using the same idea from the proof of Theorem 3.8, we have that $(T_2 L_m)(x)/m^{\alpha_2}$ and $(T_4 L_m)(x)/m^{\alpha_4}$ are bounded for any $m \in \mathbb{N}$, $m \geq m(2)$, and then we apply Corollary 2.10. $\square$

Now, we give some applications where $a_m = m$ for any $m \in \mathbb{N}$. In the following, by particularization and applying Theorems 3.8 and 3.9, we can obtain approximation theorems and Voronovskaja-type theorems for some known operators. Because every application is a simple substitute in the theorems of this section, we will not replace anything.

Application 3.10. If $a(x) = 1$ and $b(x) = e^x$, $x \in \mathbb{R}_0$, we obtain the Mirakjan-Favard-Szász operators (see [3–5]).

Application 3.11. If $a(x) = g(x) = \sum_{n=0}^{\infty} a_n x^n$ and $b(x) = e^x$, $x \in \mathbb{R}_0$, we obtain the operators considered by Jakimovski and Leviatan in the paper [1].

Application 3.12. If $a(x) = g(x) = 1$ and $b(x) = \cosh x = \sum_{k=0}^{\infty} (1/(2k)!) x^{2k}$, $x \in \mathbb{R}_0$, then we get the operators considered by Leśniewicz and Rempulska in the paper [6].

Application 3.13. If $a(x) = g(x) = 1$ and $b(x) = \sinh x = \sum_{k=0}^{\infty} (1/(2k+1)!) x^{2k+1}$, $x \in \mathbb{R}_0$, we get the operators

$$(A_m f)(x) = \begin{cases} \frac{1}{\sinh mx} \sum_{k=0}^{\infty} \frac{(mx)^{2k+1}}{(2k+1)!} f\left(\frac{2k+1}{m}\right) & \text{if } x > 0, \\ f(0) & \text{if } x = 0, \end{cases} \quad (3.28)$$

where $m \in \mathbb{N}$ and $x \in \mathbb{R}_0$. The operators of this type are introduced and studied by Rempulska and Skorupka in the paper [7].

Application 3.14. If $a(x) = b(x) = g(x) = \cosh x$, $x \in \mathbb{R}_0$, we obtain the operators studied by Ciupa in [8].

Application 3.15. If $a(x) = g(x) = \sum_{n=0}^{\infty} a_n x^n$ and $b(x) = \cosh x$, $x \in \mathbb{R}_0$, we get the operators constructed by Ciupa in the paper [9], and studied in [9, 10].

Application 3.16. If $a(x) = 1$ and $b(x) = b_m((1/m)x)$, $x \in \mathbb{R}_0$ and $m \in \mathbb{N}$, we obtain the operators studied in the paper [11].

References

- [1] A. Jakimovski and D. Leviatan, "Generalized Szász operators for the approximation in the infinite interval," *Mathematica*, vol. 11 (34), pp. 97–103, 1969.
- [2] O. T. Pop, "About some linear and positive operators defined by infinite sum," *Demonstratio Mathematica*, vol. 39, no. 2, pp. 377–388, 2006.
- [3] J. Favard, "Sur les multiplicateurs d'interpolation," *Journal de Mathématiques Pures et Appliquées*, vol. 23, pp. 219–247, 1944.
- [4] G. M. Mirakjan, "Approximation of continuous functions with the aid of polynomials," *Doklady Akademii Nauk SSSR*, vol. 31, pp. 201–205, 1941 (Russian).
- [5] O. Szász, "Generalization of S. Bernstein's polynomials to the infinite interval," *Journal of Research of the National Bureau of Standards*, vol. 45, pp. 239–245, 1950.

- [6] M. Leśniewicz and L. Rempulska, "Approximation by some operators of the Szász-Mirakjan type in exponential weight spaces," *Glasnik Matematički*, vol. 32(52), no. 1, pp. 57–69, 1997.
- [7] L. Rempulska and M. Skorupka, "Approximation theorems for some operators of the Szász-Mirakjan type in exponential weight spaces," *Publikacije Elektrotehničkog Fakulteta. Univerzitet u Beogradu*, vol. 7, pp. 36–44, 1996.
- [8] A. Ciupa, "Approximation by a generalized Szász type operator," *Journal of Computational Analysis and Applications*, vol. 5, no. 4, pp. 413–424, 2003.
- [9] A. Ciupa, "A positive linear operator for approximation in exponential weight spaces," in *Mathematical Analysis and Approximation Theory. Proceedings of the 5th Romanian-German seminar on approximation theory and Its applications*, pp. 85–96, Burg, Sibiu, Romania, 2002.
- [10] A. Ciupa, "A Voronovskaya-type theorem for a positive linear operator," *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, Article ID 42368, 7 pages, 2006.
- [11] O. T. Pop, "About some linear operators defined by infinite sum," *Analele Științifice ale Universității Ovidius Constanța*, vol. 15, no. 2, 2007.

Ovidiu T. Pop: Department of Mathematics, National College "Mihai Eminescu", 5 Mihai Eminescu Street, 440014 Satu Mare, Romania; Department of Mathematics and Informatics, Vest University "Vasile Goldiș" of Arad, Branch of Satu Mare, 26 Mihai Viteazul Street, 440030 Satu Mare, Romania
 Email address: ovidiutiberiu@yahoo.com