Research Article

Strong Convergence Theorem for Two Commutative Asymptotically Nonexpansive Mappings in Hilbert Space

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C is a bounded closed convex subset of a Hilbert space H , T and $S : C \rightarrow C$ are two asymptotically nonexpansive mappings such that $ST = TS$. We establish a strong convergence theorem for S and T in Hilbert space by hybrid method. The results generalize and unify many corresponding results.

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1. Introduction

Let C be a bounded closed convex subset of a Hilbert space H . Recall that a mapping $T : C \rightarrow C$ is said to be asymptotically nonexpansive mapping if

$$\|T^n x - T^n y\| \leq t_n \|x - y\| \quad \forall x, y \in C, \quad (1.1)$$

where $t_n \rightarrow 1$ ($n \rightarrow \infty$). We may assume that $t_n \geq 1$ for all $n = 1, 2, 3, \dots$. Denote by $F(T)$ the set of fixed points of T . Throughout this paper T and $S : C \rightarrow C$ are two commutative asymptotically nonexpansive mappings with asymptotical coefficients $\{t_n\}$ and $\{s_n\}$, respectively. Suppose that $F := F(T) \cap F(S) \neq \emptyset$ ([1, Goebel and Kirk's theorem] makes it possible). It is well known that $F(T)$ and $F(S)$ are convex and closed [1, 2], so is F . P_K denotes the metric projection from H onto a closed convex subset K of H and $\omega_w(x_n)$ denotes the weak w -limit set of $\{x_n\}$. It is well known that a Hilbert space H satisfies Opial's condition [3], that is, if a sequence $\{x_n\}$ converges weakly to an element $y \in H$ and $y \neq z$, then

$$\liminf_{n \rightarrow \infty} \|x_n - y\| < \liminf_{n \rightarrow \infty} \|x_n - z\|. \quad (1.2)$$

Up to now, fixed points iteration processes for nonexpansive and asymptotically nonexpansive mappings have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities [4–6]. There are many strong convergence theorems for nonexpansive and asymptotically nonexpansive mappings in Hilbert space [7, 8].

Especially, Shimizu and Takahashi [7] studied the following iteration process of nonexpansive mappings for arbitrary $x_0 \in C$:

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n, \quad (1.3)$$

where $\{\alpha_n\} \subseteq [0, 1]$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$. And then they proved that $\{x_n\}$ converges strongly to $P_F(x_0)$. This result was extended to two commutative asymptotically nonexpansive mappings by Shioji and Takahashi [9].

Recently, some attempts to the modified Mann iteration method are made so that strong convergence is guaranteed. And for hybrid method proposed by Haugazeau [10], Kim and Xu [8] introduced the following iteration processes for asymptotically nonexpansive mapping T :

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily,} \\ y_n &= \alpha_n x_n + (1 - \alpha_n) T^n x_n, \\ C_n &= \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ Q_n &= \{v \in C : \langle x_n - v, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned} \quad (1.4)$$

where $\theta_n = (1 - \alpha_n)(t_n^2 - 1)(\text{diam } C)^2 \rightarrow 0$ as $n \rightarrow \infty$. Then proved that $\{x_n\}$ converges strongly to $P_F(x_0)$. This result was generalized to two asymptotically nonexpansive mappings by Plubtieng and Ungchittrakool [11].

On the basis of (1.3) and (1.4), we propose a new iteration processes for two commutative asymptotically nonexpansive mappings S and T :

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily,} \\ y_n &= \alpha_n x_n + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n, \\ C_n &= \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ Q_n &= \{v \in C : \langle x_n - v, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned} \quad (1.5)$$

where $\theta_n = (1 - \alpha_n)(g_n^2 - 1)(\text{diam } C)^2$, $g_n = (2/(n+1)(n+2)) \sum_{k=0}^n \sum_{i+j=k} s_i t_j$, for every $n = 1, 2, \dots$. The purpose of this paper is to prove $\{x_n\}$ converges strongly to $P_F(x_0)$.

2. Auxiliary lemmas

This section collects some lemmas which will be used to prove the main results in the next section.

Lemma 2.1 (see [7]). *Letting $L_n = (n+1)(n+2)/2$, there holds the identity in a Hilbert space H :*

$$\|y_n - v\|^2 = \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|x_{i,j} - v\|^2 - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|x_{i,j} - y_n\|^2 \quad (2.1)$$

for $\{x_{i,j}\}_{i,j=0}^\infty \subseteq H$, $y_n = (1/L_n) \sum_{k=0}^n \sum_{i+j=k} x_{i,j} \in H$ and $v \in H$.

Lemma 2.2. *Let C be a bounded closed convex subset of a Hilbert space H , S and T two commutative asymptotically nonexpansive mappings of C into itself with asymptotical coefficients $\{s_n\}$ and $\{t_n\}$, respectively. For any $x \in C$, put $F_n(x) = (2/(n+1)(n+2)) \sum_{k=0}^n \sum_{i+j=k} S^i T^j x$. Then*

$$\begin{aligned} \lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{x \in C} \|F_n(x) - S^l F_n(x)\| &= 0, \\ \lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{x \in C} \|F_n(x) - T^l F_n(x)\| &= 0. \end{aligned} \quad (2.2)$$

Proof. Put $x_{i,j} = S^i T^j x$, $v = S^l F_n(x)$ and $L_n = (n+1)(n+2)/2$. It follows from Lemma 2.1 that

$$\begin{aligned} & \|F_n(x) - S^l F_n(x)\|^2 \\ &= \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - S^l F_n(x)\|^2 - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - F_n(x)\|^2 \\ &= \frac{1}{L_n} \sum_{k=0}^{l-1} \sum_{i+j=k} \|S^i T^j x - S^l F_n(x)\|^2 + \frac{1}{L_n} \sum_{k=l}^n \sum_{i+j=k, i \leq l-1} \|S^i T^j x - S^l F_n(x)\|^2 \\ &\quad + \frac{1}{L_n} \sum_{k=l}^n \sum_{i+j=k, i \geq l} \|S^i T^j x - S^l F_n(x)\|^2 - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - F_n(x)\|^2 \\ &\leq \frac{1}{L_n} \sum_{k=0}^{l-1} \sum_{i+j=k} \|S^i T^j x - S^l F_n(x)\|^2 + \frac{1}{L_n} \sum_{k=l}^n \sum_{i+j=k, i \leq l-1} \|S^i T^j x - S^l F_n(x)\|^2 \\ &\quad + \frac{1}{L_n} \sum_{k=l}^n \sum_{i+j=k, i \geq l} s_l^2 \|S^{i-l} T^j x - F_n(x)\|^2 - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - F_n(x)\|^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{L_n} \sum_{k=0}^{l-1} \sum_{i+j=k} \|S^i T^j x - S^l F_n(x)\|^2 + \frac{1}{L_n} \sum_{k=l}^n \sum_{i+j=k, i \leq l-1} \|S^i T^j x - S^l F_n(x)\|^2 \\
&\quad + \frac{1}{L_n} \sum_{k=0}^{n-l} \sum_{i+j=k} s_l^2 \|S^i T^j x - F_n(x)\|^2 - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - F_n(x)\|^2 \\
&\leq \frac{1}{L_n} \sum_{k=0}^{l-1} \sum_{i+j=k} \|S^i T^j x - S^l F_n(x)\|^2 + \frac{1}{L_n} \sum_{k=l}^n \sum_{i+j=k, i \leq l-1} \|S^i T^j x - S^l F_n(x)\|^2 \\
&\quad + \frac{1}{L_n} \sum_{k=0}^{n-l} \sum_{i+j=k} (s_l^2 - 1) \|S^i T^j x - F_n(x)\|^2.
\end{aligned} \tag{2.3}$$

Choose $p \in F$, then there exists a constant $M > 0$ such that

$$\begin{aligned}
\|S^i T^j x - p\| &\leq s_i t_j \|x - p\| \leq \frac{M}{2}, \\
\|F_n(x) - p\| &\leq \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - p\| \leq \frac{M}{2}, \\
\|S^l F_n(x) - p\| &\leq s_l \|F_n(x) - p\| \leq \frac{M}{2},
\end{aligned} \tag{2.4}$$

for all nonnegative integer i, j, l , and n . Hence, $\|S^i T^j x - S^l F_n(x)\| \leq M$, $\|S^i T^j x - F_n(x)\| \leq M$ for all nonnegative integer i, j, l , and n . So

$$\begin{aligned}
&\sup_{x \in C} \|F_n(x) - S^l F_n(x)\|^2 \\
&\leq \frac{(l+1)l}{(n+2)(n+1)} M^2 + \frac{2(n+1-l)l}{(n+2)(n+1)} M^2 + \frac{(s_l^2 - 1)(n+2-l)(n+1-l)}{(n+2)(n+1)} M^2 \\
&\longrightarrow 0 \quad (n \longrightarrow \infty, l \longrightarrow \infty).
\end{aligned} \tag{2.5}$$

Similarly, we can prove

$$\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{x \in C} \|F_n(x) - T^l F_n(x)\| = 0. \tag{2.6}$$

□

Remark 2.3. Lemma 2.2 extends [7, Lemma 1].

Lemma 2.4. Let S and T be two commutative asymptotically nonexpansive mappings defined on a bounded closed convex subset C of a Hilbert space H with asymptotical coefficients $\{s_n\}$ and $\{t_n\}$, respectively. Let $L_n = ((n+1)(n+2)/2)$. If $\{x_n\}$ is a sequence in C such that $\{x_n\}$ converges weakly to some $x \in C$ and $\{x_n - (1/L_n) \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n\}$ converges strongly to 0, then $x \in F(S) \cap F(T)$.

Proof. We claim that $\{S^l x\}$ converges strongly to x as $l \rightarrow \infty$. If not, there exist a positive number ε_0 and a subsequence $\{l_m\}$ of $\{l\}$ such that $\|S^{l_m} x - x\| \geq \varepsilon_0$ for all m . However, we have

$$\begin{aligned}
& \|x_n - S^{l_m} x\| \\
& \leq \left\| x_n - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n \right\| + \left\| \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n - S^{l_m} \left(\frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n \right) \right\| \\
& \quad + \left\| S^{l_m} \left(\frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n \right) - S^{l_m} x \right\| \\
& \leq \left\| x_n - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n \right\| + \left\| \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n - S^{l_m} \left(\frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n \right) \right\| \quad (2.7) \\
& \quad + s_{l_m} \left\| \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n - x \right\| \\
& \leq \left\| x_n - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n \right\| + \left\| \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n - S^{l_m} \left(\frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n \right) \right\| \\
& \quad + s_{l_m} \left\| \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n - x \right\| + s_{l_m} \|x_n - x\|.
\end{aligned}$$

By Opial's condition, for any $y \in C$ with $y \neq x$, we have

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|. \quad (2.8)$$

Let $r = \liminf_{n \rightarrow \infty} \|x_n - x\|$ and choose a positive number ρ such that

$$\rho < \sqrt{r^2 + \frac{\varepsilon_0^2}{4}} - r. \quad (2.9)$$

Then, there exists a subsequence $\{x_{n_p}\}$ of $\{x_n\}$ such that $\lim_{p \rightarrow \infty} \|x_{n_p} - x\| = r$ and $\|x_{n_p} - x\| < r + (\rho/4)$ for all p . By definition of $\{s_{l_m}\}$, there exists a positive integer m_0 such that

$$s_{l_m} \|x_{n_p} - x\| < r + \frac{\rho}{4}, \quad (2.10)$$

for all $m > m_0$. Since

$$\lim_{n \rightarrow \infty} \left\| x_n - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n \right\| = 0 \quad (2.11)$$

and $\{s_{l_m}\}$ is bounded, there exists a positive integer p_0 such that

$$\begin{aligned} \left\| x_{n_p} - \frac{1}{L_{n_p}} \sum_{k=0}^{n_p} \sum_{i+j=k} S^i T^j x_{n_p} \right\| &< \frac{\rho}{4}, \\ s_{l_m} \left\| \frac{1}{L_{n_p}} \sum_{k=0}^{n_p} \sum_{i+j=k} S^i T^j x_{n_p} - x_{n_p} \right\| &< \frac{\rho}{4} \end{aligned} \quad (2.12)$$

for all m and $p > p_0$. By $\{x_{n_p}\} \subset C$ is bounded and Lemma 2.2, there exist $m_1 > m_0$ and $p_1 > 0$ such that

$$\left\| \frac{1}{L_{n_p}} \sum_{k=0}^{n_p} \sum_{i+j=k} S^i T^j x_{n_p} - S^{l_{m_1}} \left(\frac{1}{L_{n_p}} \sum_{k=0}^{n_p} \sum_{i+j=k} S^i T^j x_{n_p} \right) \right\| < \frac{\rho}{4} \quad (2.13)$$

for all $p > p_1$. By (2.7), (2.10), (2.12), and (2.13), we have

$$\|x_{n_p} - S^{l_{m_1}} x\| < \frac{\rho}{4} + \frac{\rho}{4} + \frac{\rho}{4} + r + \frac{\rho}{4} = r + \rho \quad (2.14)$$

for all $p > \max\{p_0, p_1\}$. However,

$$\begin{aligned} \left\| x_{n_p} - \frac{S^{l_{m_1}} x + x}{2} \right\|^2 &= \frac{1}{2} \|x_{n_p} - S^{l_{m_1}} x\|^2 + \frac{1}{2} \|x_{n_p} - x\|^2 - \frac{1}{4} \|S^{l_{m_1}} x - x\|^2 \\ &< \frac{(r + \rho)^2}{2} + \frac{(r + \rho/4)^2}{2} - \frac{\varepsilon_0^2}{4} \\ &< (r + \rho)^2 - \frac{\varepsilon_0^2}{4} \\ &< r^2 \end{aligned} \quad (2.15)$$

for all $p > \max\{p_0, p_1\}$. This contradicts (2.8). So $\{S^l x\}$ converges strongly to x and then $x \in F(S)$. Similarly, we can get $x \in F(T)$. Hence, x is a common fixed point of S and T . $\square$

Lemma 2.5 (see [12]). *Let C be a bounded closed convex subset of a Hilbert space H . The set $D := \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v \rangle + b\}$ is convex and closed for given $x, y, z \in C$ and $b \in \mathbb{R}$.*

3. Main results

In this section, we prove our main theorem.

Theorem 3.1. *Let C be a bounded closed convex subset of a Hilbert H , T and $S : C \rightarrow C$ be two commutative asymptotically nonexpansive mappings with asymptotical coefficients $\{t_n\}$ and $\{s_n\}$, respectively. Suppose that $0 \leq \alpha_n \leq a$ for all n , where $0 < a < 1$. If $F := F(T) \cap F(S) \neq \emptyset$, then the sequence generated by (1.5) converges strongly to $P_F(x_0)$.*

Proof. Note that C_n is convex and closed for all $n \geq 0$ by Lemma 2.5. On the other hand, Q_n is convex and closed. So is $C_n \cap Q_n$.

By definition of $\{t_n\}$ and $\{s_n\}$, there exists $M > 0$ such that $\|s_i t_j - 1\| \leq M$ for all $i, j \geq 0$. On the other hand, for arbitrary $\varepsilon > 0$, there exists $N > 0$ such that $\|s_i t_j - 1\| < \varepsilon$ for all $i, j > N$. Hence

$$\begin{aligned}
\|g_n - 1\| &= \left\| \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} (s_i t_j - 1) \right\| \\
&\leq \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} \|s_i t_j - 1\| \\
&\leq \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k, i \leq N} \|s_i t_j - 1\| + \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k, j \leq N} \|s_i t_j - 1\| \\
&\quad + \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k, i \geq N+1, j \geq N+1} \|s_i t_j - 1\| \\
&< \frac{2(N+1)M}{(n+2)} + \frac{2(N+1)M}{(n+2)} + \varepsilon.
\end{aligned} \tag{3.1}$$

Thus $\lim_{n \rightarrow \infty} g_n = 1$. Obviously, $\lim_{n \rightarrow \infty} \theta_n = 0$.

Next, we prove that $F \subset C_n \cap Q_n$. Indeed, first of all

$$\begin{aligned}
\|y_n - p\|^2 &\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) \left\| \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n - p \right\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1 - \alpha_n) g_n^2 \|x_n - p\|^2 \\
&= \|x_n - p\|^2 + (1 - \alpha_n) (g_n^2 \|x_n - p\|^2 - \|x_n - p\|^2) \\
&\leq \|x_n - p\|^2 + \theta_n
\end{aligned} \tag{3.2}$$

for all $p \in F$. So $F \subset C_n$. It suffices to show that $F \subset Q_n$ for all $n \geq 0$. We prove this by induction. For $n = 0$, we have $F \subset C = Q_0$. Assume that $F \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, we have

$$\langle x_{n+1} - z, x_0 - x_{n+1} \rangle \geq 0 \quad \forall z \in C_n \cap Q_n \tag{3.3}$$

As $F \subset C_n \cap Q_n$, (3.3) holds for all $z \in F$, in particular. This together with the definition of Q_{n+1} implies that $F \subset Q_{n+1}$. Hence, $F \subset C_n \cap Q_n$ for all $n \geq 0$.

We will show that $\|x_{n+1} - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. By the definition of Q_n , we have that $x_n = P_{Q_n}(x_0)$. It follows from $x_{n+1} \in C_n \cap Q_n \subset Q_n$ that $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$. This shows that the sequence $\{\|x_n - x_0\|\}$ is increasing. Since C is bounded, we obtain that $\lim_{n \rightarrow \infty} \|x_n - x_0\|$ exists.

Notice again that from $x_n = P_{Q_n}(x_0)$ and $x_{n+1} \in Q_n$, we have $\langle x_{n+1} - x_n, x_n - x_0 \rangle \geq 0$. Hence

$$\begin{aligned}
 \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
 &= \|x_{n+1} - x_0\|^2 + \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_0, x_n - x_0 \rangle \\
 &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_0 - (x_n - x_0), x_n - x_0 \rangle \\
 &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\
 &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \\
 &\longrightarrow 0 \quad (n \longrightarrow \infty).
 \end{aligned} \tag{3.4}$$

Now we claim that $\|(2/(n+1)(n+2)) \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n - x_n\| \rightarrow 0$ as $n \rightarrow \infty$. By the definition of y_n , we have

$$\begin{aligned}
 &\left\| \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n - x_n \right\| \\
 &= \frac{1}{1 - \alpha_n} \|y_n - x_n\| \\
 &\leq \frac{1}{1 - \alpha_n} (\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|) \\
 &\leq \frac{1}{1 - a} (\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|).
 \end{aligned} \tag{3.5}$$

Since $x_{n+1} \in C_n$, $\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n \rightarrow 0$ as $n \rightarrow \infty$. So $\|y_n - x_{n+1}\| \rightarrow 0$ as $n \rightarrow \infty$. This implies that

$$\left\| \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n - x_n \right\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.6}$$

Since C is bounded closed convex, $\omega_w(x_n) \neq \emptyset$. It follows from (3.6) and Lemma 2.4 that $\omega_w(x_n) \subset F$. By the definition of Q_n , we have that $\|x_n - x_0\| \leq \|P_F(x_0) - x_0\|$ for all $n \geq 0$. It follows from the weak lower semi-continuity of the norm that $\|w - x_0\| \leq \|P_F(x_0) - x_0\|$ for all $w \in \omega_w(x_n)$. Since $\omega_w(x_n) \subset F$, we have $w = P_F(x_0)$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{P_F(x_0)\}$. Then, $\{x_n\}$ converges to $P_F(x_0)$ weakly. By the fact

$$\begin{aligned}
 \|x_n - P_F(x_0)\|^2 &= \|x_n - x_0 + x_0 - P_F(x_0)\|^2 \\
 &= \|x_n - x_0\|^2 + \|x_0 - P_F(x_0)\|^2 + 2\langle x_n - x_0, x_0 - P_F(x_0) \rangle \\
 &\leq 2(\|P_F(x_0) - x_0\|^2 + \langle x_n - x_0, x_0 - P_F(x_0) \rangle) \\
 &\longrightarrow 0 \quad (n \longrightarrow \infty),
 \end{aligned} \tag{3.7}$$

we have $\{x_n\}$ converges to $P_F(x_0)$ strongly. This completes the proof. $\square$

The following corollary follows from Theorem 3.1.

Corollary 3.2. *Let C be a bounded closed convex subset of a Hilbert H , T and $S : C \rightarrow C$ be two commutative nonexpansive mappings. Suppose that $0 \leq \alpha_n \leq a$ for all n , where $0 < a < 1$. If $F := F(T) \cap F(S) \neq \emptyset$, then the sequence $\{x_n\}$ generated by*

$$\begin{aligned} x_0 &\in C \text{ chosen arbitrarily,} \\ y_n &= \alpha_n x_n + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n, \\ C_n &= \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2\}, \\ Q_n &= \{v \in C : \langle x_n - v, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned} \tag{3.8}$$

converges strongly to $P_F(x_0)$.

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