Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2008, Article ID 145683, 8 pages
doi:10.1155/2008/145683

Research Article

On π-Images of Locally Separable Metric Spaces

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Received 7 June 2007; Revised 2 September 2007; Accepted 4 October 2007

Recommended by Sehie Park

We characterize π-images of locally separable metric spaces by means of covers having π-property. As its application, we obtain characterizations of compact-covering (sequence-covering, pseudo-sequence-covering, and sequentially quotient) π-images of locally sparable metric spaces.

1. Introduction

To determine what spaces are the images of "nice" spaces under "nice" mappings is one of the central questions of general topology in [1]. In the past, many noteworthy results on images of metric spaces have been obtained. For a survey in this field, see [2], for example. A characterization for a quotient compact image of a locally separable metric space is obtained in [3]. Also, such a quotient image is precisely a pseudo-sequence-covering quotient compact image of a locally separable metric space [4]. Recently, π-images of metric spaces cause attention once again in [5, 6]. It is known that a space is a compact-covering (resp., sequence-covering, pseudo-sequence-covering, sequentially-quotient) π-image of a metric space if and only if it has a point-star network consisting of *cfp*-covers (resp., *cs*-covers, *wcs*-covers, cs^*-covers) [5–7]. In a personal communication, the first author of [6] informs that it seems to be difficult to obtain "nice" characterizations of "nice" images of locally separable metric spaces (instead of metric or locally compact metric domains). Thus, we are interested in the following question.

Question 1.1. How are compact-covering (resp., sequence-covering, pseudo-sequence-covering, sequentially-quotient) π-images of locally sparable metric spaces characterized?

In this paper, we characterize compact-covering (resp., sequence-covering, pseudo-sequence-covering, sequentially-quotient) π-images of locally separable metric spaces by means of *cfp*-covers (resp., *cs*-covers, *wcs*-covers, cs^*-covers) for compact subsets (resp., convergent sequences) in a space and covers having π-property to answer Question 1.1

completely. As applications of these results, we get characterizations on quotient π-images of locally separable metric spaces.

Throughout this paper, all spaces are assumed to be regular and T_1, all mappings are assumed continuous and onto, a convergent sequence includes its limit point, $\mathbb{N}$ denotes the set of all natural numbers. Let $f : X \to Y$ be a mapping, $x \in X$, and $\mathcal{P}$ be a collection of subsets of X, we denote $\mathrm{st}(x, \mathcal{P}) = \bigcup\{P \in \mathcal{P} : x \in P\}$, $\mathcal{P}_x = \{P \in \mathcal{P} : x \in P\}$, $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, and $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$. We say that a convergent sequence $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ converging to x is *eventually* (resp., *frequently*) in A if $\{x_n : n \geq n_0\} \cup \{x\} \subset A$ for some $n_0 \in \mathbb{N}$ (resp., $\{x_{n_k} : k \in \mathbb{N}\} \cup \{x\} \subset A$ for some subsequence $\{x_{n_k} : k \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$). For terms which are not defined here, please refer to [2, 8].

Let $\mathcal{P}$ be a collection of subsets of a space X, and K be a subset of X.

$\mathcal{P}$ is a *cover for K in X*, if $K \subset \bigcup \mathcal{P}$. When $K = X$, a cover for K in X is a *cover of* X [8]. A cover $\mathcal{P}$ for X is a *compact cover* if all members of $\mathcal{P}$ are compact.

For each $x \in X$, $\mathcal{P}$ is a *network at x* if $x \in P$ for every $P \in \mathcal{P}$, and if $x \in U$ with U open in X, there exists $P \in \mathcal{P}$ such that $x \in P \subset U$.

$\mathcal{P}$ is a *k-cover for K in X*, if for each compact subset H of K there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup \mathcal{F}$. When $K = X$, a k-cover for K in X is a *k-cover for* X.

$\mathcal{P}$ is a *cfp-cover for K in X*, if for each compact subset H of K there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}$ such that $H \subset \bigcup\{C_F : F \in \mathcal{F}\}$, where C_F is closed and $C_F \subset F$ for every $F \in \mathcal{F}$. Note that such an $\mathcal{F}$ is a *full cover* in the sense of [9]. When $K = X$, a *cfp*-cover for K in X is a *cfp-cover for* X [10].

$\mathcal{P}$ is a *cs-cover for K in X* (resp., *cs*-cover for K in X*), if for each convergent sequence S in K, S is eventually (resp., frequently) in some $P \in \mathcal{P}$. When $K = X$, a *cs*-cover for K in X (resp., *cs**-cover for K in X) is a *cs-cover for* X [11] (resp., *cs*-cover for* X [12]), or a *cover satisfying condition* (c_3) (resp., (c_2)) [5].

$\mathcal{P}$ is a *wcs-cover for K in X* if for each convergent sequence S converging to x in K there exists a finite subfamily $\mathcal{F}$ of $\mathcal{P}_x$ such that S is eventually in $\bigcup \mathcal{F}$. When $K = X$, a *wcs*-cover for K in X is a *wcs-cover* [7].

It is clear that if $\mathcal{P}$ is a cover (resp., k-cover, *cfp*-cover, *cs*-cover, *wcs*-cover, *cs**-cover), then $\mathcal{P}$ is a cover (resp., k-cover, *cfp*-cover, *cs*-cover, *wcs*-cover, *cs**-cover) for K in X.

Remark 1.2. (1) Closed k-cover for K in $X \Rightarrow$ *cfp*-cover for K in $X \Rightarrow$ k-cover for K in X;

(2) *cfp*-cover for K in X, or *cs*-cover for K in $X \Rightarrow$ *wcs*-cover for K in $X \Rightarrow$ *cs**-cover for K in X.

For each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be a cover for X. $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *refinement sequence* for X if $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$ for each $n \in \mathbb{N}$. A refinement sequence for X is a *refinement* of X in the sense of [4].

Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a refinement sequence for X. $\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *point-star network* for X, if $\{\mathrm{st}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x for each $x \in X$. Note that this notion is used without the assumption of a refinement sequence in [13], and in [5] $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a *σ-strong network* for X.

A cover $\{X_\lambda : \lambda \in \Lambda\}$ for X is called to have *π-property* if each X_λ has a refinement sequence $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ of countable covers for X_λ, and for each $x \in U$ with U open in X, there is $n \in \mathbb{N}$ such that

$$\bigcup\{\mathrm{st}(x, \mathcal{P}_{\lambda,n}) : \lambda \in \Lambda \text{ with } x \in X_\lambda\} \subset U. \tag{1.1}$$

Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a point-star network for a space X. For every $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\alpha : \alpha \in A_n\}$, and A_n is endowed with discrete topology. Put

$$M = \left\{ a = (\alpha_n) \in \prod_{n\in\mathbb{N}} A_n : \{P_{\alpha_n} : n \in \mathbb{N}\} \text{ forms a network at some point } x_a \text{ in } X \right\}. \tag{1.2}$$

Then M, which is a subspace of the product space $\prod_{n\in\mathbb{N}} A_n$, is a metric space with metric d described as follows.

Let $a = (\alpha_n),\ b = (\beta_n) \in M$. If $a = b$, then $d(a,b) = 0$. If $a \neq b$, then $d(a,b) = 1/(\min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\})$.

Define $f : M \to X$ by choosing $f(a) = x_a$, then f is a mapping, and $(f, M, X, \{\mathcal{P}_n\})$ is a *Ponomarev's system* [14], and if without the assumption of a refinement sequence in the notion of point-star networks, then $(f, M, X, \{\mathcal{P}_n\})$ is a *Ponomarev's system* in the sense of [13].

Let $f : X \to Y$ be a mapping.

f is a *compact-covering mapping* [15], if every compact subset of Y is the image of some compact subset of X.

f is a *sequence-covering mapping* [16], if every convergent sequence of Y is the image of some convergent sequence of X.

f is a *pseudo-sequence-covering mapping* [5], if every convergent sequence of Y is the image of some compact subset of X.

f is a *subsequence-covering mapping* [3], if for every convergent sequence S of Y, there is a compact subset K of X such that $f(K)$ is a subsequence of S.

f is a *sequentially-quotient mapping* [17], if for every convergent sequence S of Y, there is a convergent sequence L of X such that $f(L)$ is a subsequence of S.

f is a *pseudo-open mapping* [1], if $y \in \operatorname{int} f(U)$ whenever $f^{-1}(y) \subset U$ with U open in X.

f is a *π-mapping* [1], if for every $y \in Y$ and for every neighborhood U of y in Y, $d(f^{-1}(y), X - f^{-1}(U)) > 0$, where X is a metric space with a metric d.

Let X be a space. We recall that X is *sequential* [18], if a subset A of X is closed if and only if any convergent sequence in A has a limit point in A. Also, X is *Fréchet* (or *Fréchet Urysohn*) if for each $x \in \overline{A}$, there exists a sequence in A converging to x.

Note that, for a mapping $f : X \to Y$, f is compact-covering or sequence-covering $\Rightarrow$ f is pseudo-sequence-covering $\Rightarrow$ f is subsequence-covering [4]. Also, f is quotient if and only if f is subsequence-covering for Y being sequential [3].

2. Main results

Lemma 2.1. *Let $\mathcal{P}$ be a countable cover for a convergent sequence S in a space X. Then the following are equivalent:*

(1) *$\mathcal{P}$ is a cfp-cover for S in X;*

(2) *$\mathcal{P}$ is a wcs-cover for S in X;*

(3) *$\mathcal{P}$ is a cs*-cover for S in X.*

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3). By Remark 1.2.

(3)$\Rightarrow$(1). Let H be a compact subset of S. We can assume that H is a subsequence of S. Since $\mathcal{P}$ is countable, put $\mathcal{P}_x = \{P_n : n \in \mathbb{N}\}$, where x is the limit point of S. Then H is eventually

in $\bigcup_{n\le k} P_n$ for some $k \in \mathbb{N}$. If not, then for any $k \in \mathbb{N}$, H is not eventually in $\bigcup_{n\le k} P_n$. So, for every $k \in \mathbb{N}$, there exists $x_{n_k} \in S - \bigcup_{n\le k} P_n$. We may assume $n_1 < n_2 < \cdots < n_{k-1} < n_k < n_{k+1} < \cdots$. Put $H' = \{x_{n_k} : k \in \mathbb{N}\} \cup \{x\}$, then H' is a subsequence of S. Since $\mathcal{P}$ is a cs^*-cover for S in X, there exists $m \in \mathbb{N}$ such that H' is frequently in P_m. This contradicts to the construction of H'. So H is eventually in $\bigcup_{n\le k} P_n$ for some $k \in \mathbb{N}$. It implies that $\mathcal{P}$ is a *cfp*-cover for S in X. □

The following lemma is routinely shown, so we omit the proof.

Lemma 2.2. *Let $f : X \to Y$ be a mapping.*

(1) *If $\mathcal{P}$ is a k-cover for a compact set K in X, then $f(\mathcal{P})$ is a k-cover for $f(K)$ in Y.*

(2) *If $\mathcal{P}$ is a cs-cover for a convergent sequence S in X, then $f(\mathcal{P})$ is a cs-cover for $f(S)$ in Y.*

Same as [6, Lemma 2.2(2)(ii)], we get the following.

Lemma 2.3. *Let $(f, M, X, \{\mathcal{P}_n\})$ be a Ponomarev's system. For a convergent sequence S of X, if $\mathcal{P}_n$ is a cs-cover for S in X for each $n \in \mathbb{N}$, then there exists a convergent sequence L of M such that $f(L) = S$.*

Proof. Put $S = \{x_i : i \in \mathbb{N}\} \cup \{x\}$, where x is the limit point. For each $n \in \mathbb{N}$, since $\mathcal{P}_n$ is a cs-cover for S, S is eventually in some $P_{\alpha_n} \in \mathcal{P}_n$. For each $i \in \mathbb{N}$, if $x_i \in P_{\alpha_n}$, let $\alpha_{n,i} = \alpha_n$; if $x_i \notin P_{\alpha_n}$, pick $\alpha_{n,i} \in A_n$ such that $x_i \in P_{\alpha_{n,i}}$. Thus there exists $i_n \in \mathbb{N}$ such that $\alpha_{n,i} = \alpha_n$ for all $i > i_n$. So $\{\alpha_{n,i} : i \in \mathbb{N}\}$ converges to α_n. For each $i \in \mathbb{N}$, put $a_i = (\alpha_{n,i}) \in \prod_{n\in\mathbb{N}} A_n$, and put $a = (\alpha_n) \in \prod_{n\in\mathbb{N}} A_n$. Then $a_i \in f^{-1}(x_i)$, $a \in f^{-1}(x)$, and $L = \{a_i : i \in \mathbb{N}\} \cup \{a\}$ converges to a. It implies that L is a convergent sequence of M and $f(L) = S$. □

Proposition 2.4. *The following are equivalent for a space X.*

(1) *X is a π-image of a locally separable metric space,*

(2) *X has a cover $\{X_\lambda : \lambda \in \Lambda\}$ having π-property.*

Proof. (1)⇒(2). Let $f : M \to X$ be a π-mapping from a locally separable metric space M with metric d onto X. Since M is a locally separable metric space, $M = \oplus_{\lambda\in\Lambda} M_\lambda$, where each M_λ is a separable metric space by [8, 4.4.F]. For each $\lambda \in \Lambda$, let D_λ be a countable dense subset of M_λ, and put $f_\lambda = f|_{M_\lambda}$ and $X_\lambda = f_\lambda(M_\lambda)$. For each $a \in M_\lambda$ and $n \in \mathbb{N}$, put $B(a, 1/n) = \{b \in M_\lambda : d(a,b) < 1/n\}$, $\mathcal{B}_{\lambda,n} = \{B(a, 1/n) : a \in D_\lambda\}$, and $\mathcal{P}_{\lambda,n} = f_\lambda(\mathcal{B}_{\lambda,n})$. It is clear that $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a sequence of countable covers for X_λ, and $\mathcal{P}_{\lambda,n+1}$ is a refinement of $\mathcal{P}_{\lambda,n}$ for every $n \in \mathbb{N}$. We will prove that π-property is satisfied.

For each $x \in U$ with U open in X. Since f is a π-mapping, $d(f^{-1}(x), M - f^{-1}(U)) > 2/n$ for some $n \in \mathbb{N}$. Then, for each $\lambda \in \Lambda$ with $x \in X_\lambda$, we get $d(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) > 2/n$, where $U_\lambda = U \cap X_\lambda$. Let $a \in D_\lambda$ and $x \in f_\lambda(B(a, 1/n)) \in \mathcal{P}_{\lambda,n}$. We will prove that $B(a, 1/n) \subset f_\lambda^{-1}(U_\lambda)$. In fact, if $B(a, 1/n) \not\subset f_\lambda^{-1}(U_\lambda)$, then pick $b \in B(a, 1/n) - f_\lambda^{-1}(U_\lambda)$. Note that $f_\lambda^{-1}(x) \cap B(a, 1/n) \neq \varnothing$, pick $c \in f_\lambda^{-1}(x) \cap B(a, 1/n)$, then $d(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \le d(c,b) \le d(c,a) + d(a,b) < 2/n$. It is a contradiction. So $B(a, 1/n) \subset f_\lambda^{-1}(U_\lambda)$, thus $f_\lambda(B(a, 1/n)) \subset U_\lambda$. Then $\mathrm{st}(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$. It implies that $\bigcup\{\mathrm{st}(x, \mathcal{P}_{\lambda,n}) : \lambda \in \Lambda \text{ with } x \in X_\lambda\} \subset U$.

(2)⇒(1). For each $\lambda \in \Lambda$, let $x \in U_\lambda$ with U_λ open in X_λ. We get that $U_\lambda = U \cap X_\lambda$ with some U open in X. Since $\bigcup\{\mathrm{st}(x, \mathcal{P}_{\lambda,n}) : \lambda \in \Lambda \text{ with } x \in X_\lambda\} \subset U$ for some $n \in \mathbb{N}$, $\mathrm{st}(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$. It implies that $\{\mathcal{P}_{\lambda,n} : n \in \mathbb{N}\}$ is a point-star network for X_λ. Then the Ponomarev's system $(f_\lambda, M_\lambda, X_\lambda, \{\mathcal{P}_{\lambda,n}\})$ exists. Since each $\mathcal{P}_{\lambda,n}$ is countable, M_λ is a separable metric space with

metric d_λ described as follows. For $a = (\alpha_n),\ b = (\beta_n) \in M_\lambda$, if $a = b$, then $d_\lambda(a,b) = 0$, and if $a \neq b$, then $d_\lambda(a,b) = 1/(\min\{n \in \mathbb{N} : \alpha_n \neq \beta_n\})$.

Put $M = \oplus_{\lambda\in\Lambda} M_\lambda$ and define $f : M \to X$ by choosing $f(a) = f_\lambda(a)$ for every $a \in M_\lambda$ with some $\lambda \in \Lambda$. Then f is a mapping and M is a locally separable metric space with metric d defined as follows. For $a, b \in M$, if $a, b \in M_\lambda$ for some $\lambda \in \Lambda$, then $d(a,b) = d_\lambda(a,b)$, and otherwise, $d(a,b) = 1$.

We will prove that f is a π-mapping. Let $x \in U$ with U open in X, then $\bigcup\{\mathrm{st}(x, \mathcal{P}_{\lambda,n}) : \lambda \in \Lambda$ with $x \in X_\lambda\} \subset U$ for some $n \in \mathbb{N}$. So, for each $\lambda \in \Lambda$ with $x \in X_\lambda$, we get $\mathrm{st}(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$, where $U_\lambda = U \cap X_\lambda$. It is implies that $d_\lambda(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \geq 1/n$. In fact, if $a = (\alpha_k) \in M_\lambda$ such that $d_\lambda(f_\lambda^{-1}(x), a) < 1/n$, then there is $b = (\beta_k) \in f_\lambda^{-1}(x)$ such that $d_\lambda(a,b) < 1/n$. So $\alpha_k = \beta_k$ if $k \leq n$. Note that $x \in P_{\beta_n} \subset \mathrm{st}(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$. Then $f_\lambda(a) \in P_{\alpha_n} = P_{\beta_n} \subset \mathrm{st}(x, \mathcal{P}_{\lambda,n}) \subset U_\lambda$. Hence $a \in f_\lambda^{-1}(U_\lambda)$. It implies that $d_\lambda(f_\lambda^{-1}(x), a) \geq 1/n$ if $a \in M_\lambda - f_\lambda^{-1}(U_\lambda)$. So $d_\lambda(f_\lambda^{-1}(x), M_\lambda - f_\lambda^{-1}(U_\lambda)) \geq 1/n$. Therefore

$$\begin{aligned} d\left(f^{-1}(x), M - f^{-1}(U)\right) &= \inf\left\{d(a,b) : a \in f^{-1}(x),\ b \in M - f^{-1}(U)\right\} \\ &= \min\left\{1, \inf\left\{d_\lambda(a,b) : a \in f_\lambda^{-1}(x),\ b \in M_\lambda - f_\lambda^{-1}(U_\lambda),\ \lambda \in \Lambda\right\}\right\} \quad (2.1) \\ &\geq 1/n > 0. \end{aligned}$$

It implies that f is a π-mapping. □

Theorem 2.5. *The following are equivalent for a space X.*

(1) *X is a compact-covering π-image of a locally separable metric space;*

(2) *X has a cover $\{X_\lambda : \lambda \in \Lambda\}$ having π-property satisfying that for each compact subset K of X, there is a finite subset Λ_K of Λ such that K has a finite compact cover $\{K_\lambda : \lambda \in \Lambda_K\}$, and for each $\lambda \in \Lambda_K$ and $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a k-cover for K_λ in X_λ;*

(3) *same as (2), but replace the prefix "k-" by "cfp-".*

Proof. (1)⇒(2). By using notations and arguments in the proof (1)⇒(2) of Proposition 2.4 again, X has a cover $\{X_\lambda : \lambda \in \Lambda\}$ having π-property. For each compact subset K of X, since f is compact-covering, $K = f(L)$ for some compact subset L of M. By compactness of L, $L_\lambda = L \cap M_\lambda$ is compact and $\Lambda_K = \{\lambda \in \Lambda : L_\lambda \neq \varnothing\}$ is finite. For each $\lambda \in \Lambda_K$, put $K_\lambda = f(L_\lambda)$, then $\{K_\lambda : \lambda \in \Lambda_K\}$ is a finite compact cover for K. For each $n \in \mathbb{N}$, since $\mathcal{B}_{\lambda,n}$ is a k-cover for L_λ in M_λ, $\mathcal{P}_{\lambda,n}$ is a k-cover for K_λ in X_λ by Lemma 2.2.

(2)⇒(3). For each $\lambda \in \Lambda_K$ and $n \in \mathbb{N}$, since $\mathcal{P}_{\lambda,n}$ is countable, every member of $\mathcal{P}_{\lambda,n}$ can be chosen closed in X_λ. Thus, $\mathcal{P}_{\lambda,n}$ is a *cfp*-cover for K_λ in X_λ by Remark 1.2.

(3)⇒(1). By using notations and arguments in the proof (2)⇒(1) of Proposition 2.4 again, X is a π-image of a locally separable metric space. It suffices to prove that the mapping f is compact-covering.

For each compact subset K of X, there is a finite subset Λ_K of Λ such that K has a finite compact cover $\{K_\lambda : \lambda \in \Lambda_K\}$, and for each $\lambda \in \Lambda_K$ and $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a *cfp*-cover for K_λ in X_λ. It follows from [13, Lemma 13] that $K_\lambda = f_\lambda(L_\lambda)$ with some compact subset L_λ of M_λ. Put $L = \bigcup\{L_\lambda : \lambda \in \Lambda_K\}$, then L is a compact subset of M and $f(L) = K$. It implies that f is compact-covering. □

Theorem 2.6. *The following are equivalent for a space X.*

(1) *X is a pseudo-sequence-covering π-image of a locally separable metric space;*

(2) *X has a cover $\{X_\lambda : \lambda \in \Lambda\}$ having π-property satisfying that for each convergent sequence S of X, there is a finite subset Λ_S of Λ such that S has a finite compact cover $\{S_\lambda : \lambda \in \Lambda_S\}$, and for each $\lambda \in \Lambda_S$ and $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a wcs-cover for S_λ in X_λ;*

(3) *same as (2), but replace the prefix "wcs-" by "cs^*-".*

Proof. Using Lemma 2.1, notations and arguments in the proof of Theorem 2.5, here "*pseudo-sequence-covering*" and "*convergent sequence*" play the roles of "*compact-covering*" and "*compact subset*", respectively. □

Theorem 2.7. *The following are equivalent for a space X.*

(1) *X is a sequence-covering π-image of a locally separable metric space;*

(2) *X has a cover $\{X_\lambda : \lambda \in \Lambda\}$ having π-property satisfying that for each convergent sequence S of X, there is $\lambda \in \Lambda$ such that S is eventually in X_λ, and for each $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a cs-cover for $S \cap X_\lambda$ in X_λ.*

Proof. (1)⇒(2). By using notations and arguments in the proof (1)⇒(2) of Proposition 2.4 again, X has a cover $\{X_\lambda : \lambda \in \Lambda\}$ having π-property. For each convergent sequence S of X, since f is sequence-covering, $S = f(L)$ for some convergent sequence L of M. Note that L is eventually in some M_λ. Thus, S is eventually in X_λ with some $\lambda \in \Lambda$. On the other hand, for each $n \in \mathbb{N}$, $\mathcal{B}_{\lambda,n}$ is a cs-cover for $L \cap M_\lambda$ in M_λ. It follows from Lemma 2.2 that $\mathcal{P}_{\lambda,n}$ is a cs-cover for $f(L \cap M_\lambda)$ in X_λ. Then $\mathcal{P}_{\lambda,n}$ is a cs-cover for $S \cap X_\lambda$ in X_λ.

(2)⇒(1). By using notations and arguments in the proof (2)⇒(1) of Proposition 2.4 again, X is a π-image of a locally separable metric space. It suffices to prove that the mapping f is sequence-covering.

For each convergent sequence S of X, there is $\lambda \in \Lambda$ such that S is eventually in X_λ, and for each $n \in \mathbb{N}$, $\mathcal{P}_{\lambda,n}$ is a cs-cover for $S \cap X_\lambda$ in X_λ. Since $S \cap X_\lambda$ is a convergent sequence in X_λ, there is a convergent sequence L_λ in M_λ such that $f_\lambda(L_\lambda) = S \cap X_\lambda$ by Lemma 2.3. Note that $S - X_\lambda = f(F)$ with some finite subset F of M. Put $L = F \cup L_\lambda$, then L is a convergent sequence in M and $f(L) = S$. It implies that f is sequence-covering. □

Theorem 2.8. *The following are equivalent for a space X.*

(1) *X is a subsequence-covering π-image of a locally separable metric space;*

(2) *X is a sequentially-quotient π-image of a locally separable metric space;*

(3) *X has a cover $\{X_\lambda : \lambda \in \Lambda\}$ having π-property satisfying that for each convergent sequence S of X, there is $\lambda \in \Lambda$ such that $\mathcal{P}_{\lambda,n}$ is a cs-cover for some subsequence S_λ of S in X_λ for each $n \in \mathbb{N}$;*

(4) *same as (3), but replace the prefix "cs-" by "wcs-", or "cs^*-".*

Proof. (1)⇒(2). By [4, Proposition 2.1].

(2)⇒(3). For each convergent sequence S of X, since f is sequentially-quotient, there is a convergent sequence L of M such that $f(L)$ is a subsequence of S. By the proof of (1)⇒(2) in Theorem 2.7, where $f(L)$ plays the role of S in the argument, we get (3).

(3)⇒(4). By Lemma 2.1, "wcs-" and "cs^*-" are equivalent. It is routine when "cs-" is replaced by "cs^*-".

(4)⇒(1). It suffices to assume that $\mathcal{P}_{\lambda,n}$ is a cs^*-cover for a subsequence T of S in X_λ. By the proof of (3)⇒(1) in Theorem 2.5, where T plays the role of compact subset K in the argument, we get (1). □

Based on above results, it is easy to get characterizations for quotient π-images of locally separable metric spaces as follows.

Corollary 2.9. *The following are equivalent for a space X.*

(1) *X is a compact-covering quotient π-image of a locally separable metric space;*

(2) *X is a sequential space satisfying (2), or (3) in Theorem 2.5.*

Corollary 2.10. *The following are equivalent for a space X.*

(1) *X is a pseudo-sequence-covering quotient π-image of a locally separable metric space;*

(2) *X is a sequential space satisfying (2), or (3) in Theorem 2.6.*

Corollary 2.11. *The following are equivalent for a space X.*

(1) *X is a sequence-covering quotient π-image of a locally separable metric space;*

(2) *X is a sequential space satisfying (2) in Theorem 2.7.*

Corollary 2.12. *The following are equivalent for a space X.*

(1) *X is a quotient π-image of a locally separable metric space;*

(2) *X is a sequential space satisfying (3), or (4) in Theorem 2.8.*

Remark 2.13. "*Quotient*" and "*sequential*" in above corollaries can be replaced by "*pseudo-open*" and "*Fréchet*", respectively.

In [6], the authors raised a question whether pseudo-sequence-covering quotient π-images of metric spaces and quotient π-images of metric spaces are equivalent. Similar to this question we have the following.

Question 2.14. Are pseudo-sequence-covering quotient π-images of locally separable metric spaces and quotient π-images of locally separable metric spaces equivalent?

Acknowledgments

The authors would like to thank the referee for his/her valuable comments. This is supported in part by the National Natural Science Foundation of Vietnam (Grant no. 1 008 06).

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