Research Article
On Sectional Curvatures of ($\epsilon$)-Sasakian Manifolds

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We obtain some basic results for Riemannian curvature tensor of ($\epsilon$)-Sasakian manifolds and then establish equivalent relations among $\phi$-sectional curvature, totally real sectional curvature, and totally real bisectional curvature for ($\epsilon$)-Sasakian manifolds.

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1. Introduction

The index of a metric plays significant roles in differential geometry as it generates variety of vector fields such as space-like, time-like, and light-like fields. With the help of these vector fields, we establish interesting properties on ($\epsilon$)-Sasakian manifolds, which was introduced by Bejancu and Duggal [1] and further investigated by Xufeng and Xiaoli [2]. Since Sasakian manifolds with indefinite metrics play crucial roles in physics [3], hence the study of these manifolds becomes the central theme in present scenario. Here the next section is concerned with the basic results of Riemannian curvature tensor of ($\epsilon$)-Sasakian manifolds. In Section 3, these results will be used to obtain the equivalent relations among $\phi$-sectional curvature, totally real sectional curvature, and totally real bisectional curvature. In [1], authors defined the ($\epsilon$)-Sasakian manifold as follows.

Let $M$ be a real $(2n + 1)$-dimensional differentiable manifold endowed with an almost contact structure $(\phi, \eta, \xi)$, where $\phi$ is a tensor field of type $(1, 1)$, $\eta$ is a 1-form, and $\xi$ is a vector field on $M$ satisfying

$$\phi^2X = -X + \eta(X)\xi, \quad \eta(\xi) = 1.$$ (1.1)

It follows that

$$\eta(\phi X) = 0, \quad \phi(\xi) = 0, \quad \text{rank } \phi = 2n;$$ (1.2)
then \( M \) is called an almost contact manifold. If there exists a semi-Riemannian metric \( g \) satisfying
\[
g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y) \quad \forall X, Y \in \chi(X),
\]
where \( \epsilon = \pm 1 \), then \( (\phi, \eta, \xi, g) \) is called an \( (\epsilon) \) almost contact metric structure and \( M \) is known as an \( (\epsilon) \) almost contact manifold.

For an \( (\epsilon) \) almost contact manifold we also have
\[
\eta(X) = \epsilon g(X, \xi) \quad \forall X \in \chi(X),
\]
\[
\epsilon = g(\xi, \xi),
\]
which hence \( \xi \) is never a light-like vector field on \( M \), and according to the casual character of \( \xi \), we have two classes of \( (\epsilon) \)-Sasakian manifolds. When \( \epsilon = -1 \) and the index of \( g \) is an odd number \( (v = 2s + 1) \), then \( M \) is a time-like Sasakian manifold and \( M \) is a space-like Sasakian manifold when \( \epsilon = -1 \) and \( v = 2s \). For \( \epsilon = 1 \) and \( v = 0 \), we obtain usual Sasakian manifold and for \( \epsilon = 1 \) and \( v = 1 \), \( M \) is a Lorentz-Sasakian manifold.

If \( d\eta(X, Y) = g(\phi X, Y) \), then \( M \) is said to have \( (\epsilon) \)-contact metric structure \( (\phi, \eta, \xi, g) \). If, moreover, this structure is normal, that is, if
\[
[\phi X, \phi Y] + \epsilon^2 [X, Y] - \phi[X, \phi Y] - \phi[\phi X, Y] = -2d\eta(X, Y)\xi,
\]
then the \( (\epsilon) \)-contact metric structure is called an \( (\epsilon) \)-Sasakian structure, and manifold endowed with this structure is called an \( (\epsilon) \)-Sasakian manifold.

Now, let \( \sigma \) be a plane section in tangent space \( T_p(M) \) at a point \( p \) of \( M \), and let it be spanned by vectors \( X \) and \( Y \), then the sectional curvature of \( \sigma \) is given by
\[
K(X, Y) = \frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}.
\]

A plane \( \{X, Y\} \), where \( X \) and \( Y \) are orthonormal to \( \xi \) and satisfy \( \phi(\{X, Y\}) \perp \{X, Y\} \), is called totally real section, and sectional curvature associated with this section is called a totally real sectional curvature. The totally real bisectional curvature \( B(X, Y) \) is defined as
\[
B(X, Y) = R(X, \phi X, Y, \phi Y),
\]
where \( \eta(X) = \eta(Y) = g(X, Y) = g(X, \phi Y) = 0 \).

A plane section \( \{X, \phi X\} \), where \( X \) is orthonormal to \( \xi \), is called \( \phi \)-section, and the curvature associated with this is called \( \phi \)-sectional curvature which is denoted by \( H(X) \), where
\[
H(X) = K(X, \phi X) = R(X, \phi X, X, \phi X).
\]
If a Sasakian manifold $M$ has constant $\phi$-sectional curvature $c$, then it is called a Sasakian space form and denoted by $M^{2n+1}(c)$.

### 2. Riemannian curvature tensor

**Theorem 2.1 [1].** An $(\epsilon)$ almost contact metric structure $(\phi, \eta, \xi, g)$ is $(\epsilon)$-Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \epsilon \eta(Y)X, \quad \forall X, Y \in \chi(M), \quad (2.1)$$

where $\nabla$ is the Levi-Civita connection with respect to $g$. Also one has

$$\nabla_X \xi = -\epsilon \phi X, \quad \forall X \in \chi(M). \quad (2.2)$$

For an $(\epsilon)$-Sasakian manifold, using (2.1) we have

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (2.3)$$

where $R$ denotes the Riemannian curvature tensor on $M$, and also from above we have

$$R(X, \xi)Y = -\epsilon g(X, Y)\xi + \eta(Y)X. \quad (2.4)$$

Using (2.1) and (2.2), we have

$$R(X, Y)\phi Z = \phi R(X, Y)Z + \epsilon \{g(Z, \phi X)Y - g(Z, \phi Y)X + g(X, Z)\phi Y - g(Y, Z)\phi X\}. \quad (2.5)$$

And by using (2.5), we obtain the following set of equations:

$$R(X, Y)Z = -\phi R(X, Y)\phi Z + \epsilon \{g(Y, Z)X - g(X, Z)Y + g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X\}, \quad (2.6)$$

$$g(R(X, Y)\phi Z, \phi W) = g(R(X, Y)Z, W)$$

$$+ \epsilon \{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)$$

$$- g(\phi Z, X)g(\phi W, Y) + g(\phi Z, Y)g(\phi W, X)\}, \quad (2.7)$$

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = g(R(X, Y)Z, W) + \eta(W)\eta(Y)g(X, Z)$$

$$- \eta(W)\eta(X)g(Y, Z) + \eta(Z)\eta(X)g(Y, W)$$

$$- \eta(Z)\eta(Y)g(X, W). \quad (2.8)$$

Now, we can write (2.5) as

$$g(R(X, Y)\phi Z, W) = g(\phi R(X, Y)Z, W)$$

$$+ \epsilon \{g(Z, \phi X)g(Y, W) - g(Z, \phi Y)g(X, W)$$

$$+ g(X, Z)g(\phi Y, W) - g(Y, Z)g(\phi, W)\}.$$
Replacing $Y$ to $\xi$ and using (2.8), we have

$$g(R(X, Y)\phi Z, W) = g(\phi R(X, Y)Z, W) - \epsilon P(X, Y; Z, W),$$  \hspace{1cm} (2.10)

where

$$P(X, Y; Z, W) = g(Y, Z)g(\phi X, W) - g(\phi X, Z)g(Y, W) + g(\phi Y, Z)g(X, W) - g(X, Z)g(\phi Y, W).$$  \hspace{1cm} (2.11)

Clearly $P(X, Y; Z, W) = -P(Z, W; X, Y)$, and if $\{X, Y\}$ is an orthonormal pair orthogonal to $\xi$, and if we set $g(\phi X, Y) = \cos \theta, 0 \leq \theta \leq \pi$, then

$$P(X, Y; X, \phi Y) = -\sin^2 \theta.$$  \hspace{1cm} (2.12)

If we put $D(X) = Q(X, \phi X)$ for any vector $X$ orthogonal to $\xi$ and $Q(X, Y) = g(R(X, Y)Y, X)$ for any vectors $X$ and $Y$, then we have the following lemma.

**Lemma 2.2.** For any vectors $X$ and $Y$ orthogonal to $\xi$, one obtains

$$Q(X, Y) = \frac{1}{32} \left[ 3D(X + \phi Y) + 3D(X - \phi Y) - D(X + Y) - D(X - Y) - 4D(X) - 4D(Y) - 24\epsilon P(X, Y; X, \phi Y) \right].$$  \hspace{1cm} (2.13)

**Proof.** For $X, Y$ orthogonal to $\xi$, we have

$$D(X + Y) + D(X - Y) = 2 \left[ D(X) + D(Y) + 2R(X, \phi X, Y, \phi Y) + 2R(X, \phi Y, Y, \phi X) + R(X, \phi Y, X, \phi Y) + R(Y, \phi X, Y, \phi X) \right],$$  \hspace{1cm} (2.14)

and using (2.8), we have

$$R(\phi X, \phi Y, \phi X, \phi Y) = R(X, Y, X, Y),$$

$$R(X, \phi Y, X, \phi Y) = R(Y, \phi X, Y, \phi X).$$  \hspace{1cm} (2.15)

Substituting (2.15) in (2.14), we get

$$D(X + Y) + D(X - Y) = 2 \left[ D(X) + D(Y) + 2R(X, \phi X, Y, \phi Y) + 2R(X, \phi Y, Y, \phi X) + 2Q(X, \phi Y) \right].$$  \hspace{1cm} (2.16)

Replacing $Y$ by $\phi Y$ in (2.16), we get

$$D(X + \phi Y) + D(X - \phi Y) = 2 \left[ D(X) + D(Y) - 2R(X, \phi X, \phi Y, Y) - 2R(X, Y, \phi Y, \phi X) + 2Q(X, Y) \right].$$  \hspace{1cm} (2.17)
Using (2.16) and (2.17), we have
\[3D(X + \phi Y) + 3D(X - \phi Y) - D(X + Y) - D(X - Y) - 4D(X) - 4D(Y)\]
\[= 12Q(X, Y) - 4Q(X, \phi Y) + 8R(X, \phi X, Y, \phi Y) + 12R(X, Y, \phi X, \phi Y)\]  \hspace{1cm} (2.18)
\[+ R(X, \phi Y, \phi X, Y).\]

Replacing \(W\) by \(\phi X\) and \(Z\) by \(Y\) in (2.9), we have
\[R(X, Y, \phi X, \phi Y) = R(X, Y, X, Y) + \epsilon P(X, Y; X, \phi Y).\]  \hspace{1cm} (2.19)

Again replacing \(Y\) by \(\phi Y\), \(W\) by \(Y\), and \(Z\) by \(X\) in (2.9), we have
\[R(X, \phi Y, Y, \phi X) = R(X, \phi Y, X, \phi Y) + \epsilon P(X, Y; X, \phi Y).\]  \hspace{1cm} (2.20)

By using Bianchi’s first identity (2.19) and (2.20), we have
\[R(X, \phi X, Y, \phi Y) = Q(X, Y) + Q(X, \phi Y) + 24\epsilon P(X, Y; X, \phi Y).\]  \hspace{1cm} (2.21)

Thus using the last four equations, we have the result. \(\square\)

Now, it should be noted that \(D(X) = H(X)\) if and only if \(X\) is a unit vector, and \(Q(X, Y) = K(X, Y)\) if and only if \(\{X, Y\}\) is an orthonormal pair. Then, as an application of lemma, we have the following lemma.

**Lemma 2.3.** Let \(\{X, Y\}\) be an orthonormal pair of the tangent space of an \((\epsilon)\)-Sasakian manifold \(M\) orthogonal to \(\xi\). If one puts \(g(X, \phi Y) = \cos \theta, 0 \leq \theta \leq \pi\), then
\[K(X, Y) = \frac{1}{8} \left\{ 3(1 + \cos \theta)^2 H \left( \frac{X + \phi Y}{|X + \phi Y|} \right) \right.\]
\[+ 3(1 - \cos \theta)^2 H \left( \frac{X - \phi Y}{|X - \phi Y|} \right) - H \left( \frac{X + Y}{|X + Y|} \right)\]
\[\left. - H \left( \frac{X - Y}{|X - Y|} \right) - H(X) - H(Y) + 6\epsilon \sin^2 \theta \right\}.\]  \hspace{1cm} (2.22)

**Proof.** It follows from Lemma (2.2).

Since the \(\phi\)-sectional curvature determines the curvature of a Sasakian manifold, then it can be easily verified that if the \(\phi\)-sectional curvature \(H(X)\) is independent of the choice of a vector \(X\) at any point and has value \(c\), then \(c\) is constant on \(M\) and the curvature tensor
\[ R(X,Y,Z,W) = \frac{(c+3\epsilon)}{4} \{ g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \} \]
\[ + \frac{(c-\epsilon)}{4} \{ \eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \]
\[ + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z) \]
\[ + g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) \]
\[ + 2g(X, \phi Y)g(\phi Z, W) \]. \] (2.23)

Now, our next aim of this paper is as follows.

**Theorem 2.4.** Let \((M^{2n+1}, \phi, \eta, \xi)\) be an \((\epsilon)\)-Sasakian manifold of dimension \(\geq 7\), then the following relations are equivalent.

(i) \(M\) has constant \(\phi\)-sectional curvature \(c\); that is, \(H(X)\) is constant.

(ii) \(M\) has constant totally real sectional curvature; that is, for any totally real section \(\{X, Y\}\), \(K(X, Y)\) is constant.

(iii) \(M\) has constant totally real bisectional curvature; that is, \(B(X, Y)\) is constant.

3. **Proof of the main Theorem 2.4**

In the proof, we assume that \(X, Y,\) and \(Z\) are unit vector fields.

If \(H(X)\) is constant and equal to \(c\), then for a totally real section \(\{X, Y\}\), (2.23) gives \(K(X, Y) = -(c+3\epsilon)/4\) and \(B(X, Y) = -(c+7\epsilon)/2\); this gives (i) \(\Rightarrow\) (ii) and (i) \(\Rightarrow\) (iii) respectively.

Now, let \(\{X, Y\}\) be a totally real section, then \(\{(X+Y)/\sqrt{2},(-\phi X+\phi Y)/\sqrt{2}\}\) is also a totally real section, and assume that \(M\) has constant totally real sectional curvature (say \(k\)); then

\[ K\left(\frac{X+Y}{\sqrt{2}}, \frac{-\phi X+\phi Y}{\sqrt{2}}\right) = k; \] (3.1)

this gives

\[ 4k = H(X) + H(Y) + K(X, \phi Y) + K(Y, \phi X) - 4R(X, \phi Y, Y, \phi X) - 2R(X, Y, \phi X, \phi Y), \] (3.2)

or

\[ H(X) + H(Y) = 8k + 6. \] (3.3)

Since the dimension of \(M\) is \((2n+1), n = 3\), therefore there exists a unit vector \(Z\) orthonormal to \(\{X, Y\}\) such that

\[ H(X) + H(Z) = 8k + 6. \] (3.4)
Therefore, using (3.3) and (3.4), we conclude that

\[ H(X) = H(Y). \]  

(3.5)

Thus, we have (ii) \( \Rightarrow \) (i).

Next, we prove that (iii) \( \Rightarrow \) (i).

Since

\[ B(X, Y) = R(X, \phi X, Y, \phi Y), \]  

(3.6)

where \( \eta(X) = \eta(Y) = g(X, Y) = g(X, \phi Y) = 0 \), then using (2.19) and (2.20), we have

\[ B(X, Y) = K(X, Y) + K(X, \phi Y) - 2\epsilon. \]  

(3.7)

Now, let \( M \) have constant totally real bisectional curvature (say \( t \)), then

\[ K(X, Y) + K(X, \phi Y) = t + 2\epsilon. \]  

(3.8)

Also \( \{(X + Y)/\sqrt{2}, (-\phi X + \phi Y)/\sqrt{2}\} \) is a totally real section for a totally real section \( \{X, Y\} \) then

\[ B\left( \frac{X + Y}{\sqrt{2}}, \frac{-\phi X + \phi Y}{\sqrt{2}} \right) = t; \]  

(3.9)

this gives

\[ H(X) + H(Y) + 2R(X, \phi X, Y, \phi Y) - 4R(X, \phi Y, X, \phi Y) = 4t - 2\epsilon, \]  

(3.10)

or

\[ H(X) + H(Y) - 4K(X, \phi Y) = 2t - 2\epsilon. \]  

(3.11)

Replacing \( Y \) by \( \phi Y \), we get

\[ H(X) + H(Y) - 4K(X, Y) = 2t - 2\epsilon. \]  

(3.12)

Using (3.8) in addition to (3.11) and (3.12), we have

\[ H(X) + H(Y) = 4t + 2\epsilon. \]  

(3.13)

Since there can exist a unit vector \( Z \) orthogonal to \( \{X, Y\} \), then

\[ H(X) + H(Z) = 4t + 2\epsilon. \]  

(3.14)

Using (3.13) and (3.14), we have

\[ H(X) = H(Y). \]  

(3.15)

Hence, the result is given.
References


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