Research Article

The Order of Generalized Hypersubstitutions of Type $\tau = (2)$

Wattapong Puninagool and Sorasak Leeratanavalee

Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand

Correspondence should be addressed to Sorasak Leeratanavalee, scislr@chiangmai.ac.th

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The order of hypersubstitutions, all idempotent elements on the monoid of all hypersubstitutions of type $\tau = (2)$ were studied by K. Denecke and Sh. L. Wismath and all idempotent elements on the monoid of all hypersubstitutions of type $\tau = (2, 2)$ were studied by Th. Changpas and K. Denecke. We want to study similar problems for the monoid of all generalized hypersubstitutions of type $\tau = (2)$. In this paper, we use similar methods to characterize idempotent generalized hypersubstitutions of type $\tau = (2)$ and determine the order of each generalized hypersubstitution of this type. The main result is that the order is 1, 2 or infinite.

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1. Introduction

The concept of generalized hypersubstitutions was introduced by Leeratanavalee and Denecke [1]. We use it as a tool to study strong hyperidentities and use strong hyperidentities to classify varieties into collections called *strong hypervarieties*. Varieties which are closed under arbitrary application of generalized hypersubstitutions are called *strongly solid*.

A generalized hypersubstitution of type $\tau = (n_i)_{i \in I}$, for short, a generalized hypersubstitution is a mapping σ which maps each n_i -ary operation symbol of type τ to the set $W_\tau(X)$ of all terms of type τ built up by operation symbols from $\{f_i \mid i \in I\}$ where f_i is n_i -ary and variables from a countably infinite alphabet of variables $X := \{x_1, x_2, x_3, \dots\}$ which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type τ by $\text{Hyp}_G(\tau)$. First, we define inductively the concept of *generalized superposition of terms* $S^m : W_\tau(X)^{m+1} \rightarrow W_\tau(X)$ by the following steps:

- (i) if $t = x_j$, $1 \leq j \leq m$, then $S^m(x_j, t_1, \dots, t_m) := t_j$;
- (ii) if $t = x_j$, $m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \dots, t_m) := x_j$;

(iii) if $t = f_i(s_1, \dots, s_{n_i})$, then

$$S^m(t, t_1, \dots, t_m) := f_i(S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m)). \quad (1.1)$$

We extend a generalized hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \rightarrow W_\tau(X)$ inductively defined as follows:

- (i) $\hat{\sigma}[x] := x \in X$;
- (ii) $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$, for any n_i -ary operation symbol f_i supposed that $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Then we define a binary operation $\circ_G$ on $\text{Hyp}_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in \text{Hyp}_G(\tau)$. Let σ_{id} be the hypersubstitution which maps each n_i -ary operation symbol f_i to the term $f_i(x_1, \dots, x_{n_i})$. We proved the following propositions.

Proposition 1.1 (see [1]). *For arbitrary terms $t, t_1, \dots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ one has*

- (i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \dots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \dots, t_n)]$;
- (ii) $(\hat{\sigma}_1 \circ \sigma_2)^\wedge = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

Proposition 1.2 (see [1]). *$\text{Hyp}_G(\tau) = (\text{Hyp}_G(\tau); \circ_G, \sigma_{id})$ is a monoid and the set of all hypersubstitutions of type τ forms a submonoid of $\text{Hyp}_G(\tau)$.*

The order of the element a is defined as the order of the cyclic subsemigroup $\langle a \rangle$. The order of any hypersubstitution of type $\tau = (2)$ was determined in [2].

Theorem 1.3 ([2]). *Let $\tau = (2)$ be a type. The order of any hypersubstitution of type τ is 1, 2 or infinite.*

In Section 4, we characterize the order of generalized hypersubstitutions of type $\tau = (2)$.

2. Idempotent elements in $\text{Hyp}_G(2)$

In this section, we consider especially the idempotent elements of $\text{Hyp}_G(2)$. We have only one binary operation symbol, say f . The generalized hypersubstitution σ which maps f to the term t is denoted by σ_t . For any term $t \in W_{(2)}(X)$, the set of all variables occurring in t is denoted by $\text{var}(t)$. First, we will recall the definition of an idempotent element.

Definition 2.1. For any semigroup S , an element $e \in S$ is called idempotent if $ee = e$. In general, by $E(S)$ we denote the set of all idempotent elements of S .

Proposition 2.2. *An element $\sigma_t \in \text{Hyp}_G(2)$ is idempotent if and only if $\widehat{\sigma}_t[t] = t$.*

Proof. Assume that σ_t is idempotent, that is, $\sigma_t^2 = \sigma_t$. Then

$$\widehat{\sigma}_t[t] = \widehat{\sigma}_t[\sigma_t(f)] = \sigma_t^2(f) = \sigma_t(f) = t. \quad (2.1)$$

Conversely, let $\widehat{\sigma}_t[t] = t$. We have $(\sigma_t \circ_G \sigma_t)(f) = \widehat{\sigma}_t[\sigma_t(f)] = \widehat{\sigma}_t[t] = t = \sigma_t(f)$. Thus $\sigma_t^2 = \sigma_t$. $\square$

Proposition 2.3. *For every $x_i \in X$, σ_{x_i} and σ_{id} are idempotent.*

Proof. Since for every $x_i \in X$, $\widehat{\sigma}_{x_i}[x_i] = x_i$. By Proposition 2.2 we have σ_{x_i} is idempotent. σ_{id} is idempotent because it is a neutral element. $\square$

Proposition 2.4. *For every $i, j \in \mathbb{N}$, the generalized hypersubstitutions $\sigma_{f(x_1, x_j)}$ and $\sigma_{f(x_i, x_2)}$ are idempotent.*

Proof. Let $i, j \in \mathbb{N}$. Then we have

$$\begin{aligned} \widehat{\sigma}_{f(x_1, x_j)}[f(x_1, x_j)] &= S^2(\sigma_{f(x_1, x_j)}(f), x_1, x_j) = S^2(f(x_1, x_j), x_1, x_j) = f(x_1, x_j), \\ \widehat{\sigma}_{f(x_i, x_2)}[f(x_i, x_2)] &= S^2(\sigma_{f(x_i, x_2)}(f), x_i, x_2) = S^2(f(x_i, x_2), x_i, x_2) = f(x_i, x_2). \end{aligned} \quad (2.2)$$

Note that for any $t \in W_{(2)}(X) \setminus X$ and $x_1, x_2 \notin \text{var}(t)$, σ_t is idempotent. Since there is nothing to substitute in the term $\widehat{\sigma}_t[t]$, thus $\widehat{\sigma}_t[t] = t$. $\square$

Proposition 2.5. *Let $t \in W_{(2)}(X) \setminus X$. Then the following propositions hold:*

- (i) *if $x_2 \notin \text{var}(t)$, then $\sigma_{f(x_1, t)}$ is idempotent;*
- (ii) *if $x_1 \notin \text{var}(t)$, then $\sigma_{f(t, x_2)}$ is idempotent.*

Proof. (i) Let $x_2 \notin \text{var}(t)$. Then

$$\widehat{\sigma}_{f(x_1, t)}[f(x_1, t)] = S^2(\sigma_{f(x_1, t)}(f), x_1, \widehat{\sigma}_{f(x_1, t)}[t]) = S^2(f(x_1, t), x_1, \widehat{\sigma}_{f(x_1, t)}[t]) = f(x_1, t). \quad (2.3)$$

(ii) Let $x_1 \notin \text{var}(t)$. Then

$$\widehat{\sigma}_{f(t, x_2)}[f(t, x_2)] = S^2(\sigma_{f(t, x_2)}(f), \widehat{\sigma}_{f(t, x_2)}[t], x_2) = S^2(f(t, x_2), \widehat{\sigma}_{f(t, x_2)}[t], x_2) = f(t, x_2). \quad (2.4)$$

$\square$

3. Nonidempotent elements of $\text{Hyp}_G(2)$

In this section, we characterize all elements of $\text{Hyp}_G(2)$ which are not idempotent.

Proposition 3.1. *If $i, j \in \mathbb{N}$ with $i \neq 1$ and $j \neq 2$, then $\sigma_{f(x_2, x_j)}$ and $\sigma_{f(x_i, x_1)}$ are not idempotent.*

Proof. Let $i, j \in \mathbb{N}$ with $i \neq 1$ and $j \neq 2$. Since $j \neq 2$, $\widehat{\sigma}_{f(x_2, x_j)}[f(x_2, x_j)] = S^2(f(x_2, x_j), x_2, x_j) \neq f(x_2, x_j)$. Since $i \neq 1$, $\widehat{\sigma}_{f(x_i, x_1)}[f(x_i, x_1)] = S^2(f(x_i, x_1), x_i, x_1) \neq f(x_i, x_1)$. $\square$

Proposition 3.2. *Let $t \in W_{(2)}(X) \setminus X$. Then the following propositions hold:*

- (i) *if $x_2 \in \text{var}(t)$, then $\sigma_{f(x_1,t)}$ is not idempotent;*
- (ii) *if $x_1 \in \text{var}(t)$, then $\sigma_{f(t,x_2)}$ is not idempotent;*
- (ii) *$\sigma_{f(t,x_1)}$ and $\sigma_{f(x_2,t)}$ are not idempotent;*
- (iv) *if $x_1 \in \text{var}(t)$ or $x_2 \in \text{var}(t)$, then $\sigma_{f(x_i,t)}$ and $\sigma_{f(t,x_i)}$ are not idempotent for any $i \in \mathbb{N}$ with $i > 2$.*

Proof. (i) Let $x_2 \in \text{var}(t)$. Then we have $\widehat{\sigma}_{f(x_1,t)}[f(x_1,t)] = S^2(f(x_1,t), x_1, \widehat{\sigma}_{f(x_1,t)}[t])$. Since $x_2 \in \text{var}(t)$, then we have to substitute x_2 in the term t by $\widehat{\sigma}_{f(x_1,t)}[t]$. $S^2(f(x_1,t), x_1, \widehat{\sigma}_{f(x_1,t)}[t]) \neq f(x_1,t)$.

The proofs of (ii), (iii), and (iv) are similar to (i). $\square$

Proposition 3.3. *Let $t_1, t_2 \in W_{(2)}(X) \setminus X$. If $x_1 \in \text{var}(t_1) \cup \text{var}(t_2)$ or $x_2 \in \text{var}(t_1) \cup \text{var}(t_2)$, then $\sigma_{f(t_1,t_2)}$ is not idempotent.*

Proof. The proof is similar to that of Proposition 3.2. $\square$

By Sections 2 and 3, we get $P_G(2) \cup E_{x_1}^G \cup E_{x_2}^G \cup G \cup \{\sigma_{\text{id}}\}$ is the set of all idempotent elements in $\text{Hyp}_G(2)$ where $P_G(2) := \{\sigma_{x_i} \in \text{Hyp}_G(2) \mid i \in \mathbb{N}, x_i \in X\}$, $E_{x_1}^G := \{\sigma_{f(x_1,s)} \in \text{Hyp}_G(2) \mid s \in W_{(2)}(X), x_2 \notin \text{var}(s)\}$, $E_{x_2}^G := \{\sigma_{f(s,x_2)} \in \text{Hyp}_G(2) \mid s \in W_{(2)}(X), x_1 \notin \text{var}(s)\}$, and $G := \{\sigma_s \in \text{Hyp}_G(2) \mid s \in W_{(2)}(X) \setminus X, x_1, x_2 \notin \text{var}(s)\}$.

4. The order of generalized hypersubstitutions of type $\tau = (2)$

In this section, we characterize the order of generalized hypersubstitutions of type $\tau = (2)$. First, we introduce some notations. For $s, f(c, d) \in W_{(2)}(X)$, $x_i, x_j \in X$, $i, j \in \mathbb{N}$ we denote

$vb(s) :=$ the total number of variables occurring in the term s ;

$\text{leftmost}(s) :=$ the first variable (from the left) that occurs in s ;

$\text{rightmost}(s) :=$ the last variable that occurs in s ;

$W_{(2)}^G(\{x_1\}) := \{s \in W_{(2)}(X) \mid x_1 \in \text{var}(s), x_2 \notin \text{var}(s)\}$,

$W_{(2)}^G(\{x_2\}) := \{s \in W_{(2)}(X) \mid x_2 \in \text{var}(s), x_1 \notin \text{var}(s)\}$,

$\overline{f(c, d)} :=$ the term obtained from $f(c, d)$ by interchanging all occurrences of the letters x_1 and x_2 , that is, $\overline{f(c, d)} = S^2(f(c, d), x_2, x_1)$ and $f(c, d) = S^2(\overline{f(c, d)}, x_2, x_1)$;

$f(c, d)' :=$ the term defined inductively by $x_i' = x_i$ and $f(c, d)' = f(d', c')$;

$_{xi}C[f(c, d)] :=$ the term obtained from $f(c, d)$ by replacing each of the occurrences of the letter x_1 by x_i , that is, $_{xi}C[f(c, d)] = S^2(f(c, d), x_i, x_2)$;

$C_{x_i}[f(c, d)] :=$ the term obtained from $f(c, d)$ by replacing each of the occurrences of the letter x_2 by x_i , that is, $C_{x_i}[f(c, d)] = S^2(f(c, d), x_1, x_i)$;

$_{xi}C_{x_j}[f(c, d)] :=$ the term obtained from $f(c, d)$ by replacing each of the occurrences of the letter x_1 by x_i and the letter x_2 by x_j , that is, $_{xi}C_{x_j}[f(c, d)] = S^2(f(c, d), x_i, x_j)$.

An element a in a semigroup S is idempotent if and only if the order of a is 1. Then we consider only the order of generalized hypersubstitutions of type $\tau = (2)$ which are not idempotent. We have the following lemmas and propositions.

Lemma 4.1. *Let $f(c, d), f(u, v) \in W_{(2)}(X)$ and $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_w$. Then $vb(w) > vb(f(c, d))$ unless $f(c, d)$ and $f(u, v)$ match one of the following 16 possibilities:*

- E(1) $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_{f(c,d)}$ where $\sigma_{f(c,d)} \in G$;
- E(2) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1, x_1)} = \sigma_{C_{x_1}[f(c,d)]}$;
- E(3) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2, x_2)} = \sigma_{x_2 C[f(c,d)]}$;
- E(4) $\sigma_{f(c,d)} \circ_G \sigma_{id} = \sigma_{f(c,d)}$;
- E(5) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1, x_i)} = \sigma_{C_{x_i}[f(c,d)]}$ where $x_i \in X, i > 2$;
- E(6) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2, x_1)} = \sigma_{\overline{f(c,d)}}$;
- E(7) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2, x_i)} = \sigma_{x_2 C_{x_i}[f(c,d)]}$ where $x_i \in X, i > 2$;
- E(8) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i, x_1)} = \sigma_{x_i C_{x_1}[f(c,d)]}$ where $x_i \in X, i > 2$;
- E(9) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i, x_2)} = \sigma_{x_i C[f(c,d)]}$ where $x_i \in X, i > 2$;
- E(10) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i, x_j)} = \sigma_{x_i C_{x_j}[f(c,d)]}$ where $x_i, x_j \in X, i, j > 2$;
- E(11) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1, v)} = \sigma_{f(c,d)}$ where $v \notin X, f(c, d) \in W_{(2)}^G(\{x_1\})$;
- E(12) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2, v)} = \sigma_{\overline{f(c,d)}}$ where $v \notin X, f(c, d) \in W_{(2)}^G(\{x_1\})$;
- E(13) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i, v)} = \sigma_{x_i C[f(c,d)]}$ where $x_i \in X, i > 2, v \notin X, f(c, d) \in W_{(2)}^G(\{x_1\})$;
- E(14) $\sigma_{f(c,d)} \circ_G \sigma_{f(u, x_1)} = \sigma_{\overline{f(c,d)}}$ where $u \notin X, f(c, d) \in W_{(2)}^G(\{x_2\})$;
- E(15) $\sigma_{f(c,d)} \circ_G \sigma_{f(u, x_2)} = \sigma_{f(c,d)}$ where $u \notin X, f(c, d) \in W_{(2)}^G(\{x_2\})$;
- E(16) $\sigma_{f(c,d)} \circ_G \sigma_{f(u, x_i)} = \sigma_{C_{x_i}[f(c,d)]}$ where $x_i \in X, i > 2, u \notin X, f(c, d) \in W_{(2)}^G(\{x_2\})$.

Proof. Assume that $f(c, d), f(u, v) \in W_{(2)}(X)$ and $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_w$. We want to compare $vb(w)$ with $vb(f(c, d))$. From $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_w$, it follows that $w = S^2(f(c, d), \hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v])$. If $\sigma_{f(c,d)} \in G$, then $w = f(c, d)$ and we have E(1). Assume that $\sigma_{f(c,d)} \notin G$. Then $x_1 \in \text{var}(f(c, d))$ or $x_2 \in \text{var}(f(c, d))$. We will consider the following cases.

Case 1. If $u, v \in X$, then $\hat{\sigma}_{f(c,d)}[u] = u$ and $\hat{\sigma}_{f(c,d)}[v] = v$. This gives 9 possible subcases:

- (1) $u = v = x_1$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1, x_1)} = \sigma_{C_{x_1}[f(c,d)]}$, which is E(2);
- (2) $u = v = x_2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2, x_2)} = \sigma_{x_2 C[f(c,d)]}$, which is E(3);
- (3) $u = x_1, v = x_2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{id} = \sigma_{f(c,d)}$, which is E(4);
- (4) $u = x_1, v = x_i, i > 2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1, x_i)} = \sigma_{C_{x_i}[f(c,d)]}$, which is E(5);
- (5) $u = x_2, v = x_1$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2, x_1)} = \sigma_{\overline{f(c,d)}}$, which is E(6);
- (6) $u = x_2, v = x_i, i > 2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2, x_i)} = \sigma_{x_2 C_{x_i}[f(c,d)]}$, which is E(7);
- (7) $u = x_i, v = x_1, i > 2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i, x_1)} = \sigma_{x_i C_{x_1}[f(c,d)]}$, which is E(8);
- (8) $u = x_i, v = x_2, i > 2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i, x_2)} = \sigma_{x_i C[f(c,d)]}$, which is E(9);
- (9) $u = x_i, v = x_j, i, j > 2$: then we have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i, x_j)} = \sigma_{x_i C_{x_j}[f(c,d)]}$, which is E(10).

Case 2. If $u = x_1$ and $v \notin X$, then $w = S^2(f(c, d), x_1, \hat{\sigma}_{f(c,d)}[v])$. If $f(c, d) \in W_{(2)}^G(\{x_1\})$, then $w = f(c, d)$, as in E(11). Assume that $x_2 \in \text{var}(f(c, d))$. Since $vb(\hat{\sigma}_{f(c,d)}[v]) > 1$ and we have to substitute x_2 in $f(c, d)$ by $\hat{\sigma}_{f(c,d)}[v]$, we get $vb(w) > vb(f(c, d))$.

Case 3. $u = x_2$ and $v \notin X$. In this case we get E(12) or $vb(w) > vb(f(c, d))$.

Case 4. $u = x_i, i > 2$ and $v \notin X$. In this case we get E(13) or $vb(w) > vb(f(c, d))$.

Case 5. $u \notin X$ and $v = x_1$. In this case we get E(14) or $vb(w) > vb(f(c, d))$.

Case 6. $u \notin X$ and $v = x_2$. In this case we get E(15) or $vb(w) > vb(f(c, d))$.

Case 7. $u \notin X$ and $v = x_i, i > 2$. In this case we get E(16) or $vb(w) > vb(f(c, d))$.

Case 8. If $u, v \notin X$, then $vb(\hat{\sigma}_{f(c,d)}[u]) > 1$ and $vb(\hat{\sigma}_{f(c,d)}[v]) > 1$. Since $vb(\hat{\sigma}_{f(c,d)}[u]) > 1$, $vb(\hat{\sigma}_{f(c,d)}[v]) > 1$ and we have to substitute x_1 in $f(c, d)$ by $\hat{\sigma}_{f(c,d)}[u]$ or x_2 in $f(c, d)$ by $\hat{\sigma}_{f(c,d)}[v]$, we get $vb(w) > vb(f(c, d))$. $\square$

Lemma 4.2. Let $s \in W_{(2)}(X) \setminus X$, $x_1, x_2 \in \text{var}(s)$, $t \in W_{(2)}(X)$ and $x_i \in X$ where $i \in \mathbb{N}$. If $x_i \in \text{var}(t)$, then $x_i \in \text{var}(\hat{\sigma}_s[t])$.

Proof. If $t \in X$, then $t = x_i$. So $\hat{\sigma}_s[t] = x_i$ and thus $x_i \in \text{var}(\hat{\sigma}_s[t])$. Assume that $t = f(t_1, t_2)$ for some $t_1, t_2 \in W_{(2)}(X)$. Then $x_i \in \text{var}(t_1)$ or $x_i \in \text{var}(t_2)$. Assume that $x_i \in \text{var}(t_1)$ and $x_i \in \text{var}(\hat{\sigma}_s[t_1])$. Consider $\hat{\sigma}_s[t] = \hat{\sigma}_s[f(t_1, t_2)] = S^2(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2])$. Since $x_1 \in \text{var}(s)$ and $x_i \in \text{var}(\hat{\sigma}_s[t_1])$, we get $x_i \in \text{var}(\hat{\sigma}_s[t])$. By the same way, we can show that if $x_i \in \text{var}(t_2)$, then $x_i \in \text{var}(\hat{\sigma}_s[t])$. $\square$

Lemma 4.3. Let $s \in W_{(2)}(X) \setminus X$. If $x_1, x_2 \in \text{var}(s)$, then $x_1, x_2 \in \text{var}(\sigma_s^n(f))$ for all $n \in \mathbb{N}$.

Proof. Assume that $s = f(s_1, s_2)$ for some $s_1, s_2 \in W_{(2)}(X)$. For $n = 1$, $\sigma_s^1(f) = \sigma_s(f) = s$. So $x_1, x_2 \in \text{var}(\sigma_s^1(f))$. Assume that $x_1, x_2 \in \text{var}(\sigma_s^n(f))$. Consider $\sigma_s^{n+1}(f) = (\sigma_s^n \circ_G \sigma_s)(f) = \hat{\sigma}_s^n[\sigma_s(f)] = \hat{\sigma}_s^n[s] = \hat{\sigma}_s^n[f(s_1, s_2)] = S^2(\sigma_s^n(f), \hat{\sigma}_s^n[s_1], \hat{\sigma}_s^n[s_2])$. If $x_1, x_2 \in \text{var}(s_1)$, then by Lemma 4.2 we get $x_1, x_2 \in \text{var}(\hat{\sigma}_s^n[s_1])$. Since $x_1 \in \text{var}(\sigma_s^n(f))$ and $x_1, x_2 \in \text{var}(\hat{\sigma}_s^n[s_1])$, we get $x_1, x_2 \in \text{var}(\sigma_s^{n+1}(f))$. If $s_1 \in W_{(2)}^G(\{x_1\})$, then $x_2 \in \text{var}(s_2)$. By Lemma 4.2, we get $x_1 \in \text{var}(\hat{\sigma}_s^n[s_1])$ and $x_2 \in \text{var}(\hat{\sigma}_s^n[s_2])$. Since $x_1, x_2 \in \text{var}(\sigma_s^n(f))$, we get $x_1, x_2 \in \text{var}(\sigma_s^{n+1}(f))$. If $s_1 \in W_{(2)}^G(\{x_2\})$, then by the same proof as for the case $s_1 \in W_{(2)}^G(\{x_1\})$ we get $x_1, x_2 \in \text{var}(\sigma_s^{n+1}(f))$. If $x_1, x_2 \notin \text{var}(s_1)$, then $x_1, x_2 \in \text{var}(s_2)$. By the same proof as for the case $x_1, x_2 \in \text{var}(s_1)$, we get $x_1, x_2 \in \text{var}(\sigma_s^{n+1}(f))$. $\square$

Lemma 4.4. Let $s \in W_{(2)}(X) \setminus X$ and $s \in W_{(2)}^G(\{x_1\})$. If $\text{leftmost}(s) = x_i$ where $x_i \in X$, $i > 2$, then $x_1, x_2 \notin \text{var}(\sigma_s^2(f))$.

Proof. Assume that $s = f(s_1, s_2)$ for some $s_1, s_2 \in W_{(2)}(X)$. Consider $\sigma_s^2(f) = (\sigma_s \circ_G \sigma_s)(f) = \hat{\sigma}_s[\sigma_s(f)] = \hat{\sigma}_s[s] = \hat{\sigma}_s[f(s_1, s_2)] = S^2(s, \hat{\sigma}_s[s_1], \hat{\sigma}_s[s_2])$. If $s_1 \in X$, then s_1 is the leftmost variable of s , so $s_1 = x_i$. Thus $\hat{\sigma}_s[s_1] = x_i$. Since $s \in W_{(2)}^G(\{x_1\})$, $x_1, x_2 \notin \text{var}(\hat{\sigma}_s[s_1])$ and $\sigma_s^2(f) = S^2(s, \hat{\sigma}_s[s_1], \hat{\sigma}_s[s_2])$, we get $x_1, x_2 \notin \text{var}(\sigma_s^2(f))$. Assume that $s_1 = f(s_3, s_4)$ for some $s_3, s_4 \in W_{(2)}(X)$. Consider $\hat{\sigma}_s[s_1] = \hat{\sigma}_s[f(s_3, s_4)] = S^2(s, \hat{\sigma}_s[s_3], \hat{\sigma}_s[s_4])$. If $s_3 \in X$, then s_3 is the leftmost variable of s , so $s_3 = x_i$. Thus $\hat{\sigma}_s[s_3] = x_i$. Since $s \in W_{(2)}^G(\{x_1\})$, $x_1, x_2 \notin \text{var}(\hat{\sigma}_s[s_3])$ and $\hat{\sigma}_s[s_1] = S^2(s, \hat{\sigma}_s[s_3], \hat{\sigma}_s[s_4])$, we get $x_1, x_2 \notin \text{var}(\hat{\sigma}_s[s_1])$. This implies $x_1, x_2 \notin \text{var}(\sigma_s^2(f))$. This procedure stops after finitely many steps at $\text{leftmost}(s) = x_i$. $\square$

Lemma 4.5. Let $s \in W_{(2)}(X) \setminus X$. If $\text{leftmost}(s) = x_1$, then $\text{leftmost}(\sigma_s^n(f)) = x_1$ for all $n \in \mathbb{N}$.

Proof. Assume that $s = f(s_1, s_2)$ for some $s_1, s_2 \in W_{(2)}(X)$. For $n = 1$, $\sigma_s^1(f) = \sigma_s(f) = s$. So $\text{leftmost}(\sigma_s^1(f)) = x_1$. Assume that $\text{leftmost}(\sigma_s^n(f)) = x_1$. Consider $\sigma_s^{n+1}(f) = (\sigma_s^n \circ_G \sigma_s)(f) = \hat{\sigma}_s^n[\sigma_s(f)] = \hat{\sigma}_s^n[s] = \hat{\sigma}_s^n[f(s_1, s_2)] = S^2(\sigma_s^n(f), \hat{\sigma}_s^n[s_1], \hat{\sigma}_s^n[s_2])$. If $s_1 \in X$, then s_1 is the leftmost variable of s , so $s_1 = x_1$. Thus $\hat{\sigma}_s^n[s_1] = x_1$. Since $\sigma_s^{n+1}(f) = S^2(\sigma_s^n(f), \hat{\sigma}_s^n[s_1], \hat{\sigma}_s^n[s_2])$, $\text{leftmost}(\sigma_s^n(f)) = x_1$ and $\hat{\sigma}_s^n[s_1] = x_1$, we get $\text{leftmost}(\sigma_s^{n+1}(f)) = x_1$. Assume that $s_1 = f(s_3, s_4)$ for some $s_3, s_4 \in W_{(2)}(X)$. Consider $\hat{\sigma}_s^n[s_1] = \hat{\sigma}_s^n[f(s_3, s_4)] = S^2(\sigma_s^n(f), \hat{\sigma}_s^n[s_3], \hat{\sigma}_s^n[s_4])$. If $s_3 \in X$, then s_3 is the leftmost variable of s , so $s_3 = x_1$. Thus $\hat{\sigma}_s^n[s_3] = x_1$. Since $\hat{\sigma}_s^n[s_1] = S^2(\sigma_s^n(f), \hat{\sigma}_s^n[s_3], \hat{\sigma}_s^n[s_4])$, $\text{leftmost}(\sigma_s^n(f)) = x_1$ and $\hat{\sigma}_s^n[s_3] = x_1$, we get $\text{leftmost}(\hat{\sigma}_s^n[s_1]) = x_1$. This implies $\text{leftmost}(\sigma_s^{n+1}(f)) = x_1$. This procedure stops after finitely many steps at $\text{leftmost}(s) = x_1$. $\square$

Lemma 4.6. Let $s \in W_{(2)}(X) \setminus X$ and $s \in W_{(2)}^G(\{x_2\})$. If $\text{rightmost}(s) = x_i$ where $x_i \in X$, $i > 2$, then $x_1, x_2 \notin \text{var}(\sigma_s^2(f))$.

Proof. The proof is similar to the proof of Lemma 4.4. $\square$

Lemma 4.7. Let $s \in W_{(2)}(X) \setminus X$. If $\text{rightmost}(s) = x_2$, then $\text{rightmost}(\sigma_s^n(f)) = x_2$ for all $n \in \mathbb{N}$.

Proof. The proof is similar to the proof of Lemma 4.5. $\square$

Note that $\{\sigma_{f(x_2, x_1)}^n \mid n \in \mathbb{N}\} = \{\sigma_{\text{id}}, \sigma_{f(x_2, x_1)}\}$, and that the order of $\sigma_{f(x_2, x_1)}$ is 2.

Proposition 4.8. Let $s \in W_{(2)}(X)$, $x_1, x_2 \in \text{var}(s)$, σ_s be not idempotent and not equal to $\sigma_{f(x_2, x_1)}$. Then the order of σ_s is infinite.

Proof. Let $n \in \mathbb{N}$. Let $\sigma_s^n(f) = w$. By Lemma 4.3, we get $x_1, x_2 \in \text{var}(w)$. Then the equation $\sigma_s^{n+1} = \sigma_s^n \circ_G \sigma_s$ does not fit any of E(1) to E(16), so by Lemma 4.1, we have that the term for σ_s^{n+1} is longer than w . This implies the order of σ_s is infinite. $\square$

Proposition 4.9. Let $s \in W_{(2)}^G(\{x_1\})$, and σ_s be not idempotent. If $\text{leftmost}(s) = x_1$, then the order of σ_s is infinite.

Proof. Let $n \in \mathbb{N}$. Let $\sigma_s^n(f) = w$. By Lemma 4.5, we get $\text{leftmost}(w) = x_1$. Then the equation $\sigma_s^{n+1} = \sigma_s^n \circ_G \sigma_s$ does not fit any of E(1) to E(16), so by Lemma 4.1 we have that the term for σ_s^{n+1} is longer than w . This implies the order of σ_s is infinite. $\square$

Proposition 4.10. Let $s \in W_{(2)}^G(\{x_1\})$ and σ_s be not idempotent. If $\text{leftmost}(s) = x_i$ where $x_i \in X$, $i > 2$, then the order of σ_s is 2.

Proof. Let $\sigma_s^2(f) = w$. By Lemma 4.4, we get $x_1, x_2 \notin \text{var}(w)$. This implies $\sigma_s^n = \sigma_s^2$ for all $n \in \mathbb{N}$ where $n \geq 2$. So the order of σ_s is 2. $\square$

Proposition 4.11. Let $s \in W_{(2)}^G(\{x_2\})$ and σ_s be not idempotent. If $\text{rightmost}(s) = x_2$, then the order of σ_s is infinite.

Proof. The proof is similar to the proof of Proposition 4.9. $\square$

Proposition 4.12. Let $s \in W_{(2)}^G(\{x_2\})$ and σ_s be not idempotent. If $\text{rightmost}(s) = x_i$ where $x_i \in X$, $i > 2$, the order of σ_s is 2.

Proof. The proof is similar to the proof of Proposition 4.10. $\square$

Now we have the main result.

Theorem 4.13. *The order of any generalized hypersubstitution of type $\tau = (2)$ is 1, 2 or infinite.*

Proof. Let $\sigma_t \in \text{Hyp}_G(2)$. If σ_t is idempotent, then the order of σ_t is 1. If σ_t is not idempotent, then $x_1 \in \text{var}(t)$ or $x_2 \in \text{var}(t)$. Assume that $x_1, x_2 \in \text{var}(t)$. If $\sigma_t = \sigma_{f(x_2, x_1)}$, then the order of σ_t is 2. If $\sigma_t \neq \sigma_{f(x_2, x_1)}$, then by Proposition 4.8 we get the order of σ_t is infinite. Assume that $x_1 \in \text{var}(t)$ and $x_2 \notin \text{var}(t)$. If $\text{leftmost}(t) = x_1$, then by Proposition 4.9 we get the order of σ_t is infinite. If $\text{leftmost}(t) = x_i$ where $i > 2$, then by Proposition 4.10 we get the order of σ_t is 2. By the same way we can show that if $x_2 \in \text{var}(t)$ and $x_1 \notin \text{var}(t)$, then the order of σ_t is 2 or infinite. $\square$

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References

- [1] S. Leeratanavalee and K. Denecke, "Generalized hypersubstitutions and strongly solid varieties," in *General Algebra and Applications, Proceedings of the "59th Workshop on General Algebra," "15th Conference for Young Algebraists Potsdam 2000"*, pp. 135–146, Shaker, Aachen, Germany, 2000.
- [2] K. Denecke and S. L. Wismath, "The monoid of hypersubstitutions of type (2)," in *Contributions to General Algebra, 10 (Klagenfurt, 1997)*, pp. 109–126, Heyn, Klagenfurt, Austria, 1998.