Research Article

A Connection between $C^\infty(\mathbb{T}^n)$ and $\mathcal{S}(\mathbb{R}^n)$

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We interpret $C^\infty(\mathbb{T}^n)$ as a quotient space of $\mathcal{S}(\mathbb{R}^n)$.

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In measure-theoretic sense, the n -torus $\mathbb{T}^n$ is the cube $[0, 1]^n$ with Lebesgue measure. A function f in $C^\infty(\mathbb{R}^n)$ is said to be in $C^\infty(\mathbb{T}^n)$ if $f(x + m) = f(x)$ for all $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}^n$. $\mathcal{S}(\mathbb{R}^n)$ denotes the space of rapidly decreasing functions.

Given $f \in L^1(\mathbb{R}^n)$, we denote its Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \quad \xi \in \mathbb{R}^n. \quad (1)$$

Given $f \in L^1(\mathbb{T}^n)$, we denote its Fourier coefficients by

$$\tilde{f}(m) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i m \cdot x} dx, \quad m \in \mathbb{Z}^n. \quad (2)$$

We have $\sup_{m \in \mathbb{Z}^n} |\tilde{f}(m)| \leq \|f\|_{L^1(\mathbb{T}^n)}$.

LEMMA 1. Suppose that $f, \hat{f}$ are in $L^1(\mathbb{R}^n)$, then it can be assumed that f and $\hat{f}$ are both continuous since they can be expressed in terms of each other via Fourier inversion. If they satisfy

$$|f(x)| + |\hat{f}(x)| \leq C(1 + |x|)^{-n-\delta} \quad (3)$$

for some $C, \delta > 0$, then

$$\sum_{m \in \mathbb{Z}^n} \hat{f}(m) e^{2\pi i m \cdot x} = \sum_{m \in \mathbb{Z}^n} f(x + m), \quad (4)$$

for all $x \in \mathbb{R}^n$, and in particular,

$$\sum_{m \in \mathbb{Z}^n} \hat{f}(m) = \sum_{m \in \mathbb{Z}^n} f(m). \quad (5)$$

(See [1, Theorem 3.1.17].)

LEMMA 2. Let $s \in \mathbb{Z}$ with $s \geq 0$, suppose that f is in $C^s(\mathbb{T}^n)$, then

$$|\tilde{f}(m)| \leq c_{n,s} \frac{\max(\|f\|_{L^1(\mathbb{T}^n)}, \sup_{|\alpha|=s} |\widetilde{\partial^\alpha f}(m)|)}{(1 + |m|)^s}, \quad (6)$$

for some constant $c_{n,s}$.

(See [1, Corollary 3.2.10].)

We are in the position to get the following theorem.

THEOREM 3. If ϕ is in $\mathcal{F}(\mathbb{R}^n)$ and

$$g(x) = \sum_{m \in \mathbb{Z}^n} \phi(x + m), \quad (7)$$

then $g \in C^\infty(\mathbb{T}^n)$. Conversely, for every $g \in C^\infty(\mathbb{T}^n)$, there exists $\phi \in \mathcal{F}(\mathbb{R}^n)$ such that

$$g(x) = \sum_{m \in \mathbb{Z}^n} \phi(x + m). \quad (8)$$

Proof. The proof of the first part is trivial.

Now assume that $g \in C^\infty(\mathbb{T}^n)$ and set

$$G(x) = \sum_{m \in \mathbb{Z}^n} \tilde{g}(m) \mathcal{X}_{B(m,\lambda)}(x), \quad (9)$$

where $B(m,\lambda) = \{x \in \mathbb{R}^n : |x - m| < \lambda\}$, $0 < \lambda < 2/5$, and $\mathcal{X}_{B(m,\lambda)}$ denotes the characteristic function of $B(m,\lambda)$.

According to Lemma 2, for all positive integers N , we have

$$|\tilde{g}(m)| \leq c_{n,N} \frac{\max(\|g\|_{L^1(\mathbb{T}^n)}, \sup_{|\alpha|=N} |\widetilde{\partial^\alpha g}(m)|)}{(1 + |m|)^N} \quad (10)$$

$$\leq c_{n,N} \frac{\max(\|g\|_{L^1(\mathbb{T}^n)}, \sup_{|\alpha|=N} \|\partial^\alpha g\|_{L^1(\mathbb{T}^n)})}{(1 + |m|)^N}. \quad (11)$$

So, it is easily seen that $G(x) \in L^1(\mathbb{R}^n)$.

Set

$$k(x) = \begin{cases} ce^{1/(|x|^2-1)}, & |x| \leq 1, \\ 0, & |x| > 1, \end{cases} \quad (12)$$

where c is a constant such that $\int_{\mathbb{R}^n} k(x) dx = 1$.

For $\varepsilon > 0$, set $k_\varepsilon(x) = \varepsilon^{-n} k(\varepsilon^{-1}x)$, and denote

$$G_1(x) = (G * k_{\lambda/4})(x). \quad (13)$$

Then by the property of convolution, $G_1 \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha G_1 = G * \partial^\alpha k_{\lambda/4}$.

Also, since $\partial^\gamma k_{\lambda/4}(y)$ is continuous and supported in $B(0, \lambda/4)$. So for any multi-index γ and nonnegative integer N ,

$$\begin{aligned} (1 + |x|)^N |\partial^\gamma G_1(x)| &= (1 + |x|)^N \left| \int_{\mathbb{R}^n} G(x - y) \partial^\gamma k_{\lambda/4}(y) dy \right| \\ &\leq C(1 + |x|)^N \sup_{y \in B(0, \lambda/4)} |G(x - y)| \\ &\leq C(1 + |m|)^N |\tilde{g}(m)|, \end{aligned} \quad (14)$$

here m is the only point with integer coordinates that is in $B(x, 5\lambda/4)$ (if there is one such m , otherwise $(1 + |x|)^N |\partial^\gamma G_1(x)|$ is 0). C depends only on γ and N . So by (11), G_1 is in $\mathcal{S}(\mathbb{R}^n)$.

And

$$G_1(m) = \int_{B(0, \lambda/4)} G(m - y) k_{\lambda/4}(y) dy = G(m) \int_{B(0, \lambda/4)} k_{\lambda/4}(y) dy = G(m) = \tilde{g}(m). \quad (15)$$

Suppose that ϕ is the function in $\mathcal{S}(\mathbb{R}^n)$ such that $\hat{\phi} = G_1$. Clearly, ϕ and G_1 satisfy the conditions of Lemma 1, and so we have

$$g(x) = \sum_{m \in \mathbb{Z}^n} \tilde{g}(m) e^{2\pi i m \cdot x} = \sum_{m \in \mathbb{Z}^n} G_1(m) e^{2\pi i m \cdot x} = \sum_{m \in \mathbb{Z}^n} \phi(x + m). \quad (16)$$

□

$C^\infty(\mathbb{T}^n)$ is generally topologized by the family of seminorms

$$\rho_\alpha(f) = \sup_x |\partial^\alpha f(x)|, \quad (17)$$

where α ranges over all multi-indices. In this topology, $\phi_j \rightarrow \phi$ means

$$\sup_x |\partial^\alpha \phi_j(x) - \partial^\alpha \phi(x)| \longrightarrow 0 \quad (18)$$

for all multi-indices α . $C^\infty(\mathbb{T}^n)$ is a Fréchet space and it can be regarded as a quotient space of $\mathcal{S}(\mathbb{R}^n)$ up to isomorphism of topological vector spaces.

THEOREM 4. Set

$$H = \left\{ \phi \in \mathcal{S}(\mathbb{R}^n) : \sum_{m \in \mathbb{Z}^n} \phi(x+m) \equiv 0 \right\}, \quad (19)$$

then H is a closed subspace of $\mathcal{S}(\mathbb{R}^n)$, and there is a linear one-to-one correspondence between the quotient space $\mathcal{S}(\mathbb{R}^n)/H$ and $C^\infty(\mathbb{T}^n)$ which is a homomorphism.

Proof. It is easy to see that H is closed in $\mathcal{S}(\mathbb{R}^n)$.

Define $\Lambda : \mathcal{S}(\mathbb{R}^n)/H \rightarrow C^\infty(\mathbb{T}^n)$ by

$$\Lambda(\phi + H) = \sum_{m \in \mathbb{Z}^n} \phi(x+m). \quad (20)$$

It is obvious that Λ is well defined, linear, one-to-one, and onto. It remains to prove that Λ is continuous and open.

If d is an invariant metric on $\mathcal{S}(\mathbb{R}^n)$ compatible with its topology, then

$$\rho(\phi + H, \varphi + H) = \inf \{d(\phi - \varphi, \psi) : \psi \in H\} \quad (21)$$

defines an invariant metric on $\mathcal{S}(\mathbb{R}^n)/H$ which is compatible with the quotient topology.

Suppose $\phi_j + H \rightarrow \phi + H$ ($j \rightarrow \infty$) in the quotient topology of $\mathcal{S}(\mathbb{R}^n)/H$, we have

$$\rho(\phi_j + H, \phi + H) = \inf \{d(\phi_j - \phi, \psi) : \psi \in H\} \rightarrow 0, \quad (j \rightarrow \infty). \quad (22)$$

For each j , there is $\psi_j \in H$ such that

$$d(\phi_j - \phi, \psi_j) \leq 2 \inf \{d(\phi_j - \phi, \psi) : \psi \in H\}. \quad (23)$$

So,

$$\lim_{j \rightarrow \infty} d(\phi_j - \psi_j, \phi) = \lim_{j \rightarrow \infty} d(\phi_j - \phi, \psi_j) = 0. \quad (24)$$

That is, $\phi_j - \psi_j \rightarrow \phi$ ($j \rightarrow \infty$) in $\mathcal{S}(\mathbb{R}^n)$. Hence, it is easy to see that

$$\lim_{j \rightarrow \infty} \sum_{m \in \mathbb{Z}^n} (\phi_j(x+m) + \psi_j(x+m)) = \lim_{j \rightarrow \infty} \sum_{m \in \mathbb{Z}^n} \phi_j(x+m) = \sum_{m \in \mathbb{Z}^n} \phi(x+m) \quad (25)$$

in the topology of $C^\infty(\mathbb{T}^n)$.

That is,

$$\lim_{j \rightarrow \infty} \Lambda(\phi_j + H) = \Lambda(\phi + H), \quad (26)$$

so Λ is continuous.

Since both $\mathcal{S}(\mathbb{R}^n)/H$ and $C^\infty(\mathbb{T}^n)$ are F -spaces, Λ is also open, by the open mapping theorem. This completes the proof. $\square$

The elements of the dual space $\mathcal{D}'(\mathbb{T}^n)$ of $C^\infty(\mathbb{T}^n)$ are called distributions on $\mathbb{T}^n$. The above result may shed some light on the relation between $\mathcal{D}'(\mathbb{T}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, the space of

tempered distributions on $\mathbb{R}^n$. For example, for every $u \in \mathcal{D}'(\mathbb{T}^n)$, $u \circ \Lambda \circ \pi$ is in $\mathcal{S}'(\mathbb{R}^n)$, where $\pi(\phi) = \phi + H$ is the quotient mapping from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)/H$. Hence, $\mathcal{D}'(\mathbb{T}^n)$ can be imbedded into $\mathcal{S}'(\mathbb{R}^n)$ in a natural way.

References

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