Research Article

On Some Analytic Functions Defined by a Multiplier Transformation

Khalida Inayat Noor

Received 26 July 2007; Accepted 19 November 2007

Recommended by Heinrich Begehr

We introduce and study a new class of analytic functions defined in the unit disc using a certain multiplier transformation. Some inclusion results and other interesting properties of this class are investigated.

Copyright © 2007 Khalida Inayat Noor. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Let $P_k(\eta)$ be the class of functions $p(z)$ analytic in the unit disc $E = \{z : |z| < 1\}$ satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\text{Re} p(z) - \eta}{1 - \eta} \right| d\theta \leq k\pi,$$  \hspace{1cm} (1.1)

where $z = re^{i\theta}$, $k \geq 2$, $0 \leq \eta < 1$. For $\eta = 0$, we obtain the class $P_k$ defined by Pinchuk [1], and for $\eta = 0$, $k = 2$, we have the class $P$ of functions with positive real part, whereas $P_2(\eta) = P(\eta)$ is the class of functions with positive real part greater than $\eta$. We can write (1.1) as

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\eta)ze^{-i\theta}}{1 - ze^{-i\theta}} d\mu(t),$$ \hspace{1cm} (1.2)

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\int_0^{2\pi} d\mu(t) = 2, \quad \int_0^{2\pi} |d\mu(t)| \leq k.$$ \hspace{1cm} (1.3)
We can also write (1.1), for $p \in P_k(\eta)$ in $E$, if and only if
\[ p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z), \quad p_1, p_2 \in P(\eta). \tag{1.4} \]

It is known [2] that the class $P_k(\eta)$ is a convex set. Let $A$ be the class of functions $f$, defined by
\[ f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \tag{1.5} \]
which are analytic in $E$. By $S, K, S^*$, and $C$, we denote the subclasses of $A$ which are univalent, close-to-convex, starlike, and convex in $E$, respectively. The class $A$ is closed under the Hadamard product or convolution:
\[ (f \ast g)(z) = \sum_{m=0}^{\infty} a_m b_m z^{m+1}, \tag{1.6} \]
where
\[ f(z) = \sum_{m=0}^{\infty} a_m z^{m+1}, \quad g(z) = \sum_{m=0}^{\infty} b_m z^{m+1}. \tag{1.7} \]

We define the following.

**Definition 1.1.** Let $f \in A$. Then, for $\alpha, \beta \geq 0$, $0 \leq \eta < \alpha + \beta \leq 1$, $k \geq 2$, and $z \in E$, $f \in Q_k(\alpha, \beta, \eta)$ if and only if
\[ \{ \alpha f'(z) + \beta (zf'(z))' \} \in P_k(\eta). \tag{1.8} \]

We note that, for $\beta = 0$ and $k = 2$, $f' \in P(\eta) \subset P$ for $z \in E$ and this implies that $f$ is univalent in $E$, see [3]. For any real number $s$, the multiplier transformations $I^s_\lambda$ of functions $f \in A$ are defined by
\[ I^s_\lambda f(z) = \sum_{m=2}^{\infty} \left(\frac{s}{1+\lambda}\right)^m a_m z^m \quad (\lambda > -1). \tag{1.9} \]

It is obvious that $I^s_\lambda(I^{s+t}_\lambda f(z)) = I^{s+t}_\lambda f(z)$ for all real numbers $s$ and $t$. The operator $I^s_\lambda$ has been studied by several authors for different choices of $s$ and $\lambda$, see [4–7]. It is worth noting that, for $s$ any nonnegative integer and $\lambda = 0$, the operator $I^s_\lambda$ is the differential operator defined by Sălăgean [8]. Also the operator $I^s_\lambda$ is related rather closely to the multiplier transformation discussed by Flett [9]. Using (1.9) and convolution, function $f^s_\mu$ is defined as follows:
\[ f^s_\mu(z) = \frac{z}{(1-z)^\mu}, \quad z \in E, \mu > 0. \tag{1.10} \]
Motivated essentially by Choi et al. operator [10] and Noor integral operator [11–14], Cho and Kim [15] defined the operator $I_{\lambda,\mu}^s : A \rightarrow A$ as

$$I_{\lambda,\mu}^s f(z) = f_{\lambda,\mu}^s(z) \ast f(z), \quad (1.11)$$

where $s$ is real, $\lambda > -1$, $\mu > 0$, and $f \in A$. In particular, $I_{0,0}^0 f(z) = zf'(z)$, $I_{0,2}^1 f(z) = f(z)$.

From (1.10) and (1.11), we have

$$z(I_{\lambda,\mu}^{s+1} f(z))' = (\lambda + 1)I_{\lambda,\mu}^s f(z) - \lambda I_{\lambda,\mu}^{s+1} f(z), \quad (1.12)$$
$$z(I_{\lambda,\mu}^s f(z))' = \mu I_{\lambda,\mu}^{s+1} f(z) - (\mu - 1)I_{\lambda,\mu}^s f(z). \quad (1.13)$$

We now define the following:

**Definition 1.2.** Let $f \in A$. Then, for $s$ real, $\lambda > 1$, $\mu > 0$,

$$f \in Q_k^s(\lambda, \mu, \alpha, \beta, \eta) \iff I_{\lambda,\mu}^s f(z) \in Q_k(\alpha, \beta, \eta) \text{ for } z \in E. \quad (1.14)$$

2. Preliminary results

**Lemma 2.1.** If $h(z)$ is analytic in $E$ with $h(0) = 1$ and if $\lambda_1$ is a complex number satisfying $\text{Re} \lambda_1 \geq 0$ ($\lambda_1 \neq 0$), then $\{h(z) + \lambda_1 zh'(z)\} \in P_k(\delta_1)$, $0 \leq \delta < 1$, implies $h(z) \in P_k(\delta_1 + (1 - \delta)(2\gamma - 1))$ and

$$\gamma = \int_0^1 (1 + i \text{Re} \lambda_1)^{-1} dt, \quad (2.1)$$

where $\gamma$ is an increasing function of $\text{Re} \lambda_1$ and $1/2 \leq \gamma < 1$. The estimate is sharp.

**Proof.** Let $h(z) = (k/4 + 1/2)h_1(z) - (k/4 - 1/2)h_2(z)$, $h(z)$ is analytic in $E$ with $h(0) = 1$. Then, $h(z) + \lambda_1 zh'(z) = (k/4 + 1/2)[h_1(z) + \lambda_1 zh_1'(z)] - (k/4 - 1/2)[h_2(z) + \lambda_1 zh_2'(z)]$. Since $[h(z) + \lambda_1 zh'(z)] \in P_k(\delta_1)$, we use (1.4) to have $[h_i(z) + \lambda_i zh_i'(z)] \in P(\delta), i = 1, 2$. We now apply a lemma in [16] to conclude that $h_i \in P(\delta_2), i = 1, 2$, and $\delta_1 = \delta + (1 - \delta)(2\gamma - 1)$, where $\gamma$ is given by (2.1) and it is an increasing function of $\text{Re} \lambda_1$ with $1/2 \leq \gamma < 1$. Consequently $h \in P_k(\delta_1)$ in $E$.

**Lemma 2.2** [17]. If $p(z)$ is analytic in $E$ with $p(0) = 1$, then, for any function $F$, analytic in $E$, the function $p \ast F$ takes values in the convex hull of image of $E$ under $F$.

**Lemma 2.3.** Let $\beta_1 < 1$. If the function $p$ is analytic in $E$, with $p(0) = 1$, then $p \in P_k(\beta_2)$, $\beta_2 = (2\beta_1 - 1) + 2(1 - \beta_1)\ln 2$, $z \in E$. This result is sharp.
Proof. The proof is immediate when we use (1.4) and a similar result for the class $P(\beta_2)$ in [18].

**Lemma 2.4.** For $\eta_1 \leq 1$ and $\eta_2 \leq 1$, $P_k(\eta_1) \ast P_k(\eta_2) \subset P_k(1 - 2(1 - \eta_1)(1 - \eta_2))$. This result is sharp.

**Proof.** Let $H \in P_k(\eta_1)$, $p \in P_k(\eta_2)$. Then, using (1.4), we can write
\[
(H \ast p)(z) = \left(\frac{k}{4} + \frac{1}{2}\right) [ (H_1 \ast p_1)(z) ] - \left(\frac{k}{4} - \frac{1}{2}\right) [ (H_2 \ast p_2)(z) ],
\]
(2.2)
\[H_i \in P(\eta_1), \quad p_i \in P(\eta_2), \quad i = 1, 2.
\]
Now using a result from [19], we have, for $i = 1, 2$,
\[
(H_i \ast p_i) \in P(\eta), \quad \eta = 1 - 2(1 - \eta_1)(1 - \eta_2).
\]
(2.3)
This result is shown to be sharp in [19] and consequently $(H \ast p) \in P_k(\eta)$.

### 3. Main results

**Theorem 3.1.** $Q^k_\alpha(\lambda, \mu, \alpha, \beta, \eta) \subset Q^k_\alpha(\lambda, \mu, 1, 0, \sigma)$ for
\[
\sigma = \sigma_1 + (1 - \sigma_1) (2\sigma_2 - 1), \quad \sigma_1 = \frac{\eta}{\alpha + \beta},
\]
\[
\sigma_2 = \int_0^1 (1 + t^{\beta/(\alpha + \beta)})^{-1} dt, \quad \text{with} \quad \frac{1}{2} \leq \sigma_2 \leq 1.
\]
(3.1)

**Proof.** Let $f \in Q^k_\alpha(\lambda, \mu, \alpha, \beta, \eta)$. Then, by definition it follows that
\[
\{ \alpha (I^\alpha_{\lambda, \mu} f)' + \beta (z (I^\alpha_{\lambda, \mu} f))' - \eta \} = P_k(\eta), \quad z \in E.
\]
(3.2)
Set $(I^\alpha_{\lambda, \mu} f(z))' = p(z)$. Then $p$ is analytic in $E$ with $p(0) = 1$ and for $z \in E$,
\[
\left\{ \frac{\alpha (I^\alpha_{\lambda, \mu} f(z))' + \beta (z (I^\alpha_{\lambda, \mu} f(z))') - \eta}{\alpha + \beta - \eta} \right\} = \left\{ \frac{\alpha + \beta}{\alpha + \beta - \eta} p(z) + \frac{\beta}{\alpha + \beta - \eta} z p'(z) - \frac{\eta}{\alpha + \beta - \eta} \right\} \in P_k.
\]
(3.3)
From (1.4) and (3.4), we have, for $i = 1, 2$,
\[
\left[ \frac{\alpha + \beta}{\alpha + \beta - \eta} p_i(z) + \frac{\beta}{\alpha + \beta - \eta} z p'_i(z) - \frac{\eta}{\alpha + \beta - \eta} \right] = h_i(z) \in P.
\]
(3.4)
By putting $\sigma_1 = \eta/(\alpha + \beta)$, we see that
\[
p_i(z) + \frac{\beta}{\alpha + \beta} z p'_i(z) = (1 - \sigma_1) h_i(z) + \sigma_1 = H_i(z) \in P(\sigma_1).
\]
(3.5)
Now using Lemma 2.1, we obtain $p_i \in P(\sigma)$, where $\sigma$ is given by (3.1). Therefore, $(I^\alpha_{\lambda, \mu} f)' \in P_k(\sigma)$ and consequently $f \in Q^k_\alpha(\lambda, \mu, 1, 0, \sigma)$ in $E$. □
Remark 3.2. By writing $\sigma_1 = \eta/(\alpha + \beta)$, $\alpha_1 = \alpha/(\alpha + \beta)$, we can deduce from Definition 1.2 that $f \in Q_k^\lambda(\lambda, \mu, \alpha, \beta, \eta)$, if and only if, for $0 \leq \alpha_1 \leq 1$,

$$
\left[ \alpha_1 (I_{\lambda, \mu}^i f)' + (1 - \alpha_1) (z(I_{\lambda, \mu}^i f)' \right]' \right] \in P_k(\sigma_1), \quad z \in E. \tag{3.6}
$$

In this case, we say that $f \in Q_k^\lambda(\lambda, \mu, \alpha_1, \sigma_1)$ in $E$.

Theorem 3.3. Let $s$ be real, $\lambda > -1$, $\mu > 0$. Then,

$$
Q_k^\lambda(\lambda, \mu + 1, \alpha_1, \sigma_1) \subset Q_k^\lambda(\lambda, \mu, \alpha_1, \delta_1) \subset Q_k^{\lambda + 1}(\lambda, \mu, \alpha_1, \delta_2), \tag{3.7}
$$

where $\alpha_1$ and $\sigma_1$ are as defined in Remark 3.2 and

$$
\delta_1 = \sigma_1 + (1 - \sigma_1)(2\eta_1 - 1), \quad \eta_1 = \int_0^1 (1 + t^{1/\mu})^{-1} dt, \tag{3.8}
$$

$$
\delta_2 = \delta_1 + (1 - \delta_1)(2\eta_2 - 1), \quad \eta_2 = \int_0^1 (1 + t^{1/(\lambda + 1)})^{-1} dt. \tag{3.9}
$$

Proof. We first show that $Q_k^\lambda(\lambda, \mu + 1, \alpha_1, \sigma_1) \subset Q_k^\lambda(\lambda, \mu, \alpha_1, \delta_1)$.

Let $f \in Q_k^\lambda(\lambda, \mu + 1, \alpha_1, \sigma_1)$ and set

$$
p(z) = \alpha_1 [ (I_{\lambda, \mu}^i f(z))' ] + (1 - \alpha_1) [(zI_{\lambda, \mu}^i f(z)')']. \tag{3.10}
$$

From (1.13) and (3.10), we have, for $z \in E$,

$$
\left\{ \alpha_1 (I_{\lambda, \mu+1}^i f(z))' + (1 - \alpha_1) (z(I_{\lambda, \mu+1}^i f(z))')' \right\} \in P_k(\sigma_1) \tag{3.11}
$$

and, on using (1.4), it follows that $\text{Re} \{ p_i(z) + (1/\mu)zp_i'(z) \} > \sigma_1$, $z \in E$, $i = 1, 2$.

Now, applying Lemma 2.1, we have $\text{Re} \ p_i(z) > \delta_1$, $i = 1, 2$, where $\delta_1$ is given by (3.8).

This implies $p \in P_k(\delta_1)$ for $z \in E$ and hence $f \in Q_k^\lambda(\lambda, \mu, \alpha_1, \delta_1)$ in $E$. To prove $Q_k^\lambda(\lambda, \mu, \alpha_1, \delta_1) \subset Q_k^{\lambda + 1}(\lambda, \mu, \alpha_1, \delta_2)$, we proceed as follows. Set

$$
\left\{ \alpha_1 (I_{\lambda, \mu+1}^{i+1} f(z))' + (1 - \alpha_1) (z(I_{\lambda, \mu+1}^{i+1} f(z))')' \right\} = h(z). \tag{3.12}
$$

Then, using (1.12), we have

$$
\left\{ \alpha_1 (I_{\lambda, \mu}^i f(z))' + (1 - \alpha_1) (z(I_{\lambda, \mu}^i f(z))')' \right\} = \left\{ h(z) + \frac{1}{\lambda + 1} z h'(z) \right\} \in P_k(\delta_1). \tag{3.13}
$$

With similar argument as detailed above, we obtain the required result. \hfill \square

Theorem 3.4. The class $Q_k^\lambda(\lambda, \mu, \alpha_1, \sigma_1)$ is closed under the convolution with a convex function. That is, if $f \in Q_k^\lambda(\lambda, \mu, \alpha_1, \sigma_1)$ and $\phi \in C$ for $z \in E$, then $(\phi * f) \in Q_k^\lambda(\lambda, \mu, \alpha_1, \sigma_1)$.  

Proof. Let \( f \in Q_k^\alpha(\lambda, \mu, \alpha_1, \sigma_1) \). Consider
\[
\alpha_1(I_{\lambda, \mu}^i(\phi \ast f)(z))' + (1 - \alpha_1)(z(I_{\lambda, \mu}^i(\phi \ast f)(z))')'
\]
\[
= \alpha_1(I_{\lambda, \mu}^i(z)(\phi \ast f)(z))' + (1 - \alpha_1)(z(f(I_{\lambda, \mu}^i(\phi \ast f)(z)))')'
\]
\[
= \alpha_1(\phi(z) \ast f(I_{\lambda, \mu}^i(z) \ast f(z)))' + (1 - \alpha_1)(z(\phi(z) \ast f(I_{\lambda, \mu}^i(z) \ast f(z)))')'
\]
\[
= \frac{\phi(z)}{z} \{ \alpha_1(I_{\lambda, \mu}^i f(z))' + (1 - \alpha_1)(z(I_{\lambda, \mu}^i f(z))') \}
\]
\[
= \left( \frac{k}{4} + \frac{1}{2} \right) \left[ \frac{\phi(z)}{z} \ast h_1(z) \right] - \left( \frac{k}{4} - \frac{1}{2} \right) \left[ \frac{\phi(z)}{z} \ast h_2(z) \right],
\]
where \( \phi(z)/z \in P(1/2) \) and \( h_i \in P(\sigma_1) \). Using Lemma 2.2, we see that \( [(\phi(z)/z) \ast h_i(z)] \in P(\sigma_1) \) and consequently \( h \in P_k(\sigma_1) \), which implies that \( \phi \ast f \in Q_k^\alpha(\lambda, \mu, \alpha_1, \sigma_1) \); the proof is complete. \( \square \)

Corollary 3.5. The class \( Q_k^\alpha(\lambda, \mu, \alpha_1, \sigma_1) \) is invariant under the following integral operators:
(i) \( f_1(z) = \int_0^z (f(t)/t) \, dt \),
(ii) \( f_2(z) = (2/z) \int_0^z f(t) \, dt \) (Libera’s operator [20]),
(iii) \( f_3(z) = \int_0^z (f(t) - f(xt)/(t - xt)) \, dt, |x| \leq 1, x \neq 1 \),
(iv) \( f_4(z) = ((1 + c)/z^c) \int_0^z t^c - 1 f(t) \, dt, \Re c > 0 \).

One may write (see [21, 22])
\[
f_1(z) = f(z) \ast \phi_1(z), \quad f_2(z) = f(z) \ast \phi_2(z),
\]
\[
f_3(z) = f(z) \ast \phi_3(z), \quad f_4(z) = f(z) \ast \phi_4(z),
\]
where \( \phi_i, i = 1, 2, 3, 4 \), are convex and
\[
\phi_1(z) = -\log(1 - z) = \sum_{n=1}^\infty \frac{1}{n} z^n,
\]
\[
\phi_2(z) = -2 \left[ \frac{z + \log(1 - z)}{z} \right] = \sum_{n=1}^\infty \frac{2}{n+1} z^n,
\]
\[
\phi_3(z) = \frac{1}{1 - x} \log \left[ \frac{1 - xz}{1 - z} \right] = \sum_{n=1}^\infty \frac{1 - x^n}{(1 - x)^n} z^n, \quad |x| \leq 1, x \neq 1,
\]
\[
\phi_4(z) = \sum_{n=1}^\infty \frac{1 + c}{n + c} z^n, \quad \Re c > 0.
\]

Now, the result follows by applying Theorem 3.4. Let \( \mu_1 \) and \( \mu_2 \) be linear operators defined on the class \( S \) as follows:
\[
\mu_1(f(z)) = zf'(z), \quad \mu_2(f(z)) = \frac{[f(z) + zf'(z)]}{2} \quad \text{(Livingston’s operator [23])}.
\]
(3.17)
Then, both of these operators can be written as a convolution operator [21], given by
\[ \mu_i(f) = h_i * f, \quad i = 1, 2, \]
where
\[
\begin{align*}
 h_1(z) &= \sum_{n=1}^{\infty} nz^n = \frac{z}{1-z}, \\
 h_2(z) &= \sum_{n=1}^{\infty} \frac{n+1}{2} z^n = \frac{z-2z^2/2}{(1-z)^2}.
\end{align*}
\tag{3.18}
\]
It can easily be verified that the radius of convexity \( r_c(h_1) = 2 - \sqrt{3} \) and \( r_c(h_2) = 1/2 \).

These facts together with Theorem 3.4 yield the following.

**Theorem 3.6.** Let \( f \in Q_k^\varepsilon(\lambda, \mu, \alpha_1, \sigma_1) \). Then,
\[
\begin{align*}
 \mu_1(f) &= (f * h_1) \in Q_k^\varepsilon(\lambda, \mu, \alpha_1, \sigma_1), \quad \text{for } |z| < 2 - \sqrt{3}, \\
 \mu_2(f) &= (f * h_2) \in Q_k^\varepsilon(\lambda, \mu, \alpha_1, \sigma_1), \quad \text{for } |z| < \frac{1}{2}.
\end{align*}
\tag{3.19}
\]

**Theorem 3.7.** Let \( 0 \leq \alpha_1 < \alpha_2 \). Then, \( Q_k^\varepsilon(\lambda, \mu, \alpha_1, \sigma_1) \subset Q_k^\varepsilon(\lambda, \mu, \alpha_2, \sigma_1) \).

**Proof.** If \( \alpha_1 = 0 \), the result is obvious. Therefore, we assume that \( \alpha_1 > 0 \) and \( f \in Q_k^\varepsilon(\lambda, \mu, \alpha_2, \sigma_1) \). Let \( (I_{\lambda, \mu}^i f(z))' = H_1(z) \). Then, by Theorem 3.1, \( H_1 \in P_k(\sigma_1) \). Also, let
\[
\left\{ \alpha_1(I_{\lambda, \mu}^i f(z))' + (1 - \alpha_1)(z(I_{\lambda, \mu}^i f(z)))' \right\} = H_2(z), \quad H_2 \in P_k(\sigma_1) \text{ in } E.
\tag{3.20}
\]
Now,
\[
\alpha_2(I_{\lambda, \mu}^i f(z))' + (1 - \alpha_2)(z(I_{\lambda, \mu}^i f(z)))' = \frac{\alpha_2 - \alpha_1}{(1 - \alpha_1)} H_1(z) + \frac{1 - \alpha_2}{(1 - \alpha_1)} H_2(z)
\]
\[
= \frac{(\alpha_2 - \alpha_1)}{(1 - \alpha_1)} H_1(z) + \left(1 - \frac{\alpha_2 - \alpha_1}{(1 - \alpha_1)} \right) H_2(z).
\tag{3.21}
\]
Since \( H_1, H_2 \in P_k(\sigma_1) \) and \( P_k(\sigma_1) \) is a convex set, see [2], we obtain the required result. \( \square \)

**Theorem 3.8.** Let \( f_i \in Q_k^\varepsilon(\lambda, \mu, \alpha_1, \zeta_i), \quad i = 1, 2, \) and let \( \Psi = f_1 * f_2 \). Then, \( \Psi(z)/z \in Q_k^\varepsilon(\lambda, \mu, 1, \zeta_i) \) for \( z \in E \), where \( \zeta_i = 1 - \delta(1 - \delta_1)(1 - \delta_2)(\ln 2 - 1)^2 \) and
\[
\delta_i = \zeta_i + (1 - \zeta_i)(2m - 1).
\tag{3.22}
\]

**Proof.** Since \( f_i \in Q_k^\varepsilon(\lambda, \mu, \alpha_1, \zeta_i) \), it follows from Theorem 3.1 that \( f_i \in Q_k^\varepsilon(\lambda, \mu, 1, \delta_i) \), \( \delta_i = \zeta_i + (1 - \zeta_i)(2m - 1) \), and
\[
m = \int_0^1 (1 + t^{1-a})^{-1} \, dt.
\tag{3.23}
\]
Now,
\[
(z(I_{\lambda, \mu}^i \Psi(z))')' = I_{\lambda, \mu}^i \left[ (\Psi'(z) + z\Psi''(z)) \right] = (z(I_{\lambda, \mu}^i (f_1 * f_2)(z))')'
\]
\[
= I_{\lambda, \mu}^i \left[ (f_1'(z) * f_2'(z)) \right] = (I_{\lambda, \mu}^i f_1(z))' * (I_{\lambda, \mu}^i f_2(z))'.
\tag{3.24}
\]
Since \( f_i \in Q_k^s(\lambda, \mu, 1, \delta_i) \), it follows, by Lemma 2.4, that \( \{ \Psi'(z) + z\Psi''(z) \} \in Q_k^s(\lambda, \mu, 1, \delta) \), where
\[
\delta = 1 - 2(1 - \delta_1)(1 - \delta_2).
\] (3.25)

From (3.25) and Lemma 2.3, we have
\[
\Psi'(z) \in Q_k^s(\lambda, \mu, 1, \{1 + 4(1 - \delta_1)(1 - \delta_2)(\ln 2 - 1)\}).
\] (3.26)

From (3.26) and Lemma 2.3, again, we have
\[
\frac{\Psi(z)}{z} \in Q_k^s(\lambda, \mu, 1, \{1 - \delta(1 - \delta_1)(1 - \delta_2)(\ln 2 - 1)^2\}), \quad z \in E.
\] (3.27)

We now consider the converse case of Theorem 3.1 as follows. \(\Box\)

**Theorem 3.9.** Let \( f \in Q_k^s(\lambda, \mu, 1, \sigma) \). Then, \( f \in Q_k^s(\lambda, \mu, \alpha_1, \sigma), 0 < \alpha_1 \leq 1, \) for \( |z| < r_{\alpha_1} \) \( \alpha_1 \neq 1/2 \), where
\[
r_{\alpha_1} = \frac{1}{2(1 - \alpha_1) + \sqrt{4\alpha_1^2 - 6\alpha_1 + 3}}.
\] (3.28)

This result is sharp.

**Proof.** Let \( \phi_{\alpha_1}(z) = \alpha_1(I_{\lambda, \mu}^s f(z))' + (1 - \alpha_1)(z(I_{\lambda, \mu}^s f(z))')' \). Then,
\[
\phi_{\alpha_1}(z) = \frac{k_{\alpha_1}(z)}{z} \ast (I_{\lambda, \mu}^s f(z))', \quad \text{where } k_{\alpha_1}(z) = \alpha_1 \frac{z}{1 - z} + (1 - \alpha_1) \frac{z}{(1 - z)^2}.
\] (3.29)

It is known [23] that the function \( k_{\alpha_1} \) is convex for \( |z| < r_{\alpha_1} \), where \( r_{\alpha_1} \) is given by (3.28) and this radius is sharp and consequently, for \( |z| < r_{\alpha_1} \), by a well-known result, \( k_{\alpha_1} \in P(1/2) \). Thus, using Lemma 2.2, and the given fact that \( f \in Q_k^s(\lambda, \mu, 1, \sigma) \), we obtain the required result. \(\Box\)

**Acknowledgment**

The author would like to thank Dr. S. M. Junaid Zaidi, (Rector CIIT), who generously supports scientific research in all aspects.

**References**


Khalida Inayat Noor: Mathematics Department, COMSATS Institute of Information Technology, Sector H-8/1, Islamabad 44000, Pakistan

*Email addresses: khalidanoor@hotmail.com; khalidainayat@comsats.edu.pk*
Submit your manuscripts at
http://www.hindawi.com