Research Article

Second Hankel Determinant for a Class of Analytic Functions Defined by Fractional Derivative

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By making use of the fractional differential operator $\Omega^\lambda_\alpha$ due to Owa and Srivastava, a class of analytic functions $R_{\lambda}(\alpha, \rho)$ $(0 \leq \rho \leq 1, 0 \leq \lambda < 1, |\alpha| < \pi/2)$ is introduced. The sharp bound for the nonlinear functional $|a_2a_4 - a_2^3|$ is found. Several basic properties such as inclusion, subordination, integral transform, Hadamard product are also studied.

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1. Introduction

Let $A$ denote the class of functions analytic in the open unit disc

$$A := \{ z : z \in \mathbb{C}, |z| < 1 \}$$

and let $A_0$ be the class of functions $f$ in $A$ given by the normalized power series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in A).$$

Also let $S$, $S^*(\beta)$, $CU(\beta)$, and $K$ denote, respectively, the subclasses of $A_0$ consisting of functions which are univalent, starlike of order $\beta$, convex of order $\beta$ (cf. [1]), and close-to-convex (cf. [2]) in $A$. In particular, $S^*(0) = S^*$ and $CU(0) = CU$ are the familiar classes of starlike and convex functions in $A$ (cf. [2]).

Given $f$ and $g$ in $A$, the function $f$ is said to be subordinate to $g$ in $A$ if there exits a function $\omega \in A$ satisfying the conditions of the Schwarz Lemma such that $f(z) = g(\omega(z))$, $(z \in A)$. We denote the subordination by

$$f(z) \prec g(z) \quad (z \in A) \text{ or } f \prec g \text{ in } A.$$
It is well known [2] that if $g$ is univalent in $\mathcal{U}$, then $f \prec g$ in $\mathcal{U}$ is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

For the functions $f$ and $g$ given by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n \quad (z \in \mathcal{U}),$$

their Hadamard product (or convolution), denoted by $f \ast g$, is defined by

$$(f \ast g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g \ast f)(z) \quad (z \in \mathcal{U}).$$

Note that $f \ast g \in \mathcal{A}$.

By making use of the Hadamard product, Carlson-Shaffer [3] defined the linear operator $\mathcal{L}(a, c) : \mathcal{A} \to \mathcal{A}$ by

$$(\mathcal{L}(a, c)f)(z) := \Phi(a, c; z) \ast f(z) \quad (z \in \mathcal{U}, f \in \mathcal{A}),$$

where

$$\Phi(a, c; z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \quad (z \in \mathcal{U}, c \notin \mathbb{Z}_0^- = [0] \cup \{-1, -2, -3, \ldots\})$$

and $(\lambda)_k$ is the Pochhammer symbol (or shifted factorial) defined in terms of the gamma function by

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k = 0), \\ \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + k - 1) & (k \in \mathbb{N} := \{1, 2, \ldots\}) \end{cases}.$$

It can be readily verified that $\mathcal{L}(a, a)$ ($a \notin \mathbb{Z}_0^-$) is the identity operator; the operators $\mathcal{L}(a, b), \mathcal{L}(c, d)$ commute, where $b, d \notin \mathbb{Z}_0^-$, that is,

$$\mathcal{L}(a, b) \mathcal{L}(c, d)f = \mathcal{L}(c, d) \mathcal{L}(a, b)f \quad (f \in \mathcal{A}),$$

and the transitive property, that is,

$$\mathcal{L}(a, b) \mathcal{L}(b, c)f = \mathcal{L}(a, c)f \quad (b, c \notin \mathbb{Z}_0^-, f \in \mathcal{A}),$$

holds. Each of the following definitions will also be required in our present investigation.

**Definition 1.1** (cf. [4, 5], see also [6]). Let the function $f$ be analytic in a simply connected region of the $z$-plane containing the origin. The fractional derivative of $f$ of order $\lambda$ is defined by

$$\left(D^{1}_{\lambda}f\right)(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_{0}^{z} \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (0 \leq \lambda < 1),$$

where the multiplicity of $(z - \zeta)^1$ is removed by requiring $\log(z - \zeta)$ to be real when $(z - \zeta) > 0$. 
Using Definition 1.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [5] introduced the \( \text{fractional differintegral operator} \, \Omega^1_\lambda : \mathcal{A}_0 \to \mathcal{A}_0 \) defined by

\[
(\Omega^1_\lambda f)(z) = \Gamma(2-\lambda)z^{1}(D^1_\lambda f)(z) \quad (\lambda \neq 2,3, \ldots, z \in \mathcal{H}).
\]

(1.12)

Note that \( \Omega^0_\lambda f(z) = f(z), \Omega^1_\lambda f(z) = zf'(z), \) and

\[
(\Omega^1_\lambda f)(z) = (\mathcal{L}(2,2-\lambda)f)(z) \quad (0 \leq \lambda < 1, \, z \in \mathcal{H}).
\]

(1.13)

**Definition 1.2** (cf. [7]). For the function \( f \) given by (1.2) and \( q \in \mathbb{N} := \{1,2,3, \ldots\} \), the \( q \)th Hankel determinant of \( f \) is defined by

\[
\begin{vmatrix}
 a_n & a_{n+1} & \cdots & a_{n+q-1} \\
 a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}.
\]

(1.14)

We now introduce the following class of functions.

**Definition 1.3.** The function \( f \in \mathcal{A}_0 \) is said to be in the class \( \mathcal{R}_\lambda(a,\rho) \) \((0 \leq \lambda < 1, \, |a| < \pi/2, \, 0 \leq \rho \leq 1) \) if it satisfies the inequality

\[
\Re \left\{ e^{ia} \frac{\Omega^1_\lambda f(z)}{z} \right\} > \rho \cos \alpha \quad (z \in \mathcal{H}).
\]

(1.15)

Write

\[
\mathcal{R}_\lambda(0,\rho) := \mathcal{R}_\lambda(\rho).
\]

(1.16)

Let \( \mathcal{D} \) be the family of functions \( p \in \mathcal{A} \) satisfying \( p(0) = 1 \) and \( \Re(p(z)) > 0 \) \( (z \in \mathcal{H}). \)

It follows from (1.15) that

\[
f \in \mathcal{R}_\lambda(a,\rho) \iff e^{ia} \frac{\Omega^1_\lambda f(z)}{z} = \left[(1-\rho)p(z) + \rho\right] \cos \alpha + i \sin \alpha,
\]

(1.17)

where \( \alpha \) is real, \( |a| < \pi/2, \) and \( p(z) \in \mathcal{D}. \)

We note that

\[
\mathcal{R}_0(a,\rho) := \left\{ f \in \mathcal{A}_0 \mid \Re \left\{ e^{ia} f(z) \right\} > \rho \cos \alpha \right\},
\]

\[
\mathcal{R}_1(a,\rho) := \left\{ f \in \mathcal{A}_0 \mid \Re \left\{ e^{ia} f'(z) \right\} > \rho \cos \alpha \right\},
\]

(1.18)

and the class \( \mathcal{R}_\lambda(\rho) \) has been studied in [8].

It is well known (cf. [2]) that for \( f \in \mathcal{S} \) and given by (1.2), the sharp inequality \( |a_3 - a_2^2| \leq 1 \) holds. This corresponds to the Hankel determinant with \( q = 2 \) and \( n = 1 \). For a given family \( \mathcal{F} \) of functions in \( \mathcal{A}_0 \), the more general problem of finding sharp estimates for \( |\mu a_2^2 - a_3| \) \((\mu \in \mathbb{R} \) or \( \mu \in \mathbb{C}) \) is popularly known as the Fekete-Szegö problem for \( \mathcal{F} \). The Fekete-Szegö problem for the families \( \mathcal{S}, \mathcal{S}^*, \mathcal{C}, \mathcal{K} \) has been completely solved by many authors including [9–12].

In the present paper, we consider the Hankel determinant for \( q = 2 \) and \( n = 2 \) and we find the sharp bound for the functional \( |a_2 a_4 - a_3^2| \) \( (f \in \mathcal{R}_1(a,\rho)) \). We also obtain some basic properties of the class \( \mathcal{R}_1(a,\rho) \). Our investigation includes a recent result of Janteng et al. [13]. We also generalize some results of Ling and Ding [8].
2. Preliminaries

To establish our results, we recall the following.

Lemma 2.1 (see [2]). Let the function \( p \in \mathcal{P} \) and be given by the series
\[
p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathcal{U}).
\]

Then, the sharp estimate
\[
|c_k| \leq 2 \quad (k \in \mathbb{N})
\]
holds.

Lemma 2.2 (cf. [14, page 254], see also [15]). Let the function \( p \in \mathcal{P} \) be given by the power series (2.1). Then,
\[
2c_2 = c_1^2 + x(4 - c_1^2)
\]
for some \( x \), \( |x| \leq 1 \), and
\[
4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z
\]
for some \( z \), \( |z| \leq 1 \).

Lemma 2.3 (see [16]). Let \( F \) and \( G \) be univalent convex functions in \( \mathcal{U} \). Then, the Hadamard product \( F \ast G \) is also a univalent convex function in \( \mathcal{U} \).

Lemma 2.4 (see [17]). Let \( F \) and \( G \) be univalent convex functions in \( \mathcal{U} \). Also let \( f < F \) and \( g < G \) in \( \mathcal{U} \). Then, \( f \ast g < F \ast G \) in \( \mathcal{U} \).

Lemma 2.5 (see [16], also see [8]). Let \( f \) and \( g \) be starlike of order \( 1/2 \). Then, for each function \( F(z) \), satisfying \( \Re(F(z)) > \alpha \ (0 \leq \alpha < 1, \ z \in \mathcal{U}) \), one has
\[
\Re\left( \frac{f(z) \ast F(z) g(z)}{f(z) \ast g(z)} \right) > \alpha \quad (z \in \mathcal{U}).
\]

Lemma 2.6 (see [8]). Let the function \( h(z) = 1 + h_1 z + h_2 z^2 + \cdots \) be univalent convex in \( \mathcal{U} \). For \( 0 \leq \lambda < 1 \) if \( \Omega_{\lambda}^1 f(z) / z < h(z) \ (z \in \mathcal{U}) \), then
\[
\frac{f(z)}{z} < \left\{ \mathcal{L}(2 - \lambda, 2) [zh(z)] \right\} \quad (z \in \mathcal{U}).
\]

3. Main results

We prove the following.

Theorem 3.1. Let the function \( f \) given by (1.2) be in the class \( \mathcal{R}_\lambda(\alpha, \rho) \) \( (0 \leq \lambda < 1, -\pi/2 < \alpha < \pi/2, \) and \( 0 \leq \rho \leq 1) \). Then,
\[
|a_2 a_4 - a_3^2| \leq \frac{(1 - \rho)^2(2 - \lambda)^2(3 - \lambda)^2 \cos^2 \alpha}{9}.
\]

The estimate (3.1) is sharp.
Proof. Let \( f \in \mathcal{R}_2(\alpha, \rho) \) \((0 \leq \lambda < 1, -\pi/2 < \alpha < \pi/2, \) and \(0 \leq \rho \leq 1)\). Then, by (1.17),

\[
e^{ia} \frac{\Omega^1zf(z)}{z} = [(1 - \rho)p(z) + \rho] \cos \alpha + i \sin \alpha \quad (z \in \mathcal{U}),
\]

where \( p \in \mathcal{P} \) and is given by (2.1). Using (1.6), (1.7), and (1.13), we write

\[
\Omega^1zf(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n, \quad (z \in \mathcal{U}).
\]

Comparing the coefficients, we get

\[
e^{ia} \frac{2}{2-\lambda} a_2 = (1 - \rho)c_1 \cos \alpha,
\]

\[
e^{ia} \frac{6}{(2-\lambda)(3-\lambda)} a_3 = (1 - \rho)c_2 \cos \alpha,
\]

\[
e^{ia} \frac{24}{(2-\lambda)(3-\lambda)(4-\lambda)} a_4 = (1 - \rho)c_3 \cos \alpha.
\]

Therefore, (4.4) yields

\[
|a_2a_4 - a_3^2| = \frac{(1 - \rho)^2(2 - \lambda)^2}{12} \left| \left( \frac{4 - \lambda}{4} \right) c_1 c_3 - \left( \frac{3 - \lambda}{3} \right) c_2^2 \right|. \tag{3.5}
\]

Since the functions \( p(z) \) and \( p(e^{i\theta}z) \), \((\theta \in \mathbb{R})\) are members of the class \( \mathcal{P} \) simultaneously, we assume without loss of generality that \( c_1 > 0 \). For convenience of notation, we take \( c_1 = c \) \((c \in [0, 2])\).

Using (2.3) along with (2.4), we get

\[
|a_2a_4 - a_3^2| = \frac{(1 - \rho)^2(2 - \lambda)^2(3 - \lambda)(\cos^2 \alpha)}{12} \left[ \left( \frac{4 - \lambda}{4} \right) c^3 + 2(4 - c^2) c x - c(4 - c^2) x^2 + 2(4 - c^2)(1 - |x|^2) z \right] \left[ \left( \frac{4 - \lambda}{4} \right) c^4 + \left( \frac{4 - \lambda}{4} \right) c^3 x \left( \frac{4 - \lambda}{4} \right) c^2 \right. \left. - 2c^2(3 - \lambda)(4 - c^2) \right] x
\]

\[
- \left( \frac{4 - \lambda}{4} \right) c^2 \left( \frac{4 - \lambda}{4} \right) c \left( \frac{4 - \lambda}{4} \right) c \left( 1 - |x|^2 \right) z \right] \left( \frac{4 - \lambda}{4} \right) c^2 + \left( \frac{4 - \lambda}{4} \right) c \left( 1 - |x|^2 \right) z \right] \left( \frac{4 - \lambda}{4} \right) c^2 + \left( \frac{4 - \lambda}{4} \right) c \left( 1 - |x|^2 \right) z \right] \right|.
\]

\[
= \frac{1 - \rho^2(2 - \lambda)^2}{12} \left( \frac{4 - \lambda}{4} \right) c^4 + \left( \frac{4 - \lambda}{4} \right) c^3 x \left( \frac{4 - \lambda}{4} \right) c^2 \right. \left. - 2c^2(3 - \lambda)(4 - c^2) \right] x
\]

\[
- \left( \frac{4 - \lambda}{4} \right) c^2 \left( \frac{4 - \lambda}{4} \right) c \left( 1 - |x|^2 \right) z \right] \left( \frac{4 - \lambda}{4} \right) c^2 + \left( \frac{4 - \lambda}{4} \right) c \left( 1 - |x|^2 \right) z \right] \right|.
\]

\[
= \frac{1 - \rho^2(2 - \lambda)^2(3 - \lambda)(\cos^2 \alpha)}{12} \left[ \left( \frac{4 - \lambda}{4} \right) c^4 + \left( \frac{4 - \lambda}{4} \right) c^3 x \left( \frac{4 - \lambda}{4} \right) c^2 \right. \left. - 2c^2(3 - \lambda)(4 - c^2) \right] x
\]

\[
- \left( \frac{4 - \lambda}{4} \right) c^2 \left( \frac{4 - \lambda}{4} \right) c \left( 1 - |x|^2 \right) z \right] \left( \frac{4 - \lambda}{4} \right) c^2 + \left( \frac{4 - \lambda}{4} \right) c \left( 1 - |x|^2 \right) z \right] \right|.
\]
An application of triangle inequality and replacement of $|x|$ by $\mu$ give

\[ |a_2a_4 - a_3^2| \leq \frac{(1 - \rho)^2(2 - \lambda)^2(3 - \lambda)(\cos^2\alpha)}{48} \times \left[ \frac{\lambda c^4}{12} + \frac{(4 - c^2)c^2\mu}{6} + \frac{(4 - c^2)(48 - \lambda(16 - c^2))\mu^2}{12} + \frac{(4 - \lambda)(4 - c^2)c}{2} \right] \]

\[ = (1 - \rho)^2(2 - \lambda)^2(3 - \lambda)(\cos^2\alpha) \times \left[ \frac{\lambda c^4}{12} + \frac{(4 - \lambda)(4 - c^2)c + \lambda(4 - c^2)c^2\mu}{2} \times \frac{\lambda[2c^2 - 6(4 - \lambda)c/\lambda + 16(3 - \lambda)/\lambda](4 - c^2)\mu^2}{6} \right] \]

\[ = (1 - \rho)^2(2 - \lambda)^2(3 - \lambda)(\cos^2\alpha) \times \left[ \frac{\lambda c^4}{12} + \frac{(4 - \lambda)(4 - c^2)c}{2} \times \frac{\lambda(4 - c^2)c^2\mu}{6} + \frac{1(c - \beta_1)(c - \beta_2)(4 - c^2)\mu^2}{12} \right] \]

\[ := F(c, \mu) \text{ (say),} \]

where

\[ \beta_1 = 2, \quad \beta_2 = \frac{8(3 - \lambda)}{\lambda}, \quad 0 \leq c \leq 2, \quad 0 \leq \mu \leq 1. \]  

We next maximize the function $F(c, \mu)$ on the closed square $[0,2] \times [0,1]$. Since

\[ \frac{\partial F}{\partial \mu} = \frac{(1 - \rho)^2(2 - \lambda)^2(3 - \lambda)\cos^2\alpha}{48} \left[ \frac{\lambda(4 - c^2)c^2}{6} + \frac{\lambda(4 - \lambda)(4 - c^2)c - 8(3 - \lambda)/\lambda\mu}{6} \right], \]

$c - 2 < 0$, and $c - 8(3 - \lambda)/\lambda < 0$, we have $\partial F/\partial \mu > 0$ for $0 < c < 2$, $0 < \mu < 1$. Thus $F(c, \mu)$ cannot have a maximum in the interior of the closed square $[0,2] \times [0,1]$. Moreover, for fixed $c \in [0,2], \max_{0 \leq \mu \leq 1} F(c, \mu) = F(c, 1) = G(c)$ (say).

Next,

\[ G'(c) = \frac{-(1 - \rho)^2(2 - \lambda)^2(3 - \lambda)(c^2 - (7\lambda - 12))\cos^2\alpha}{72}, \]

so that $G'(c) < 0$ for $0 < c < 2$ and has real critical point at $c = 0$. Also $G(c) > G(2)$. Therefore, $\max_{0 \leq c \leq 2}$ occurs at $c = 0$. Therefore, the upper bound of (3.7) corresponds to $\mu = 1$ and $c = 0$. Hence,

\[ |a_2a_4 - a_3^2| \leq \frac{(1 - \rho)^2(2 - \lambda)^2(3 - \lambda)^2\cos^2\alpha}{9} \]  

(3.12)
where $R$.

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Proof. where the function

$$f(z) = \Phi(2 - \lambda, 2; z) e^{-i\alpha}\left[ z\left(1 + (1 - 2\rho)z^2 \frac{1}{1 - z^2}\cos\alpha + i\sin\alpha\right)\right].$$

(3.13)

The proof of Theorem 3.1 is complete.

The choice of $\alpha = 0$ yields what follows.

**Corollary 3.2.** Let the function $f$ given by (1.2) be a member of the class $R_\lambda(\rho)$. Then,

$$|a_2a_4 - a_3^2| \leq \frac{(1 - \rho)^2(2 - \lambda)^2(3 - \lambda)^2}{9}.$$  (3.14)

Equality holds for the function

$$f(z) = \mathcal{L}(2 - \lambda, 2) \frac{z(1 + (1 - 2\rho)z^2)}{1 - z^2}.$$  (3.15)

**Remark 3.3.** Taking $\lambda \to 1$, $\alpha = 0$, and $\rho = 0$, we get a recent result due to Janteng et al. [13].

**Theorem 3.4.** Suppose $-\pi/2 < \alpha < \pi/2$, $0 \leq \rho < 1$, and $0 \leq \mu < \lambda < 1$. Then,

$$R_\lambda(\alpha, \rho) \subset R_\mu(\alpha, \rho).$$  (3.16)

**Proof.** Let

$$f \in R_\lambda(\alpha, \rho) \quad \left(0 \leq \mu < \lambda < 1, \quad -\frac{\pi}{2} < \alpha < \frac{\pi}{2}, \quad 0 \leq \rho \leq 1\right).$$  (3.17)

Using the associative and commutative properties of the operator $\mathcal{L}$, we write

$$\Omega_2^\mu f(z) = \mathcal{L}(2, 2 - \mu) f(z)$$

$$= \mathcal{L}(2 - \lambda, 2) \mathcal{L}(2, 2 - \lambda) \mathcal{L}(2, 2 - \mu) f(z)$$

$$= \mathcal{L}(2 - \lambda, 2 - \mu) \Omega_2^\mu f(z)$$

$$= \Phi(2 - \lambda, 2 - \mu; z) \Omega_2^\mu f(z),$$  (3.18)

where the function $\Phi$ is defined by (1.7). Therefore,

$$\frac{e^{i\alpha}\Omega_2^\mu f(z)}{z} = \frac{\Phi(2 - \lambda, 2 - \mu; z) (e^{i\alpha}\Omega_2^\mu f(z)/z) - z}{\Phi(2 - \lambda, 2 - \mu; z) * z}$$

$$= \frac{f(z) * F(z) g(z)}{f(z) g(z)}.$$  (3.19)

where $f(z) = \Phi(2 - \lambda, 2 - \mu; z)$, $g(z) = z$, $F(z) = e^{i\alpha}\Omega_2^\mu f(z)/z$. We note that $g \in S^*(1/2)$, and $\Re(F(z)) > \rho \cos\alpha (0 \leq \rho \leq 1, \quad -\pi/2 < \alpha < \pi/2)$. Moreover, it is well known (cf. [18]) that
Hence, \( f(z) \in R_\mu(\alpha, \rho) \), and the proof of Theorem 3.4 is complete.

**Theorem 3.5.** Let \( f \in S^*(1/2) \) and \( g \in R_\lambda(\alpha, \rho) \) \((0 \leq \rho \leq 1, -\pi/2 < \alpha < \pi/2, 0 \leq \lambda < 1)\). Then the Hadamard product
\[
f * g \in R_\lambda(\alpha, \rho).
\]

**Proof.** Since the Hadamard product is associative and commutative, we have
\[
\Omega_1^1(f*g)(z) = f(z) * \Omega_1^1 g(z).
\]
Therefore,
\[
e^{ia}\Omega_1^1(f*g)(z) = \frac{\Omega_1^1(f(z))(z)}{f(z) * z}.
\]
Now applying Lemma 2.5, we get
\[
\Re\left(\frac{e^{ia}\Omega_1^1(f*g)(z)}{z}\right) > \rho \cos \alpha.
\]
Hence, \( f * g \in R_\lambda(\alpha, \rho) \), and the proof of Theorem 3.5 is complete.

**Theorem 3.6.** Let \( f \in R_\lambda(\alpha, \rho) \) \((0 \leq \lambda < 1, -\pi/2 < \alpha < \pi/2, 0 \leq \rho \leq 1)\). Then, the function \( \mathcal{O}(f) \) defined by the integral transform
\[
\mathcal{O}(f)(z) = \frac{z + 1}{z^\gamma} \int_0^\infty t^{-\gamma} f(t) dt \quad (z \in \mathcal{U}, \gamma > -1)
\]
is also in \( R_\lambda(\alpha, \rho) \).

**Proof.** The Integral transform \( \mathcal{O}(f) \) can be written in terms of Carlson-Shaffer operator as
\[
(\mathcal{O}(f))(z) = (\mathcal{L}(\gamma + 1, \gamma + 2) f)(z).
\]
Hence,
\[
(\Omega_2^1 \mathcal{O}(f))(z) = \mathcal{L}(\gamma + 1, \gamma + 2) \Omega_2^1 f(z) = \Phi(\gamma + 1, \gamma + 2; z) * \Omega_2^1 f(z).
\]
Therefore,
\[
e^{ia}(\Omega_2^1 \mathcal{O}(f))(z) = \frac{\Phi(\gamma + 1, \gamma + 2; z) * (e^{ia}\Omega_2^1 f(z)/z)}{\Phi(\gamma + 1, \gamma + 2; z) * z}.
\]
Using a result of Bernardi [19], it can be verified that \( \Phi(\gamma + 1, \gamma + 2; z) \in S^*(1/2) \). Thus by applying Lemma 2.5, the proof of Theorem 3.6 is complete.
Theorem 3.7. Let $f \in R_\lambda(\alpha, \rho)$, $(0 \leq \lambda < 1$, $-\pi/2 < \alpha < \pi/2$, $0 \leq \rho \leq 1)$. Then,

$$\frac{\tilde{f}(z)}{z} < G(z) \quad (z \in \mathcal{H}), \quad (3.29)$$

where

$$G(z) = \frac{e^{-i\alpha}}{z} \{ \Phi(2 - \lambda, 2; z) \ast [zh(z)] \},$$

$$h(z) = \left( \frac{1 + (1 - 2\rho)z}{1 - z} \cos \alpha + i \sin \alpha \right), \quad (3.30)$$

and $\Phi$ is defined by (1.7). Moreover, $G$ is a univalent convex function in $\mathcal{H}$.

Proof. Since $\Omega^2 f(z)/z < e^{-i\alpha}h(z)$, by an application of Lemma 2.6, we get

$$\frac{\tilde{f}(z)}{z} < \frac{e^{-i\alpha}}{z} \{ \mathcal{L}(2 - \lambda, 2) \ast [zh(z)] \} = G(z). \quad (3.31)$$

The assertion (3.29) is proved.

It is well known (cf. [18]) that $\Phi(2 - \lambda, 2; z)/z$ is a univalent convex function. Therefore, by Lemma 2.4, $G(z)$ is univalent convex function.

Remark 3.8. For $\alpha = 0$, Theorem 3.7(i) gives a result of Ling and Ding [8, Theorem 2].

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References


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