Research Article

Starlike and Convex Properties for Hypergeometric Functions

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The purpose of the present paper is to give some characterizations for a (Gaussian) hypergeometric function to be in various subclasses of starlike and convex functions. We also consider an integral operator related to the hypergeometric function.

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1. Introduction

Let $\mathcal{T}$ be the class consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad (1.1)$$

that are analytic and univalent in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Let $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the subclasses of $\mathcal{T}$ consisting of starlike and convex functions of order α ($0 \leq \alpha < 1$), respectively [1].

Recently, Bharati et al. [2] introduced the following subclasses of starlike and convex functions.

Definition 1.1. A function f of the form (1.1) is in $\mathcal{S}_p\mathcal{T}(\alpha, \beta)$ if it satisfies the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} \geq \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta, \quad \alpha \geq 0, \quad 0 \leq \beta < 1, \quad (1.2)$$

and $f \in \mathcal{MC}\mathcal{T}(\alpha, \beta)$ if and only if $zf' \in \mathcal{S}_p\mathcal{T}(\alpha, \beta)$.

Definition 1.2. A function f of the form (1.1) is in $\mathcal{PT}(\alpha)$ if it satisfies the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} + \alpha \geq \left| \frac{zf'(z)}{f(z)} - \alpha \right|, \quad 0 < \alpha < \infty, \quad (1.3)$$

and $f \in \mathcal{CP}\mathcal{T}(\alpha)$ if and only if $zf' \in \mathcal{PT}(\alpha)$.

Bharati et al. [2] showed that $\mathcal{S}_p\mathcal{T}(\alpha, \beta) = \mathcal{T}^*((\alpha + \beta)/(1 + \alpha))$, $\mathcal{MCT}(\alpha, \beta) = \mathcal{C}((\alpha + \beta)/(1 + \alpha))$, $\mathcal{PT}(\alpha) = \mathcal{T}^*(1 - \alpha)$ ($0 < \alpha \leq 1$), and $\mathcal{CP}\mathcal{T}(\alpha) = \mathcal{C}(1 - \alpha)$ ($0 < \alpha \leq 1$). In particular, we note that $\mathcal{MCT}(1, 0)$ is the class of uniformly convex functions given by Goodman [3] (also see [4–6]).

Let $F(a, b; c; z)$ be the (Gaussian) hypergeometric function defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \quad (1.4)$$

where $c \neq 0, -1, -2, \dots$, and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1 & \text{if } n = 0, \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & \text{if } n \in \mathbb{N} = \{1, 2, \dots\}. \end{cases} \quad (1.5)$$

We note that $F(a, b; c; 1)$ converges for $\operatorname{Re}(c - a - b) > 0$ and is related to the Gamma function by

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \quad (1.6)$$

Silverman [7] gave necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$, and also examined a linear operator acting on hypergeometric functions. For the other interesting developments for $zF(a, b; c; z)$ in connection with various subclasses of univalent functions, the readers can refer to the works of Carlson and Shaffer [8], Merkes and Scott [9], and Ruscheweyh and Singh [10].

In the present paper, we determine necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $\mathcal{S}_p\mathcal{T}(\alpha, \beta)$, $\mathcal{MCT}(\alpha, \beta)$, $\mathcal{PT}(\alpha)$, and $\mathcal{CP}\mathcal{T}(\alpha)$. Furthermore, we consider an integral operator related to the hypergeometric function.

2. Results

To establish our main results, we need the following lemmas due to Bharati et al. [2].

Lemma 2.1. (i) A function f of the form (1.1) is in $\mathcal{S}_p\mathcal{T}(\alpha, \beta)$ if and only if it satisfies

$$\sum_{n=2}^{\infty} (n(1 + \alpha) - (\alpha + \beta)) a_n \leq 1 - \beta. \quad (2.1)$$

(ii) A function f of the form (1.1) is in $\mathcal{MCT}(\alpha, \beta)$ if and only if it satisfies

$$\sum_{n=2}^{\infty} n(n(1 + \alpha) - (\alpha + \beta)) a_n \leq 1 - \beta. \quad (2.2)$$

Lemma 2.2. (i) A function f of the form (1.1) is in $\mathcal{PT}(\alpha)$ if and only if it satisfies

$$\sum_{n=2}^{\infty} (n-1+\alpha) a_n \leq \alpha. \quad (2.3)$$

(ii) A function f of the form (1.1) is in $\mathcal{CPT}(\alpha)$ if and only if it satisfies

$$\sum_{n=2}^{\infty} n(n-1+\alpha) a_n \leq \alpha. \quad (2.4)$$

Theorem 2.3. (i) If $a, b > -1$, $c > 0$, and $ab < 0$, then $zF(a, b; c; z)$ is in $\mathcal{S}_p\mathcal{T}(\alpha, \beta)$ if and only if

$$c \geq a + b + 1 - \frac{(1+\alpha)ab}{(1-\beta)}. \quad (2.5)$$

(ii) If $a, b > 0$ and $c > a + b + 1$, then $F_1(a, b; c; z) = z(2 - F(a, b; c; z))$ is in $\mathcal{S}_p\mathcal{T}(\alpha, \beta)$ if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(1 + \frac{(1+\alpha)ab}{(1-\beta)(c-a-b-1)} \right) \leq 2. \quad (2.6)$$

Proof. (i) Since

$$zF(a, b; c; z) = z + \frac{ab}{c} \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n = z - \left| \frac{ab}{c} \right| \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n, \quad (2.7)$$

according to (i) of Lemma 2.1, we must show that

$$\sum_{n=2}^{\infty} (n(1+\alpha) - (\alpha+\beta)) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left| \frac{c}{ab} \right| (1-\beta). \quad (2.8)$$

Noting that $(\lambda)_n = \lambda(\lambda+1)_{n-1}$ and then applying (1.6), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} ((n+2)(1+\alpha) - (\alpha+\beta)) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\ &= (1+\alpha) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (1-\beta) \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} \\ &= (1+\alpha) \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + (1-\beta) \frac{c}{ab} \left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right). \end{aligned} \quad (2.9)$$

Hence, (2.8) is equivalent to

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left(1 + \alpha + (1-\beta) \frac{c-a-b-1}{ab} \right) \leq (1-\beta) \left(\frac{c}{|ab|} + \frac{c}{ab} \right) = 0. \quad (2.10)$$

Thus, (2.10) is valid if and only if $1 + \alpha + (1-\beta)(c-a-b-1)/(ab) \leq 0$ or, equivalently, $c \geq a+b+1 - (1+\alpha)ab/(1-\beta)$.

(ii) Since

$$F_1(a, b; c; z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n, \quad (2.11)$$

by (i) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty} (n(1+\alpha) - (\alpha+\beta)) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq 1-\beta. \quad (2.12)$$

Now,

$$\begin{aligned} \sum_{n=2}^{\infty} (n(1+\alpha) - (\alpha+\beta)) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} &= (1+\alpha) \sum_{n=1}^{\infty} \frac{n(a)_n(b)_n}{(c)_n(1)_n} (1-\beta) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} \\ &= \frac{(1+\alpha)ab}{c} \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} + (1-\beta) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(\frac{(1+\alpha)ab}{c-a-b-1} + 1 - \beta \right) - (1-\beta). \end{aligned} \quad (2.13)$$

But this last expression is bounded above by $1-\beta$ if and only if (2.6) holds. $\square$

Theorem 2.4. (i) If $a, b > -1$, $ab < 0$, and $c > a+b+2$, then $zF(a, b; c; z)$ is in $\mathcal{UCT}(\alpha, \beta)$ if and only if

$$(1+\alpha)(a)_2(b)_2 + (3+2\alpha-\beta)ab(c-a-b-2) + (1-\beta)(c-a-b-2)_2 \geq 0. \quad (2.14)$$

(ii) If $a, b > 0$ and $c > a+b+2$, then $F_1(a, b; c; z) = z(2 - F(a, b; c; z))$ is in $\mathcal{UCT}(\alpha, \beta)$ if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(\frac{(1+\alpha)(a)_2(b)_2}{(1-\beta)(c-a-b-2)_2} + \left(\frac{3+2\alpha-\beta}{1-\beta} \right) \left(\frac{ab}{c-a-b-1} \right) + 1 \right) \leq 2. \quad (2.15)$$

Proof. (i) Since zF has the form (2.7), we see from (ii) of Lemma 2.1 that our conclusion is equivalent to

$$\sum_{n=2}^{\infty} n(n(1+\alpha) - (\alpha+\beta)) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \frac{c}{|ab|} (1-\beta). \quad (2.16)$$

Writing $(n+2)((n+2)(1+\alpha) - (\alpha+\beta)) = (1+\alpha)(n+1)^2 + (2+\alpha-\beta)(n+1) + (1-\beta)$, we see that

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+2)((n+2)(1+\alpha) - (\alpha+\beta)) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\ &= (1+\alpha) \sum_{n=0}^{\infty} (n+1) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (2+\alpha-\beta) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \\ & \quad + (1-\beta) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\ &= \frac{(1+\alpha)(a+1)(b+1)}{c+1} \sum_{n=0}^{\infty} \frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} + (3+2\alpha-\beta) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \\ & \quad + (1-\beta) \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} \\ &= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} \left((1+\alpha)(a+1)(b+1) + (3+2\alpha-\beta)(c-a-b-2) \right. \\ & \quad \left. + \frac{1-\beta}{ab} (c-a-b-2)_2 \right) - \frac{(1-\beta)c}{ab}. \end{aligned} \quad (2.17)$$

This last expression is bounded above by $(1-\beta)c/|ab|$ if and only if

$$(1+\alpha)(a+1)(b+1) + (3+2\alpha-\beta)(c-a-b-2) + \frac{1-\beta}{ab} (c-a-b-2)_2 \leq 0, \quad (2.18)$$

which is equivalent to (2.14).

(ii) In view of (ii) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty} n(n(1+\alpha) - (\alpha+\beta)) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq 1-\beta. \quad (2.19)$$

Now,

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+2)((n+2)(1+\alpha) - (\alpha+\beta)) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\ &= (1+\alpha) \sum_{n=0}^{\infty} (n+2)^2 \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} - (\alpha+\beta) \sum_{n=0}^{\infty} (n+2) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}. \end{aligned} \quad (2.20)$$

Writing $n + 2 = (n + 1) + 1$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} (n+2) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} &= \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}, \\ \sum_{n=0}^{\infty} (n+2)^2 \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} &= \sum_{n=0}^{\infty} (n+1) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + 2 \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}} + 3 \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}. \end{aligned} \quad (2.21)$$

Substituting (2.21) into the right-hand side of (2.20), we obtain

$$(1 + \alpha) \sum_{n=0}^{\infty} \frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_n} + (3 + 2\alpha - \beta) \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + (1 - \beta) \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}. \quad (2.22)$$

Since $(a)_{n+k} = (a)_k(a+k)_n$, we write (2.22) as

$$\begin{aligned} &\frac{(1 + \alpha)(a)_2(b)_2}{(c)_2} \frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} + (3 + 2\alpha - \beta) \frac{ab}{c} \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \\ &+ (1 - \beta) \left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right). \end{aligned} \quad (2.23)$$

By simplification, we see that the last expression is bounded above by $1 - \beta$ if and only if (2.15) holds. $\square$

Theorem 2.5. (i) If $a, b > -1$, $c > 0$, and $ab < 0$, then $zF(a, b; c; z)$ is in $\mathcal{PT}(\alpha)$ if and only if

$$c \geq a + b + 1 - \frac{ab}{\alpha}. \quad (2.24)$$

(ii) If $a, b > 0$ and $c > a + b + 1$, then $F_1(a, b; c; z) = z(2 - F(a, b; c; z))$ is in $\mathcal{PT}(\alpha)$ if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(1 + \frac{ab}{\alpha(c-a-b-1)} \right) \leq 2. \quad (2.25)$$

Proof. (i) Since

$$zF(a, b; c; z) = z + \frac{ab}{c} \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n = z - \frac{|ab|}{c} \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n, \quad (2.26)$$

according to (i) of Lemma 2.2, we must show that

$$\sum_{n=2}^{\infty} (n-1 + \alpha) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \frac{c}{|ab|} \alpha. \quad (2.27)$$

Noting that $(\lambda)_n = \lambda(\lambda+1)_{n-1}$ and then applying (1.6), we have

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1+\alpha) \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_{n+1}} &= \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_n} + \alpha \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} \\ &= \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \alpha \frac{c}{ab} \left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right). \end{aligned} \quad (2.28)$$

Hence, (2.27) is equivalent to

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} \left(1 + \alpha \frac{c-a-b-1}{ab} \right) \leq \alpha \left(\frac{c}{|ab|} - \frac{c}{ab} \right) = 0. \quad (2.29)$$

Thus, (2.29) is valid if and only if $1 + \alpha(c-a-b-1)/ab \leq 0$ or, equivalently, $c \geq a+b+1 - ab/\alpha$.
(ii) Since

$$F_1(a, b; c; z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} z^n, \quad (2.30)$$

by (i) of Lemma 2.2, we need only to show that

$$\sum_{n=2}^{\infty} (n-1+\alpha) \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} \leq \alpha. \quad (2.31)$$

Now,

$$\begin{aligned} \sum_{n=2}^{\infty} (n-1+\alpha) \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} &= \sum_{n=1}^{\infty} \frac{n(a)_n (b)_n}{(c)_n (1)_n} + \alpha \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} \\ &= \frac{ab}{c} \sum_{n=1}^{\infty} \frac{(a+1)_{n-1} (b+1)_{n-1}}{(c+1)_{n-1} (1)_{n-1}} + \alpha \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(\frac{ab}{c-a-b-1} + \alpha \right) - \alpha. \end{aligned} \quad (2.32)$$

But this last expression is bounded above by α if and only if (2.25) holds. $\square$

Theorem 2.6. (i) If $a, b > -1$, $ab < 0$, and $c > a+b+2$, then $zF(a, b; c; z)$ is in $\mathcal{CP}\mathcal{T}(\alpha)$ if and only if

$$(a)_2(b)_2 + (2+\alpha)ab(c-a-b-2) + \alpha(c-a-b-2)_2 \geq 0. \quad (2.33)$$

(ii) If $a, b > 0$ and $c > a+b+2$, then $F_1(a, b; c; z) = z(2 - F(a, b; c; z))$ is in $\mathcal{CP}\mathcal{T}(\alpha)$ if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(\frac{(a)_2(b)_2}{\alpha(c-a-b-2)_2} + \left(\frac{2+\alpha}{\alpha} \right) \left(\frac{ab}{c-a-b-1} + 1 \right) \right) \leq 2. \quad (2.34)$$

Proof. (i) Since zF has the form (2.26), we see from (ii) of Lemma 2.2 that our conclusion is equivalent to

$$\sum_{n=2}^{\infty} n(n-1+\alpha) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \frac{c}{|ab|} \alpha. \quad (2.35)$$

Writing $(n+2)(n+1+\alpha) = (n+1)^2 + (1+\alpha)(n+1) + \alpha$, we see that

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+2)(n+1+\alpha) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\ &= \sum_{n=0}^{\infty} (n+1) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + (1+\alpha) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \alpha \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\ &= \frac{(a+1)(b+1)}{c+1} \sum_{n=0}^{\infty} \frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} + (2+\alpha) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \alpha \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} \\ &= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} \left((a+1)(b+1) + (2+\alpha)(c-a-b-2) + \frac{\alpha}{ab}(c-a-b)_2 \right) - \frac{\alpha c}{ab}. \end{aligned} \quad (2.36)$$

This last expression is bounded above by $\alpha c/|ab|$ if and only if $(a+1)(b+1) + (2+\alpha)(c-a-b-2) + (\alpha/ab)(c-a-b-1)_2 \leq 0$, which is equivalent to (2.33).

(ii) In view of (ii) of Lemma 2.2, we need only to show that

$$\sum_{n=2}^{\infty} n(n-1+\alpha) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq \alpha. \quad (2.37)$$

Now,

$$\sum_{n=0}^{\infty} (n+2)(n+2-(1-\alpha)) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} = \sum_{n=0}^{\infty} (n+2)^2 \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} - (1-\alpha) \sum_{n=0}^{\infty} (n+2) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}. \quad (2.38)$$

Substituting (2.21) into the right-hand side of (2.38), we obtain

$$\sum_{n=0}^{\infty} \frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_n} + (2+\alpha) \sum_{n=0}^{\infty} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n} + \alpha \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}. \quad (2.39)$$

Since $(a)_{n+k} = (a)_k(a+k)_n$, we may write (2.39) as

$$\frac{(a)_2(b)_2}{(c)_2} \frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)} + (2+\alpha) \frac{ab}{c} \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \alpha \left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right). \quad (2.40)$$

By simplification, we see that the last expression is bounded above by α if and only if (2.34) holds. $\square$

3. An integral operator

In the next theorems, we obtain similar-type results in connection with a particular integral operator $G(a, b; c; z)$ acting on $F(a, b; c; z)$ as follows:

$$G(a, b; c; z) = \int_0^z F(a, b; c; t) dt. \quad (3.1)$$

Theorem 3.1. *Let $a, b > -1$, $ab < 0$, and $c > \max\{0, a + b\}$. Then,*

(i) $G(a, b; c; z)$ defined by (3.1) is in $\mathcal{S}_p\mathcal{T}(\alpha, \beta)$ if and only if

$$\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(\frac{(1+\alpha)}{ab} - \frac{(\alpha+\beta)(c-a-b)}{(a-1)_2(b-1)_2} \right) + \frac{(\alpha+\beta)(c-1)_2}{(a-1)_2(b-1)_2} \leq 0; \quad (3.2)$$

(ii) $G(a, b; c; z)$ defined by (3.1) is in $\mathcal{PT}(\alpha)$ if and only if

$$\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(\frac{1}{ab} + \frac{(\alpha-1)(c-a-b)}{(a-1)_2(b-1)_2} \right) - \frac{(\alpha-1)(c-a)_2}{(a-1)_2(b-1)_2} \leq 0. \quad (3.3)$$

Proof. (i) Since

$$G(a, b; c; z) = z - \frac{|ab|}{c} \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n} z^n, \quad (3.4)$$

by (i) of Lemma 2.1, we need only to show that

$$\sum_{n=2}^{\infty} (n(1+\alpha) - (\alpha+\beta)) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n} \leq (1-\beta) \frac{c}{|ab|}. \quad (3.5)$$

Now,

$$\begin{aligned} & \sum_{n=0}^{\infty} ((n+2)(1+\alpha) - (\alpha+\beta)) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}} \\ &= (1+\alpha) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} - (\alpha+\beta) \frac{c}{ab} \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} \\ &= \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(\frac{1+\alpha}{ab} - \frac{(\alpha+\beta)(c-a-b)}{(a-1)_2(b-1)_2} \right) \\ & \quad + \frac{(\alpha+\beta)(c-1)_2}{(a-1)_2(b-1)_2} - (1-\beta) \frac{c}{ab} \\ &\leq (1-\beta) \frac{c}{|ab|}, \end{aligned} \quad (3.6)$$

which is equivalent to (3.2).

(ii) According to (i) of Lemma 2.2, it is sufficient to show that

$$\sum_{n=2}^{\infty} (n-1+\alpha) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n} \leq \alpha \frac{c}{|ab|}. \quad (3.7)$$

Now,

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+1+\alpha) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}} \\ &= \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + (\alpha-1) \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}} \\ &= \frac{c}{ab} \left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right) \\ & \quad + (\alpha-1) \frac{c}{ab} \left(\frac{(c-1)}{(a-1)(b-1)} \left(\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right) - 1 \right) \\ &= \frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(\frac{1}{ab} + \frac{(\alpha-1)(c-a-b)}{(a-1)_2(b-1)_2} \right) - \frac{(\alpha-1)(c-1)_2}{(a-1)_2(b-1)_2} - \alpha \frac{c}{ab} \\ &\leq \alpha \frac{c}{|ab|}, \end{aligned} \quad (3.8)$$

which is equivalent to (3.3).

Now, we observe that $G(a, b; c; z) \in \mathcal{UCT}(\alpha, \beta)(\mathcal{CP}\mathcal{T}(\alpha))$ if and only if $zF(a, b; c; z) \in \mathcal{S}_p\mathcal{T}(\alpha, \beta)(\mathcal{PT}(\alpha))$. Thus, any result of functions belonging to the class $\mathcal{S}_p\mathcal{T}(\alpha, \beta)(\mathcal{PT}(\alpha))$ about zF leads to that of functions belonging to the class $\mathcal{UCT}(\alpha, \beta)(\mathcal{CP}\mathcal{T}(\alpha))$. Hence, we obtain the following analogues to Theorems 2.3 and 2.5. $\square$

Theorem 3.2. Let $a, b > -1$, $ab < 0$, and $c > a + b + 2$. Then,

(i) $G(a, b; c; z)$ defined by (3.1) is in $\mathcal{UCT}(\alpha, \beta)$ if and only if

$$c \geq a + b + 1 - \frac{(1+\alpha)ab}{(1-\beta)}; \quad (3.9)$$

(ii) $G(a, b; c; z)$ defined by (3.1) is in $\mathcal{CP}\mathcal{T}(\alpha)$ if and only if

$$c \geq a + b + 1 - \frac{ab}{\alpha}. \quad (3.10)$$

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