Research Article

Fixed Points for Multivalued Mappings in Uniformly Convex Metric Spaces

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The purpose of this paper is to ensure the existence of fixed points for multivalued nonexpansive weakly inward nonself-mappings in uniformly convex metric spaces. This extends a result of Lim (1980) in Banach spaces. All results of Dhompongsa et al. (2005) and Chaoha and Phon-on (2006) are also extended.

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1. Introduction

In 1974, Lim [1] developed a result concerning the existence of fixed points for multivalued nonexpansive self-mappings in uniformly convex Banach spaces. This result was extended to nonself-mappings satisfying the inwards condition independently by Downing and Kirk [2] and Reich [3]. This result was extended to weak inward mappings independently by Lim [4] and Xu [5]. Recently, Dhompongsa et al. [6] presented an analog of Lim-Xu’s result in CAT(0) spaces. In this note, we extend the result to uniformly convex metric spaces which improve results of both Lim-Xu and Dhompongsa et al. In addition, we also give a new proof of a result of Lim [7] by using Caristi’s theorem [8]. Finally, we give some basic properties of fixed point sets for quasi-nonexpansive mappings for these spaces.

2. Preliminaries

A concept of convexity in metric spaces was introduced by Takahashi [9].

Definition 2.1. Let \((X, d)\) be a metric space and \(I = [0, 1]\). A mapping \(W : X \times X \times I \to X\) is said to be a convex structure on \(X\) if for each \((x, y, \lambda) \in X \times X \times I\) and \(z \in X\),
A metric space \((X,d)\) together with a convex structure \(W\) is called a convex metric space which will be denoted by \((X,d,W)\).

**Definition 2.2.** A convex metric space \((X,d,W)\) is said to be uniformly convex [10] if for any \(\varepsilon > 0\), there exists \(\delta = \delta(\varepsilon) > 0\) such that for all \(r > 0\) and \(x,y,z \in X\) with \(d(z,x) \leq r, d(z,y) \leq r\) and \(d(x,y) \geq r\varepsilon\),

\[d\left(z, W\left(x,y, \frac{1}{2}\right)\right) \leq r(1 - \delta).\]  

(2.2)

Obviously, uniformly convex Banach spaces are uniformly convex metric spaces. By using the (CN) inequality [11], it is easy to see that CAT(0) spaces are also uniformly convex.

For \(x,y \in X\), \(C \subseteq X\), and \(\lambda \in I\), we denote \(W(x,y,\lambda) := \lambda x \oplus (1-\lambda)y\), \([x,y] := \{\lambda x \oplus (1-\lambda)y : \lambda \in I\}\), \((x,y) := [x,y] \setminus \{x\}\), and \(\lambda x \oplus (1-\lambda)C := \{\lambda x \oplus (1-\lambda)z : z \in C\}\). So we can define the inward set \(I_C(x)\) of \(x\) as follows:

\[I_C(x) := \{x\} \cup \{z : (x,z) \cap C \neq \emptyset\}.\]  

(2.3)

Let \(C\) be a nonempty subset of a metric space \(X\). Then \(C\) is called convex if for \(x,y \in C\), \([x,y] \subseteq C\). We will denote by \(\mathcal{F}(C)\) the family of nonempty closed subsets of \(C\), by \(\mathcal{K}(C)\) the family of nonempty compact subsets of \(C\), and by \(\mathcal{KC}(C)\) the family of nonempty compact convex subsets of \(C\). Let \(H(\cdot,\cdot)\) be the Hausdorff distance on \(\mathcal{F}(X)\). That is,

\[H(A,B) = \max \left\{ \sup_{a \in A} \text{dist}(a,B), \sup_{b \in B} \text{dist}(b,A) \right\}, \quad A,B \in \mathcal{F}(X).\]  

(2.4)

**Definition 2.3.** A multivalued mapping \(T : C \rightarrow \mathcal{F}(X)\) is said to be inward on \(C\) if for some \(p \in C\),

\[\lambda p \oplus (1-\lambda)Tx \subset I_C(x) \quad \forall x \in C, \forall \lambda \in [0,1],\]

(2.5)

and weakly inward on \(C\) if for some \(p \in C\),

\[\lambda p \oplus (1-\lambda)Tx \subset \overline{I_C(x)} \quad \forall x \in C, \forall \lambda \in [0,1],\]

(2.6)

where \(\overline{A}\) denotes the closure of a subset \(A\) of \(X\). In a Banach space setting, if \(C\) is convex, then so is \(I_C(x)\). Therefore, the conditions above can be replaced by \(Tx \subset I_C(x)\) and \(Tx \subset \overline{I_C(x)}\), respectively.

**Definition 2.4.** A multivalued mapping \(T : C \rightarrow \mathcal{F}(X)\) satisfying

\[H(Tx,Ty) \leq kd(x,y), \quad x,y \in C,\]

(2.7)

is called a contraction if \(k \in [0,1)\) and nonexpansive if \(k = 1\). A point \(x\) is a fixed point of \(T\) if \(x \in Tx\).
Given a metric space $X$, one way to describe a metric space ultrapower $\tilde{X}$ of $X$ is to first embed $X$ as a closed subset of a Banach space $E$ (see, e.g., [12, page 129]). Let $\tilde{E}$ denote a Banach space ultrapower of $E$ relative to some nontrivial ultrafilter $\mathcal{U}$ (see, e.g., [13]). Then take

$$\tilde{X} := \{ \tilde{x} = [(x_n)] \in \tilde{E} : x_n \in X \ \forall n \}. \quad (2.8)$$

One can then let $\tilde{d}$ denote the metric on $\tilde{X}$ inherited from the ultrapower norm $\| \cdot \|_{\mathcal{U}}$ in $\tilde{E}$. If $X$ is complete, then so is $\tilde{X}$ since $\tilde{X}$ is a closed subset of the Banach space $\tilde{E}$. In particular, the metric $\tilde{d}$ on $\tilde{X}$ is given by

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{\mathcal{U}} \| x_n - y_n \| = \lim_{\mathcal{U}} d(x_n, y_n), \quad (2.9)$$

with $\{ u_n \} \in [(x_n)]$ if and only if $\lim_{\mathcal{U}} \| x_n - u_n \| = 0$.

If $(X, d, W)$ is a convex metric space, we consider a metric space ultrapower $(\tilde{X}, \tilde{d})$ of $(X, d)$ and define a function $\tilde{W} : \tilde{X} \times \tilde{X} \times I \to \tilde{X}$ by

$$\tilde{W}(\tilde{x}, \tilde{y}, \lambda) = [W(x_n, y_n, \lambda)]. \quad (2.10)$$

In order to show that the function $\tilde{W}$ is well defined, we need the following condition. For each $p, x, y \in X$ and $\lambda \in [0, 1]$,

$$d((1 - \lambda)p \oplus \lambda x, (1 - \lambda)p \oplus \lambda y) \leq \lambda d(x, y), \quad (2.11)$$

which is equivalent to

$$d((1 - \lambda)p \oplus \lambda x, (1 - \lambda)q \oplus \lambda y) \leq \lambda d(x, y) + (1 - \lambda)d(p, q), \quad (2.12)$$

for all $p, q, x, y \in X$ and $\lambda \in [0, 1]$.

By using condition (2.12), it is easy to see that $\tilde{W}$ is a convex structure on $\tilde{X}$. This implies that $(\tilde{X}, \tilde{d}, \tilde{W})$ is a convex metric space.

Example 2.5. Every Banach space satisfies condition (2.11).

Example 2.6. Condition (2.11) is satisfied for spaces of hyperbolic type (for more details of these spaces see [14]). This is also true for CAT(0) spaces and $\mathbb{R}$-trees.

Example 2.7. Let $H$ be a hyperconvex metric space. Then there exists a nonexpansive retract $R : l_\infty(H) \to H$ (see, e.g., [15] for more on this). For any $x, y \in H$ and $t \in [0, 1]$, we let

$$tx \oplus (1 - t)y = R(tx + (1 - t)y). \quad (2.13)$$

Since $l_\infty(H)$ is a Banach space, $H$ also satisfies condition (2.11).

Let $\mathcal{U}$ be a nontrivial ultrafilter on the natural number $\mathbb{N}$. If $(X, d, W)$ is a uniformly convex metric space satisfying condition (2.11), then the metric space ultrapower
Then there are some representatives \((x_n)\) and \((y_n)\) of \(\tilde{x}\) and \(\tilde{y}\) and a set \(I \in \mathcal{U}\) such that
\[
d(z_n, x_n) \leq r, \quad d(z_n, y_n) \leq r, \quad d(x_n, y_n) \geq r\varepsilon \quad \forall n \in I.
\] (2.15)
For such \(n, d(z_n, W(x_n, y_n, 1/2)) \leq r(1 - \delta)\). This implies \(\tilde{d}(\tilde{x}, \tilde{W}(\tilde{x}, \tilde{y}, 1/2)) \leq r(1 - \delta)\).

Let \(N = \{x \in X : d(x, 0) = \text{dist}(x_0, C)\}\). For each \(n\), we define
\[
C_n := \left\{ y \in C : d(x_0, y) \leq \text{dist}(x_0, C) + \frac{1}{n} \right\}.
\] (2.16)
Then \((C_n)\) is a decreasing sequence of nonempty bounded closed convex subsets of \(C\). Moreover,
\[
N = \bigcap_{n=1}^{\infty} C_n,
\]
which is nonempty by the above observation. The uniqueness follows from the uniform convexity of \(X\).

3. Main results

We first establish the following lemma.

**Lemma 3.1.** Let \(X\) be a complete uniformly convex metric space satisfying condition (2.11), \(C\) a nonempty closed convex subset of \(X\), \(x \in X\), and \(p(x)\) the unique nearest point of \(x\) in \(C\). Then
\[
d(x, p(x)) < d(x, y) \quad \forall y \in \overline{I_C(p(x))} \setminus \{p(x)\}.
\] (3.1)

**Proof.** Let \(y \in \overline{I_C(p(x))} \setminus \{p(x)\}\). Then there is a sequence \((y_n)\) in \(I_C(p(x))\) and \(y_n \to y\). Choose \(n_0 \in \mathbb{N}\) such that \((p(x), y_n) \cap C \neq \emptyset\) for all \(n \geq n_0\). For such \(n\), let \(z_n \in (p(x), y_n) \cap C\) and write \(z_n = (1 - \alpha_n)p(x) + \alpha_n y_n\), \(\alpha_n \in (0, 1]\). Then
\[
d(x, p(x)) \leq d(x, z_n) \leq (1 - \alpha_n)d(x, p(x)) + \alpha_n d(x, y_n).
\] (3.2)
This implies
\[
d(x, p(x)) \leq d(x, y).
\] (3.3)
If \(d(x, p(x)) = d(x, y)\), we let \(u = (1/2)p(x) + (1/2)y\). By the uniform convexity of \(X\), we have \(d(x, u) < d(x, p(x))\). On the other hand, for each \(n \geq n_0\), let \(u_n = (1/2)p(x) + (1/2)y_n\). We will show that \(u_n \in I_C(p(x))\).
Case 1. \( \alpha_n = 1/2 \). We are done.

Case 2. \( 1/2 < \alpha_n \). Let \( v_n = (1 - 1/2\alpha_n)p(x) \oplus (1/2\alpha_n)z_n \). This implies

\[
d(p(x), v_n) = \frac{1}{2\alpha_n}d(p(x), z_n) = \frac{1}{2}d(p(x), y_n) = d(p(x), u_n),
\]

which is a contradiction. Hence \( u_n = v_n \in [p(x), z_n] \) and so \( u_n \in I_C(p(x)) \) by the convexity of \( C \).

Case 3. \( \alpha_n < 1/2 \). Let \( v_n = (1 - 2\alpha_n)p(x) \oplus 2\alpha_n u_n \). By the same arguments in the proof of Case 2, we can show that \( z_n \in (p(x), u_n) \). This means \( u_n \in I_C(p(x)) \).

By condition (2.11), \( \lim_n u_n = u \), which implies \( u \in \overline{I_C(p(x)) \setminus \{p(x)\}} \). By the same arguments in the first part of the proof, \( d(x, p(x)) \leq d(x, u) \) which is a contradiction. Hence \( d(x, p(x)) < d(x, y) \) as desired. \( \square \)

From [6, Lemma 3.4], we observe that the space is not necessarily assumed to be a CAT(0) space since the proof is only involved with condition (2.11) which is weaker than the (CN) inequality (see [11, Lemma 3]). Therefore, we can obtain the following lemma.

**Lemma 3.2.** Let \( X \) be a complete convex metric space satisfying condition (2.11), \( C \) a nonempty closed subset of \( X \), and \( T : C \to \mathcal{F}(X) \) a contraction mapping satisfying, for all \( x \in C \),

\[
Tx \subset \overline{I_C(x)}.
\]

Then \( T \) has a fixed point.

This lemma was first proved in Banach spaces by Lim [7], using transfinite induction, while we apply directly Caristi’s theorem. For completeness, we include the details.
Proof of Lemma 3.2. Let $0 \leq k < 1$ be the contraction constant of $T$ and let $\varepsilon > 0$ be such that $\varepsilon + (k + 2\varepsilon)(1 + \varepsilon) < 1$. Let $M = \{(x, z) : z \in Tx, x \in C\}$ and define a metric $\rho$ on $M$ by $\rho((x, z), (u, v)) = \max \{d(x, u), d(z, v)\}$. It is easy to see that $(M, \rho)$ is a complete metric space.

Now define $\psi : M \to [0, \infty)$ by $\psi(x, z) = d(x, z) / \varepsilon$. Then $\psi$ is continuous on $M$. Suppose that $T$ has no fixed points, that is, $\text{dist}(x, Tx) > 0$ for all $x \in C$. Let $(x, z) \in M$. By (3.9), we can find $z' \in I_C(x)$ satisfying $d(z, z') < \varepsilon \text{ dist}(x, Tx)$. Now choose $u \in (x, z') \cap C$ and write $u = (1 - \delta)x \oplus \delta z'$ for some $0 < \delta \leq 1$. For such $\delta$, we have

$$\delta \varepsilon + (1 - \delta) + (k + 2\varepsilon)\delta(1 + \varepsilon) < 1. \quad (3.10)$$

Since $d(x, u) > 0$, we can find $v \in Tu$ satisfying

$$d(z, v) \leq H(Tx, Tu) + \varepsilon d(x, u) \leq (k + \varepsilon)d(x, u). \quad (3.11)$$

Now we define a mapping $g : M \to M$ by $g(x, z) = (u, v)$ for all $(x, z) \in M$. We claim that $g$ satisfies

$$\rho((x, z), g(x, z)) < \psi(x, z) - \psi(g(x, z)) \quad \forall (x, z) \in M. \quad (3.12)$$

Caristi’s theorem [8] then implies that $g$ has a fixed point, which contradicts to the strict inequality (3.12) and the proof is complete. So it remains to prove (3.12). In fact, it is enough to show that

$$\rho((x, z), (u, v)) < \frac{1}{\varepsilon}(d(x, z) - d(u, v)). \quad (3.13)$$

But $d(z, v) \leq d(x, u)$, it only needs to prove that $d(x, u) < (1/\varepsilon)(d(x, z) - d(u, v))$.

Now,

$$d(x, u) = \delta d(x, z') \leq \delta (d(x, z) + \varepsilon \text{ dist}(x, Tx)) \leq \delta (d(x, z) + \varepsilon d(x, z)) \leq \delta(1 + \varepsilon)d(x, z). \quad (3.14)$$

Therefore

$$d(x, u) \leq \delta(1 + \varepsilon)d(x, z). \quad (3.15)$$

It follows that

$$d(z, v) \leq (k + \varepsilon)d(x, u) \leq (k + \varepsilon)\delta(1 + \varepsilon)d(x, z). \quad (3.16)$$

We now let $y = (1 - \delta)x \oplus \delta z$, then by the condition (2.11),

$$d(u, v) \leq d(u, y) + d(y, z) + d(z, v) \leq \delta d(z, z') + (1 - \delta)d(x, z) + (k + \varepsilon)\delta(1 + \varepsilon)d(x, z) \leq \delta \varepsilon d(x, z) + ((1 - \delta) + (k + \varepsilon)\delta(1 + \varepsilon))d(x, z). \quad (3.17)$$

Thus

$$d(u, v) \leq (\delta \varepsilon + (1 - \delta) + (k + \varepsilon)\delta(1 + \varepsilon))d(x, z). \quad (3.18)$$

Inequalities (3.15), (3.18), and (3.10) imply that

$$\varepsilon d(x, u) + d(u, v) \leq \varepsilon \delta(1 + \varepsilon)d(x, z) + \delta \varepsilon + (1 - \delta) + (k + \varepsilon)\delta(1 + \varepsilon))d(x, z) \leq (\delta \varepsilon + (1 - \delta) + (k + 2\varepsilon)\delta(1 + \varepsilon))d(x, z) < d(x, z). \quad (3.19)$$

Therefore $d(x, u) < (1/\varepsilon)(d(x, z) - d(u, v))$ as desired. \qed
By Lemmas 3.1 and 3.2 with the same arguments in the proof of Theorem 3.3 of [6], we can obtain the following theorem which extends [4, Theorem 8] by Lim and [6, Theorem 3.3] by Dhomponsa et al.

**Theorem 3.3.** Let $X$ be a complete uniformly convex metric space satisfying condition (2.11), $C$ a nonempty bounded closed convex subset of $X$, and $T : C \to \mathcal{K}(X)$ a nonexpansive weakly inward mapping. Then $T$ has a fixed point.

As an immediate consequence of Theorem 3.3, we obtain the following corollary.

**Corollary 3.4.** Let $X$ be a complete uniformly convex metric space satisfying condition (2.11), $C$ a nonempty bounded closed convex subset of $X$, and $T : C \to \mathcal{K}(C)$ a nonexpansive mapping. Then $T$ has a fixed point.

In fact, this corollary is a special case of [10, Theorem 2] in which condition (2.11) was not assumed. An interesting question is whether condition (2.11) in Theorem 3.3 can be dropped.

Let $C$ be a nonempty subset of a metric space $X$. Recall that a single-valued mapping $t : C \to C$ and a multivalued mapping $T : C \to 2^C \setminus \emptyset$ are said to be commuting if $ty \in Ttx$ for all $y \in Tx$ and $x \in C$. If $t : C \to C$ is nonexpansive with $C$ being bounded closed convex and $X$ complete uniformly convex satisfying condition (2.11), then $\text{Fix}(t)$ is nonempty by the above corollary. Moreover, by a standard argument, we can show that it is also closed and convex. So we can obtain a common fixed point theorem in uniformly convex metric spaces as [6, Theorem 4.1] (see also [16, Theorem 4.2] for a related result in Banach spaces).

**Theorem 3.5.** Let $X$ be a complete uniformly convex metric space satisfying condition (2.11), let $C$ be a nonempty bounded closed convex subset of $X$, and let $t : C \to C$ and $T : C \to \mathcal{K}(C)$ be nonexpansive. Assume that for some $p \in \text{Fix}(t)$,

$$ap \oplus (1 - \alpha)Tx \text{ is convex } \forall x \in C, \forall \alpha \in [0, 1].$$

(3.20)

If $t$ and $T$ are commuting, then there exists a point $z \in C$ such that $tz = z \in Tz$.

### 4. Fixed point sets of quasi-nonexpansive mappings

Let $X$ be a metric space. Recall that a mapping $f : X \to X$ is said to be quasi-nonexpansive if $d(f(x), p) \leq d(x, p)$ for all $x \in X$ and $p \in \text{Fix}(f)$. In this case, we will assume that $\text{Fix}(f) \neq \emptyset$.

In [17], Chaoha and Phon-on showed that if $X$ is a CAT(0) space, then $\text{Fix}(f)$ is closed convex. Furthermore, they gave an explicit construction of a continuous function defined on $X$ whose fixed point set is any prescribed closed subset of $X$. In this section, we extend these results to uniformly convex metric spaces.

We begin by proving the following lemma.

**Lemma 4.1.** Let $X$ be a uniformly convex metric space, and let $x, y, z \in X$ for which

$$d(x, z) + d(z, y) = d(x, y).$$

(4.1)

Then $z \in [x, y]$.
Proof. Let \( u \in [x, y] \) be such that \( d(x, u) = d(x, z) \). Then \( d(x, y) = d(x, u) + d(u, y) \) and also \( d(z, y) = d(u, y) \) by (4.1). We will show that \( z = u \). Suppose not, we let \( v = (1/2)z \oplus (1/2)u \) and \( r = d(x, u) = d(x, z) \). Since \( d(z, u) > 0 \), choose \( \epsilon > 0 \) so that \( d(z, u) > r \epsilon \). By the uniform convexity of \( X \), there exists \( \delta > 0 \) such that
\[
d(x, v) \leq r(1 - \delta) < r = d(x, z).
\] (4.2)
By using the same arguments, we can show that \( d(y, v) < d(y, z) \). Therefore
\[
d(x, y) < d(x, v) + d(y, v) < d(x, z) + d(y, z) = d(x, y),
\] (4.3)
which is a contradiction.

By using the above lemma with the proof of Theorem 1.3 of [17], we obtain the following result.

**Theorem 4.2.** Let \( X \) be a convex subset of a uniformly convex metric space and \( f : X \rightarrow X \) a quasi-nonexpansive mapping whose fixed point set is nonempty. Then \( \text{Fix}(f) \) is closed convex.

In [17], the authors constructed a continuous function defined on a CAT(0) space \( X \) whose fixed point set is any prescribed closed subset of \( X \) by using the following two implications of the (CN) inequality:
\[
d((1 - t)x \oplus ty, (1 - s)x \oplus sy) = |t - s|d(x, y) \quad \forall x, y \in X, t, s \in [0, 1],
\] (4.4)
\[
d((1 - t)x \oplus ty, (1 - t)x \oplus tz) \leq d(y, z) \quad \forall x, y, z \in X, t \in [0, 1].
\] (4.5)

In fact, condition (4.4) holds in uniformly convex metric spaces as the following lemma shows.

**Lemma 4.3.** Condition (4.4) holds in uniformly convex metric spaces.

**Proof.** We first note that the conclusion holds if \( s = 0 \) or \( t = 0 \). We now let \( X \) be a uniformly convex metric space, \( x, y \in X \), and \( t, s \in (0, 1] \). Let \( u = (1 - t)x \oplus ty \) and \( z = (1 - s)x \oplus sy \). Without loss of generality, we can assume that \( t \leq s \). Let \( v = (1 - t/s)x \oplus (t/s)z \), then
\[
d(x, v) = \frac{t}{s}d(x, z) = td(x, y),
\] (4.6)
\[
d(v, y) \leq \left(1 - \frac{t}{s}\right)d(x, y) + \frac{t}{s}d(z, y) = (1 - t)d(x, y).
\]
If \( u \neq v \), we let \( w = (1/2)u \oplus (1/2)v \). Then by the uniform convexity of \( X \), we can show that \( d(x, w) < d(x, u) \) and \( d(y, w) < d(y, u) \). This implies
\[
d(x, y) < d(x, u) + d(u, y) = d(x, y),
\] (4.7)
which is a contradiction, hence \( u = v \). Therefore
\[
d(z, u) = d(z, v) = \left(1 - \frac{t}{s}\right)d(x, z) = |s - t|d(x, y).\] (4.8)

It is unclear that condition (4.5) holds for uniformly convex metric spaces. However, the following theorem is a generalization of [17, Theorem 2.1], we omit the proof because it is similar to the one given in [17].
Theorem 4.4. Let $A$ be a nonempty subset of a uniformly convex metric space $X$ satisfying condition (4.5). Then there exists a continuous function $f : X \to X$ such that $\text{Fix}(f) = A$.

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