Research Article

On $\pi$-Images of Locally Separable Metric Spaces

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We characterize $\pi$-images of locally separable metric spaces by means of covers having $\pi$-property. As its application, we obtain characterizations of compact-covering (sequence-covering, pseudo-sequence-covering, and sequentially quotient) $\pi$-images of locally separable metric spaces.

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1. Introduction

To determine what spaces are the images of “nice” spaces under “nice” mappings is one of the central questions of general topology in [1]. In the past, many noteworthy results on images of metric spaces have been obtained. For a survey in this field, see [2], for example. A characterization for a quotient compact image of a locally separable metric space is obtained in [3]. Also, such a quotient image is precisely a pseudo-sequence-covering quotient compact image of a locally separable metric space [4]. Recently, $\pi$-images of metric spaces cause attention once again in [5, 6]. It is known that a space is a compact-covering (resp., sequence-covering, pseudo-sequence-covering, sequentially-quotient) $\pi$-image of a metric space if and only if it has a point-star network consisting of $\text{cfp}$-covers (resp., $\text{cs}$-covers, $\text{wcs}$-covers, $\text{cs}^*$-covers) [5–7]. In a personal communication, the first author of [6] informs that it seems to be difficult to obtain “nice” characterizations of “nice” images of locally separable metric spaces (instead of metric or locally compact metric domains). Thus, we are interested in the following question.

Question 1.1. How are compact-covering (resp., sequence-covering, pseudo-sequence-covering, sequentially-quotient) $\pi$-images of locally separable metric spaces characterized?

In this paper, we characterize compact-covering (resp., sequence-covering, pseudo-sequence-covering, sequentially-quotient) $\pi$-images of locally separable metric spaces by means of $\text{cfp}$-covers (resp., $\text{cs}$-covers, $\text{wcs}$-covers, $\text{cs}^*$-covers) for compact subsets (resp., convergent sequences) in a space and covers having $\pi$-property to answer Question 1.1
completely. As applications of these results, we get characterizations on quotient \( \pi \)-images of locally separable metric spaces.

Throughout this paper, all spaces are assumed to be regular and \( T_1 \), all mappings are assumed continuous and onto, a convergent sequence includes its limit point, \( \mathbb{N} \) denotes the set of all natural numbers. Let \( f : X \rightarrow Y \) be a mapping, \( x \in X \), and \( \mathcal{P} \) be a collection of subsets of \( X \), we denote \( \text{st}(x, \mathcal{P}) = \bigcup \{ P \in \mathcal{P} : x \in P \} \), \( \mathcal{P}_x = \{ P \in \mathcal{P} : x \in P \} \), and \( f(\mathcal{P}) = \{ f(P) : P \in \mathcal{P} \} \). We say that a convergent sequence \( \{ x_n : n \in \mathbb{N} \} \cup \{ x \} \) converging to \( x \) is eventually (resp., frequently) in \( A \) if \( \{ x_n : n \geq n_0 \} \cup \{ x \} \subset A \) for some \( n_0 \in \mathbb{N} \) (resp., \( \{ x_{n_k} : k \in \mathbb{N} \} \cup \{ x \} \subset A \) for some subsequence \( \{ x_{n_k} : k \in \mathbb{N} \} \) of \( \{ x_n : n \in \mathbb{N} \} \)). For terms which are not defined here, please refer to [2, 8].

Let \( \mathcal{P} \) be a collection of subsets of a space \( X \), and \( K \) be a subset of \( X \).

\( \mathcal{P} \) is a cover for \( K \) in \( X \), if \( K \subset \bigcup \mathcal{P} \). When \( K = X \), a cover for \( K \) in \( X \) is a cover of \( X \) [8]. A cover \( \mathcal{P} \) for \( X \) is a compact cover if all members of \( \mathcal{P} \) are compact.

For each \( x \in X \), \( \mathcal{P} \) is a network at \( x \) if \( x \in P \) for every \( P \in \mathcal{P} \), and if \( x \in U \) with \( U \) open in \( X \), there exists \( P \in \mathcal{P} \) such that \( x \in P \subset U \).

\( \mathcal{P} \) is a k-cover for \( K \) in \( X \), if for each compact subset \( H \) of \( K \) there exists a finite subfamily \( \mathcal{F} \) of \( \mathcal{P} \) such that \( H \subset \bigcup \mathcal{F} \). When \( K = X \), a k-cover for \( K \) in \( X \) is a k-cover for \( X \).

\( \mathcal{P} \) is a cf-p-cover for \( K \) in \( X \), if for each compact subset \( H \) of \( K \) there exists a finite subfamily \( \mathcal{F} \) of \( \mathcal{P} \) such that \( H \subset \bigcup \{ P \cap F : P \in \mathcal{F} \} \), where \( C_F \) is closed and \( C_F \subset F \) for every \( F \in \mathcal{F} \).

Note that such an \( \mathcal{F} \) is a full cover in the sense of [9]. When \( K = X \), a cf-p-cover for \( K \) in \( X \) is a cf-p-cover for \( X \) [10].

\( \mathcal{P} \) is a cs-cover for \( K \) in \( X \) (resp., cs*-cover for \( K \) in \( X \)), if for each convergent sequence \( S \) in \( K, S \) is eventually (resp., frequently) in some \( P \in \mathcal{P} \). When \( K = X \), a cs-cover for \( K \) in \( X \) (resp., cs*-cover for \( K \) in \( X \)) is a cs-cover for \( X \) [11] (resp., cs*-cover for \( X \) [12]), or a cover satisfying condition \((c_3)\) (resp., \((c_2)\)) [5].

\( \mathcal{P} \) is a wcs-cover for \( K \) in \( X \) if for each convergent sequence \( S \) converging to \( x \) in \( K \) there exists a finite subfamily \( \mathcal{F} \) of \( \mathcal{P}_x \) such that \( S \) is eventually in \( \bigcup \mathcal{F} \). When \( K = X \), a wcs-cover for \( K \) in \( X \) is a wcs-cover [7].

It is clear that if \( \mathcal{P} \) is a cover (resp., k-cover, cf-p-cover, cs-cover, wcs-cover, cs*-cover), then \( \mathcal{P} \) is a cover (resp., k-cover, cf-p-cover, cs-cover, wcs-cover, cs*-cover) for \( K \) in \( X \).

Remark 1.2. (1) Closed k-cover for \( K \) in \( X \) \( \Rightarrow \) cf-p-cover for \( K \) in \( X \) \( \Rightarrow \) k-cover for \( K \) in \( X \);

(2) cf-p-cover for \( K \) in \( X \), or cs-cover for \( K \) in \( X \) \( \Rightarrow \) wcs-cover for \( K \) in \( X \) \( \Rightarrow \) cs*-cover for \( K \) in \( X \).

For each \( n \in \mathbb{N} \), let \( \mathcal{P}_n \) be a cover for \( X \). \( \{ \mathcal{P}_n : n \in \mathbb{N} \} \) is a refinement sequence for \( X \) if \( \mathcal{P}_{n+1} \) is a refinement of \( \mathcal{P}_n \) for each \( n \in \mathbb{N} \). A refinement sequence for \( X \) is a refinement of \( X \) in the sense of [4].

Let \( \{ \mathcal{P}_n : n \in \mathbb{N} \} \) be a refinement sequence for \( X \). \( \{ \mathcal{P}_n : n \in \mathbb{N} \} \) is a point-star network for \( X \), if \( \{ \text{st}(x, \mathcal{P}_n) : n \in \mathbb{N} \} \) is a network at \( x \) for each \( x \in X \). Note that this notion is used without the assumption of a refinement sequence in [13], and in [5] \( \bigcup \{ \mathcal{P}_n : n \in \mathbb{N} \} \) is a \( \sigma \)-strong network for \( X \).

A cover \( \{ X_\lambda : \lambda \in \Lambda \} \) for \( X \) is called to have \( \pi \)-property if each \( X_\lambda \) has a refinement sequence \( \{ \mathcal{P}_{\lambda,n} : n \in \mathbb{N} \} \) of countable covers for \( X_\lambda \), and for each \( x \in U \) with \( U \) open in \( X \), there is \( n \in \mathbb{N} \) such that

\[
\bigcup \{ \text{st}(x, \mathcal{P}_{\lambda,n}) : \lambda \in \Lambda \text{ with } x \in X_\lambda \} \subset U.
\]
Let \( \mathcal{P}_n : n \in \mathbb{N} \) be a point-star network for a space \( X \). For every \( n \in \mathbb{N} \), put \( \mathcal{P}_n = \{ P_a : a \in A_n \} \), and \( A_n \) is endowed with discrete topology. Put

\[
M = \left\{ a = (a_n) \in \prod_{n \in \mathbb{N}} A_n : \{ P_{a_n} : n \in \mathbb{N} \} \text{ forms a network at some point } x_a \text{ in } X \right\}. \tag{1.2}
\]

Then \( M \), which is a subspace of the product space \( \prod_{n \in \mathbb{N}} A_n \), is a metric space with metric \( d \) described as follows.

Let \( a = (a_n), \ b = (\beta_n) \in M \). If \( a = b \), then \( d(a, b) = 0 \). If \( a \neq b \), then \( d(a, b) = 1/\left( \min\{n \in \mathbb{N} : a_n \neq \beta_n \} \right) \).

Define \( f : M \to X \) by choosing \( f(a) = x_a \), then \( f \) is a mapping, and \( (f, M, X, \{ \mathcal{P}_n \}) \) is a Pomomarev's system [14], and if without the assumption of a refinement sequence in the notion of point-star networks, then \( (f, M, X, \{ \mathcal{P}_n \}) \) is a Pomomarev's system in the sense of [13].

Let \( f : X \to Y \) be a mapping.

\( f \) is a compact-covering mapping [15], if every compact subset of \( Y \) is the image of some compact subset of \( X \).

\( f \) is a sequence-covering mapping [16], if every convergent sequence of \( Y \) is the image of some convergent sequence of \( X \).

\( f \) is a subsequence-covering mapping [3], if for every convergent sequence \( S \) of \( Y \), there is a compact subset \( K \) of \( X \) such that \( f(K) \) is a subsequence of \( S \).

\( f \) is a sequentially-quotient mapping [17], if for every convergent sequence \( S \) of \( Y \), there is a convergent sequence \( L \) of \( X \) such that \( f(L) \) is a subsequence of \( S \).

\( f \) is a pseudo-open mapping [1], if \( y \in \text{int } f(U) \) whenever \( f^{-1}(y) \subseteq U \) with \( U \) open in \( X \).

\( f \) is a \( \pi \)-mapping [1], if for every \( y \in Y \) and for every neighborhood \( U \) of \( y \) in \( Y \), \( d(f^{-1}(y), X - f^{-1}(U)) > 0 \), where \( X \) is a metric space with a metric \( d \).

Let \( X \) be a space. We recall that \( X \) is sequential [18], if a subset \( A \) of \( X \) is closed if and only if any convergent sequence in \( A \) has a limit point in \( A \). Also, \( X \) is Fréchet (or Fréchet Urysohn) if for each \( x \in \overline{A} \), there exists a sequence in \( A \) converging to \( x \).

Note that, for a mapping \( f : X \to Y \), \( f \) is compact-covering or sequence-covering \( \Rightarrow \) \( f \) is pseudo-sequence-covering \( \Rightarrow \) \( f \) is subsequence-covering [4]. Also, \( f \) is quotient if and only if \( f \) is subsequence-covering for \( Y \) being sequential [3].

2. Main results

Lemma 2.1. Let \( \mathcal{P} \) be a countable cover for a convergent sequence \( S \) in a space \( X \). Then the following are equivalent:

1. \( \mathcal{P} \) is a \( \text{cfp-cover} \) for \( S \) in \( X \);
2. \( \mathcal{P} \) is a \( \text{tocs-cover} \) for \( S \) in \( X \);
3. \( \mathcal{P} \) is a \( \text{cs-cover} \) for \( S \) in \( X \).

Proof. \((1) \Rightarrow (2) \Rightarrow (3) \). By Remark 1.2.

\((3) \Rightarrow (1) \). Let \( H \) be a compact subset of \( S \). We can assume that \( H \) is a subsequence of \( S \). Since \( \mathcal{P} \) is countable, put \( \mathcal{P}_x = \{ P_n : n \in \mathbb{N} \} \), where \( x \) is the limit point of \( S \). Then \( H \) is eventually
Lemma 2.2. Let \( f : X \to Y \) be a mapping.

1. If \( \mathcal{D} \) is a \( k \)-cover for a compact set \( K \) in \( X \), then \( f(\mathcal{D}) \) is a \( k \)-cover for \( f(K) \) in \( Y \).
2. If \( \mathcal{D} \) is a cs\(^*\)-cover for a convergent sequence \( S \) in \( X \), then \( f(\mathcal{D}) \) is a cs\(^*\)-cover for \( f(S) \) in \( Y \).

Same as [6, Lemma 2.2(2)(ii)], we get the following.

Lemma 2.3. Let \( (f, M, X, \{\mathcal{D}_n\}) \) be a Ponomarev's system. For a convergent sequence \( S \) of \( X \), if \( \mathcal{D}_n \) is a cs-cover for \( S \) in \( X \) for each \( n \in \mathbb{N} \), then there exists a convergent sequence \( L \) of \( M \) such that \( f(L) = S \).

Proof. Put \( S = \{x_i : i \in \mathbb{N}\} \cup \{x\} \), where \( x \) is the limit point. For each \( n \in \mathbb{N} \), since \( \mathcal{D}_n \) is a cs-cover for \( S \), \( S \) is eventually in some \( P_n \in \mathcal{D}_n \). For each \( i \in \mathbb{N} \), if \( x_i \in P_m \), let \( a_{n,i} = a_a \); if \( x_i \notin P_n \), pick \( a_{n,i} \in A_n \) such that \( x_i \in P_{m+i} \). Thus there exists \( i_n \in \mathbb{N} \) such that \( a_{n,i_n} = a_a \) for all \( i > i_n \). So \( \{a_{n,i} : i \in \mathbb{N}\} \) converges to \( a_a \). For each \( i \in \mathbb{N} \), put \( a^i = (a_{n,i}) \in \prod_{n \in \mathbb{N}} A_n \), and put \( a = (a^i) \in \prod_{n \in \mathbb{N}} A_n \). Then \( a^i \in f^{-1}(x_i) \), \( a \in f^{-1}(x) \), and \( L = \{a^i : i \in \mathbb{N}\} \cup \{a\} \) converges to \( a \). It implies that \( L \) is a convergent sequence of \( M \) and \( f(L) = S \).

Proposition 2.4. The following are equivalent for a space \( X \).

1. \( X \) is a \( \pi \)-image of a locally separable metric space,
2. \( X \) has a cover \( \{X_\lambda : \lambda \in \Lambda\} \) having \( \pi \)-property.

Proof. (1)\( \Rightarrow \) (2). Let \( f : M \to X \) be a \( \pi \)-mapping from a locally separable metric space \( M \) with metric \( d \) onto \( X \). Since \( M \) is a locally separable metric space, \( M = \bigoplus_{\lambda \in \Lambda} M_\lambda \), where each \( M_\lambda \) is a separable metric space by [8, 4.4.F]. For each \( \lambda \in \Lambda \), let \( D_\lambda \) be a countable dense subset of \( M_\lambda \), and put \( f_\lambda = f|_{M_\lambda} \) and \( X_\lambda = f_\lambda(M_\lambda) \). For each \( a \in M_\lambda \) and \( n \in \mathbb{N} \), put \( B(a, 1/n) = \{b \in M_\lambda : d(a, b) < 1/n\} \), \( B_{\lambda,n} = \{B(a, 1/n) : a \in D_\lambda\} \), and \( D_{\lambda,n} = f_\lambda(B_{\lambda,n}) \). It is clear that \( \{D_{\lambda,n} : n \in \mathbb{N}\} \) is a sequence of countable covers for \( X_\lambda \), and \( D_{\lambda,n+1} \) is a refinement of \( D_{\lambda,n} \) for every \( n \in \mathbb{N} \). We will prove that \( \pi \)-property is satisfied.

For each \( x \in U \) with \( U \) open in \( X \). Since \( f \) is a \( \pi \)-mapping, \( d(f^{-1}(x), M - f^{-1}(U)) > 2/n \) for some \( n \in \mathbb{N} \). Then, for each \( \lambda \in \Lambda \) with \( x \in X_\lambda \), we get \( d(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) > 2/n \), where \( U_\lambda = U \cap X_\lambda \). Let \( a \in D_\lambda \) and \( x \in f_\lambda(B(a, 1/n)) \) \( D_{\lambda,n} \). We will prove that \( B(a, 1/n) \subset f_\lambda^{-1}(U_\lambda) \). In fact, if \( B(a, 1/n) \notin f_\lambda^{-1}(U_\lambda) \), then pick \( b \in B(a, 1/n) - f_\lambda^{-1}(U_\lambda) \). Note that \( f_\lambda^{-1}(x) \cap B(a, 1/n) \neq \emptyset \), pick \( c \in f_\lambda^{-1}(x) \cap B(a, 1/n) \), then \( d(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \leq d(c, b) \leq d(c, a) + d(a, b) < 2/n \). It is a contradiction. So \( B(a, 1/n) \subset f_\lambda^{-1}(U_\lambda) \), thus \( f_\lambda(B(a, 1/n)) \subset U_\lambda \). Then \( s(t(x, D_{\lambda,n}) \subset U_\lambda \). It implies that \( \bigcup_{t(x, D_{\lambda,n}) : \lambda \in \Lambda \text{ with } x \in X_\lambda} \subset U \).

(2)\( \Rightarrow \) (1). For each \( \lambda \in \Lambda \), let \( x \in U_\lambda \) with \( U_\lambda \) open in \( X_\lambda \). We get that \( U_\lambda = U \cap X_\lambda \) with some \( U \) open in \( X \). Since \( \bigcup_{t(x, D_{\lambda,n})} \subset U \) for some \( n \in \mathbb{N} \), \( s(x, D_{\lambda,n}) \subset U_\lambda \). It implies that \( \{D_{\lambda,n} : n \in \mathbb{N}\} \) is a point-star network for \( X_\lambda \). Then the Ponomarev’s system \( (f_\lambda, M_\lambda, X_\lambda, \{D_{\lambda,n}\}) \) exists. Since each \( D_{\lambda,n} \) is countable, \( M_\lambda \) is a separable metric space with
metric $d_1$ described as follows. For $a = (\alpha_n)$, $b = (\beta_n) \in M_1$, if $a = b$, then $d_1(a, b) = 0$, and if $a \neq b$, then $d_1(a, b) = 1/(\min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\})$.

Put $M = \bigoplus_{\lambda \in \Lambda} M_1$ and define $f : M \to X$ by choosing $f(a) = f_1(a)$ for every $a \in M_1$ with some $\lambda \in \Lambda$. Then $f$ is a mapping and $M$ is a locally separable metric space with metric $d$ defined as follows. For $a, b \in M$, if $a, b \in M_1$ for some $\lambda \in \Lambda$, then $d(a, b) = d_1(a, b)$, and otherwise, $d(a, b) = 1$.

We will prove that $f$ is a $\pi$-mapping. Let $x \in U$ with $U$ open in $X$, then $\bigcup \{\text{st}(x, \rho_{\lambda, n}) : \lambda \in \Lambda, x \in X_1\} \subset U$ for some $n \in \mathbb{N}$. So, for each $\lambda \in \Lambda$ with $x \in X_1$, we get $\text{st}(x, \rho_{\lambda, n}) \subset U_1$, where $U_1 = U \cap X_1$. It is implies that $d_1(f^{-1}_1(x), M_1 - f^{-1}_1(U_1)) \geq 1/n$. In fact, if $a = (\alpha_k) \in M_1$ such that $d_1(f^{-1}_1(x), \alpha_k) < 1/n$, then there is $b = (\beta_k) \in f^{-1}_1(x)$ such that $d_1(a, b) < 1/n$. So $\alpha_k = \beta_k$ if $k \leq n$. Note that $x \in P_{\lambda, n} \subset \text{st}(x, \rho_{\lambda, n}) \subset U_1$. Then $f_1(a) \in P_{\lambda, n} \subset \text{st}(x, \rho_{\lambda, n}) \subset U_1$. Hence $a \in f^{-1}_1(U_1)$. It implies that $d_1(f^{-1}(x), a) \geq 1/n$ if $a \in M_1 - f^{-1}_1(U_1)$. So $d_1(f^{-1}(x), M_1 - f^{-1}_1(U_1)) \geq 1/n$. Therefore

$$d(f^{-1}(x), M - f^{-1}(U)) = \inf \{d(a, b) : a \in f^{-1}(x), b \in M - f^{-1}(U)\}$$

$$= \min \left\{1, \inf \left\{d_1(a, b) : a \in f^{-1}_1(x), b \in M_1 - f^{-1}_1(U_1), \lambda \in \Lambda\right\}\right\} (2.1) \geq 1/n > 0.$$ 

It implies that $f$ is a $\pi$-mapping.

**Theorem 2.5.** The following are equivalent for a space $X$.

1. $X$ is a compact-covering $\pi$-image of a locally separable metric space;
2. $X$ has a cover $\{X_{\lambda} : \lambda \in \Lambda\}$ having $\pi$-property satisfying that for each compact subset $K$ of $X$, there is a finite subset $\Lambda_K$ of $\Lambda$ such that $K$ has a finite compact cover $\{K_{\lambda} : \lambda \in \Lambda_K\}$, and for each $\lambda \in \Lambda_K$ and $n \in \mathbb{N}$, $\rho_{\lambda, n}$ is a $k$-cover for $K_{\lambda}$ in $X_1$;
3. same as (2), but replace the prefix "$k$-" by "$\text{cfp}$-".

**Proof.** (1)$\Rightarrow$(2). By using notations and arguments in the proof (1)$\Rightarrow$(2) of Proposition 2.4 again, $X$ has a cover $\{X_{\lambda} : \lambda \in \Lambda\}$ having $\pi$-property. For each compact subset $K$ of $X$, since $f$ is compact-covering, $K = f(L)$ for some compact subset $L$ of $M$. By compactness of $L$, $L_1 = L \cap M_1$ is compact and $\Lambda_K = \{\lambda \in \Lambda : L_1 \neq \emptyset\}$ is finite. For each $\lambda \in \Lambda_K$, put $K_{\lambda} = f(L_1)$, then $\{K_{\lambda} : \lambda \in \Lambda_K\}$ is a finite compact cover for $K$. For each $n \in \mathbb{N}$, since $\beta_{1, n}$ is a $k$-cover for $L_1$ in $M_1$, $\rho_{\lambda, n}$ is a $k$-cover for $K_{\lambda}$ in $X_1$ by Lemma 2.2.

(2)$\Rightarrow$(3). For each $\lambda \in \Lambda_K$ and $n \in \mathbb{N}$, since $\rho_{\lambda, n}$ is countable, every member of $\rho_{\lambda, n}$ can be chosen closed in $X_1$. Thus, $\rho_{\lambda, n}$ is a $\text{cfp}$-cover for $K_{\lambda}$ in $X_1$ by Remark 1.2.

(3)$\Rightarrow$(1). By using notations and arguments in the proof (2)$\Rightarrow$(1) of Proposition 2.4 again, $X$ is a $\pi$-image of a locally separable metric space. It suffices to prove that the mapping $f$ is compact-covering.

For each compact subset $K$ of $X$, there is a finite subset $\Lambda_K$ of $\Lambda$ such that $K$ has a finite compact cover $\{K_{\lambda} : \lambda \in \Lambda_K\}$, and for each $\lambda \in \Lambda_K$ and $n \in \mathbb{N}$, $\rho_{\lambda, n}$ is a $\text{cfp}$-cover for $K_{\lambda}$ in $X_1$. It follows from [13, Lemma 13] that $K_{\lambda} = f_1(L_1)$ with some compact subset $L_1$ of $M_1$. Put $L = \bigcup \{L_1 : \lambda \in \Lambda_K\}$, then $L$ is a compact subset of $M$ and $f(L) = K$. It implies that $f$ is compact-covering.
Theorem 2.6. The following are equivalent for a space $X$.

1. $X$ is a pseudo-sequence-covering $\pi$-image of a locally separable metric space;
2. $X$ has a cover $\{X_\lambda : \lambda \in \Lambda\}$ having $\pi$-property satisfying that for each convergent sequence $S$ of $X$, there is a finite subset $\Lambda_S$ of $\Lambda$ such that $S$ has a finite compact cover $\{S_\lambda : \lambda \in \Lambda_S\}$, and for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, $D_{\lambda,n}$ is a wcs-cover for $S_\lambda$ in $X_{\lambda,n}$;
3. same as (2), but replace the prefix “wcs” by “cs”.

Proof. Using Lemma 2.1, notations and arguments in the proof of Theorem 2.5, here “pseudo-sequence-covering” and “convergent sequence” play the roles of “compact-covering” and “compact subset”, respectively.

Theorem 2.7. The following are equivalent for a space $X$.

1. $X$ is a sequence-covering $\pi$-image of a locally separable metric space;
2. $X$ has a cover $\{X_\lambda : \lambda \in \Lambda\}$ having $\pi$-property satisfying that for each convergent sequence $S$ of $X$, there is $\lambda \in \Lambda$ such that $S$ is eventually in $X_{\lambda,n}$, and for each $n \in \mathbb{N}$, $D_{\lambda,n}$ is a cs-cover for $S \cap X_{\lambda}$ in $X_{\lambda,n}$.

Proof. $(1) \Rightarrow (2)$. By using notations and arguments in the proof $(1) \Rightarrow (2)$ of Proposition 2.4 again, $X$ has a cover $\{X_\lambda : \lambda \in \Lambda\}$ having $\pi$-property. For each convergent sequence $S$ of $X$, since $f$ is sequence-covering, $S = f(L)$ for some convergent sequence $L$ of $M$. Note that $L$ is eventually in some $M_1$. Thus, $S$ is eventually in $X_{\lambda}$ with some $\lambda \in \Lambda$. On the other hand, for each $n \in \mathbb{N}$, $B_{\lambda,n}$ is a cs-cover for $L \cap M_{\lambda}$ in $M_{\lambda}$. It follows from Lemma 2.2 that $D_{\lambda,n}$ is a cs-cover for $f(L \cap M_{\lambda})$ in $X_{\lambda,n}$. Then $D_{\lambda,n}$ is a cs-cover for $S \cap X_{\lambda}$ in $X_{\lambda,n}$.

$(2) \Rightarrow (1)$. By using notations and arguments in the proof $(2) \Rightarrow (1)$ of Proposition 2.4 again, $X$ is a $\pi$-image of a locally separable metric space. It suffices to prove that the mapping $f$ is sequence-covering.

For each convergent sequence $S$ of $X$, there is $\lambda \in \Lambda$ such that $S$ is eventually in $X_{\lambda,n}$, and for each $n \in \mathbb{N}$, $D_{\lambda,n}$ is a cs-cover for $S \cap X_{\lambda}$ in $X_{\lambda,n}$. Since $S \cap X_{\lambda}$ is a convergent sequence in $X_{\lambda}$, there is a convergent sequence $L_{\lambda}$ in $M_{\lambda}$ such that $f_{\lambda}(L_{\lambda}) = S \cap X_{\lambda}$ by Lemma 2.3. Note that $S - X_{\lambda} = f(F)$ with some finite subset $F$ of $M$. Put $L = F \cup L_{\lambda}$, then $L$ is a convergent sequence in $M$ and $f(L) = S$. It implies that $f$ is sequence-covering.

Theorem 2.8. The following are equivalent for a space $X$.

1. $X$ is a subsequence-covering $\pi$-image of a locally separable metric space;
2. $X$ is a sequentially-quotient $\pi$-image of a locally separable metric space;
3. $X$ has a cover $\{X_\lambda : \lambda \in \Lambda\}$ having $\pi$-property satisfying that for each convergent sequence $S$ of $X$, there is $\lambda \in \Lambda$ such that $D_{\lambda,n}$ is a cs-cover for some subsequence $S_\lambda$ of $S$ in $X_{\lambda}$ for each $n \in \mathbb{N}$;
4. same as (3), but replace the prefix “cs” by “wcs”, or “cs$^*$”.

Proof. $(1) \Rightarrow (2)$. By [4, Proposition 2.1].

$(2) \Rightarrow (3)$. For each convergent sequence $S$ of $X$, since $f$ is sequentially-quotient, there is a convergent sequence $L$ of $M$ such that $f(L)$ is a subsequence of $S$. By the proof of $(1) \Rightarrow (2)$ in Theorem 2.7, where $f(L)$ plays the role of $S$ in the argument, we get (3).
By Lemma 2.1, “wcs-” and “cs∗-” are equivalent. It is routine when “cs-” is replaced by “cs∗-”.

By Lemma 2.1, “wcs-” and “cs∗-” are equivalent. It is routine when “cs-” is replaced by “cs∗-”.

(3)⇒(4). By Lemma 2.1, “wcs-” and “cs∗-” are equivalent. It is routine when “cs-” is replaced by “cs∗-”.

(4)⇒(1). It suffices to assume that $p_{A,n}$ is a cs∗-cover for a subsequence $T$ of $S$ in $X_1$. By the proof of (3)⇒(1) in Theorem 2.5, where $T$ plays the role of compact subset $K$ in the argument, we get (1).

Based on above results, it is easy to get characterizations for quotient $\pi$-images of locally separable metric spaces as follows.

**Corollary 2.9.** The following are equivalent for a space $X$.

1. $X$ is a compact-covering quotient $\pi$-image of a locally separable metric space;
2. $X$ is a sequential space satisfying (2), or (3) in Theorem 2.5.

**Corollary 2.10.** The following are equivalent for a space $X$.

1. $X$ is a pseudo-sequence-covering quotient $\pi$-image of a locally separable metric space;
2. $X$ is a sequential space satisfying (2), or (3) in Theorem 2.6.

**Corollary 2.11.** The following are equivalent for a space $X$.

1. $X$ is a sequence-covering quotient $\pi$-image of a locally separable metric space;
2. $X$ is a sequential space satisfying (2) in Theorem 2.7.

**Corollary 2.12.** The following are equivalent for a space $X$.

1. $X$ is a quotient $\pi$-image of a locally separable metric space;
2. $X$ is a sequential space satisfying (3), or (4) in Theorem 2.8.

**Remark 2.13.** “Quotient” and “sequential” in above corollaries can be replaced by “pseudo-open” and “Fréchet”, respectively.

In [6], the authors raised a question whether pseudo-sequence-covering quotient $\pi$-images of metric spaces and quotient $\pi$-images of metric spaces are equivalent. Similar to this question we have the following.

**Question 2.14.** Are pseudo-sequence-covering quotient $\pi$-images of locally separable metric spaces and quotient $\pi$-images of locally separable metric spaces equivalent?

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**References**


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