Research Article

Norm Attaining Multilinear Forms on $L_1(\mu)$

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Received 5 November 2007; Revised 23 March 2008; Accepted 9 June 2008

Recommended by Manfred Moller Moller

Given an arbitrary measure $\mu$, this study shows that the set of norm attaining multilinear forms is not dense in the space of all continuous multilinear forms on $L_1(\mu)$. However, we have the density if and only if $\mu$ is purely atomic. Furthermore, the study presents an example of a Banach space $X$ in which the set of norm attaining operators from $X$ into $X^*$ is dense in the space of all bounded linear operators $L(X,X^*)$. In contrast, the set of norm attaining bilinear forms on $X$ is not dense in the space of continuous bilinear forms on $X$.

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1. Introduction

The Bishop-Phelps theorem [1] asserts that the set of norm attaining linear functionals on a Banach space $X$ is dense in the dual space $X^*$. Some authors have considered the question of the density of norm attaining multilinear forms. To present the problem more precisely, given real Banach spaces $X_1, \ldots, X_N$, we denote by $\mathcal{L}^N(X_1, \ldots, X_N)$ the space of all continuous $N$-linear mappings from $X_1 \times \cdots \times X_N$ into the scalar field. We say that $\varphi \in \mathcal{L}^N(X_1, \ldots, X_N)$ attains its norm if there is $x_i \in B_{X_i}$ (the unit ball of $X_i$) for $i = 1, 2, \ldots, N$, such that

$$|\varphi(x_1, \ldots, x_N)| = \|\varphi\| = \sup \{|\varphi(y_1, \ldots, y_N)| : (y_1, \ldots, y_N) \in B_{X_1} \times \cdots \times B_{X_N}\},$$

(1.1)

and we denote by $\mathcal{A}\mathcal{L}^N(X_1, \ldots, X_N)$ the set of all norm attaining $N$-linear forms. In the case where $X_1 = \cdots = X_N = X$, we write simply $\mathcal{L}^N(X)$ and $\mathcal{A}\mathcal{L}^N(X)$.

Aron et al. [2] posed the question of when $\mathcal{A}\mathcal{L}^N(X)$ is dense in $\mathcal{L}^N(X)$, and gave sufficient conditions for this density to hold. The first example of a Banach space $X$ such that $\mathcal{A}\mathcal{L}^2(X)$ is not dense in $\mathcal{L}^2(X)$ was given in [3]. Shortly after, Choi [4] showed that $\mathcal{A}\mathcal{L}^2(L_1[0,1])$ is not dense in $\mathcal{L}^2(L_1[0,1])$. For additional results on this problem, we refer the reader to [5–9].
In this paper, we give some improvements on the results in [10]. More concretely, it was shown in that study that given an arbitrary finite measure $\mu$, $\mathcal{A}\mathcal{L}^2(L_1(\mu))$ is dense in $\mathcal{L}^2(L_1(\mu))$ if and only if $\mu$ is purely atomic. In this note, we extend the above result to an arbitrary measure. Namely, we proved that, given any arbitrary measure $\mu$, $\mathcal{A}\mathcal{L}^N(L_1(\mu))$ is dense in $\mathcal{L}^N(L_1(\mu))$ if and only if $\mu$ is purely atomic. Also, we present a new example of a Banach space $X$ such that the set of norm attaining operators from $X$ into $X^*$ is dense in the space of all bounded linear operators from $X$ into $X^*$, but the set $\mathcal{A}\mathcal{L}^2(X)$ is not dense in $\mathcal{L}^2(X)$. This can be shown by relating the main result in our work to the following theorem.

**Theorem 1.1** (see [11, Theorem 1]). Given an arbitrary measure $\mu$ and a localizable measure $\nu$, the set of norm attaining operators from $L_1(\mu)$ into $L_\infty(\nu)$ is dense in the space $L(L_1(\mu), L_\infty(\nu))$.

**2. The results**

We begin by recalling the isometric classification of $L_1$-spaces and a technical lemma which deals with the density of norm attaining bilinear forms on arbitrary $l_1$-sums of Banach spaces in order to reduce the proof of our problem to the case where $\mu$ is a finite measure. Recall that if $\mu$ is an arbitrary measure, $L_1(\mu)$ can be decomposed in the form

$$L_1(\mu) \cong \left( \oplus_{i \in I} L_1(\mu_i) \right)_{\ell_1} \quad (2.1)$$

where $\mu_i$ is a finite measure for all $i \in I$ (see, e.g., [12, Appendix B]). On the other hand, if $\nu$ is a localizable measure we have that $L_\infty(\nu) = L_1(\nu)^*$, and we get a set of finite measures $\{\nu_j : j \in J\}$ such that

$$L_\infty(\nu) \cong \left( \oplus_{j \in J} L_\infty(\nu_j) \right)_{\ell_\infty}. \quad (2.2)$$

In what follows, we may assume without loss of generality that $(\Omega, \mathcal{A}, \mu)$ is a finite measure space. The well-known representation of the space $\mathcal{L}^2(L_1(\mu))$ is nothing but $L_\infty(\mu \otimes \mu)$ “the space of all essential bounded measurable functions,” where $\mu \otimes \mu$ denotes the product measure on $\Omega \times \Omega$. More concretely,

$$\mathcal{L}^2(L_1(\mu)) \cong L(L_1(\mu), L_1(\mu)^*) \cong L(L_1(\mu), L_\infty(\mu)) \cong L_\infty(\mu \otimes \mu); \quad (2.3)$$

see [12, Example 3.27]. In view of the above, we get the integral representation for the continuous bilinear form $\tilde{h}$ on $\mathcal{L}^2(L_1(\mu))$ as follows:

$$\tilde{h}(f, g) = \int_{\Omega \times \Omega} h(x, y) f(x) g(y) d\mu(x) d\mu(y), \quad (2.4)$$

for $f, g \in L_1(\mu), x, y \in \Omega$, and $h \in L_\infty(\mu \otimes \mu)$. Moreover, the application $h \mapsto \tilde{h}$ is linear isometric bijection from $L_\infty(\mu \otimes \mu)$ onto $\mathcal{L}^2(L_1(\mu))$; see [4].

To make the vision more comprehensive, we state the following technical lemmas that will be needed later. To simplify the notation, we consider the case $N = 2$. The proof for the general case is exactly the same.

**Lemma 2.1** (see [10, Lemma 2.1]). Let $\nu$ be an arbitrary nonzero finite measure and $\mu = \nu \otimes m$, where $m$ denotes Lebesgue measure on $I = [0, 1]$. Then $\mathcal{A}\mathcal{L}^2(L_1(\mu))$ is not dense in $\mathcal{L}^2(L_1(\mu))$. 

The other technical lemma deals with $l_1$-sums of Banach spaces. By $Y \oplus Z$ we denote the $\ell_1$-sum of two Banach spaces $Y$ and $Z$, that is, $\|y + z\| = \|y\| + \|z\|$ for arbitrary $y \in Y, z \in Z$.

**Lemma 2.2** (see [10, Lemma 2.2]). Let $Y, Z$ be Banach spaces and $X = Y \oplus Z$. If $\mathcal{A}L^2(X)$ is dense in $L^2(X)$, then $\mathcal{A}L^2(Y)$ is dense in $L^2(Y)$.

Our first result of this paper is a characterization of those functions $h \in L_\infty(\mu \otimes \mu)$, where $h$ is its corresponding bilinear form in $L^2(L_1(\mu))$ that attains its norm (see [4]).

**Proposition 2.3.** Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, fixed $h \in L_\infty(\mu \otimes \mu)$, and let $\widehat{h}$ be its corresponding bilinear form as defined in (2.4)

1. There exist sets $A, B \in \mathcal{A}$ with $\mu(A) > 0, \mu(B) > 0$ and a scalar $t$ with $|t| = 1$ such that

\[ th(x, y) = \|h\|_\infty \quad (2.5) \]

for $[\mu \otimes \mu]$-almost every $(x, y) \in A \times B$.

2. There are sets $A, B$ like in (1) and measurable functions $\varphi, \psi$ on $\Omega$ such that

\[ |\varphi(w)| = |\psi(w)| = 1, \quad (2.6) \]

where $w \in \Omega$ and $\varphi(x)\psi(y)h(x, y) = \|h\|_\infty$, for $[\mu \otimes \mu]$-almost every $(x, y) \in A \times B$.

3. The bilinear form $\widehat{h} \in L^2(L_1(\mu))$ corresponding to $h \in L_\infty(\mu \otimes \mu)$ attains its norm.

Then (1) $\Rightarrow$ (2) $\iff$ (3).

Moreover, in the real case all three statements are equivalent.

**Proof.** (1) $\Rightarrow$ (2) is clear, just take $\varphi = t$ and $\psi = 1$.

For (2) $\Rightarrow$ (3), just consider the functions $f = \varphi X_A/\mu(A), g = \psi X_A/\mu(B)$ where $f, g$ are in the unit sphere of $L_1(\mu), X_A, X_B$ denote the characteristic functions on $A$ and $B$, respectively, and

\[ \widehat{h}(f, g) = \frac{1}{\mu(A)\mu(B)} \int_{A \times B} h(x, y)\varphi(x)\psi(y)d\mu(x)d\mu(y) = \frac{1}{\mu(A)\mu(B)} \int_{A \times B} \|h\|_\infty d(\mu \otimes \mu) = \|h\|_\infty. \quad (2.8) \]

(3) $\Rightarrow$ (2) Let $f, g \in L_1(\mu)$ be such that $\|f\|_1 = \|g\|_1 = 1$ and $\widehat{h}(f, g) = \|h\|_\infty$. Take

\[ A = \{x \in \Omega : f(x) \neq 0\}, \quad B = \{y \in \Omega : g(y) \neq 0\} \quad (2.9) \]

to be two measurable sets in $\Omega$ with $\mu(A) > 0, \mu(B) > 0$, and write $f, g$ in the forms $f = \varphi|f|$, $g = \psi|g|$ where $\varphi, \psi$ are measurable functions on $\Omega$ with $|\varphi| = 1, |\psi| = 1$, then we have

\[ \|h\|_\infty = \widehat{h}(f, g) = \int_{A \times B} h(x, y)\varphi(x)|f(x)|\psi(y)|g(y)|d\mu(x)d\mu(y) \leq \|h\|_\infty\|f\|_1\|g\|_1 = \|h\|_\infty, \quad (2.10) \]
from which we conclude that

\[ h(x, y)\varphi(x)\varphi(y) = \|h\|_\infty \]  

(2.11)

for \([\mu \otimes \mu]\)-almost every \((x, y) \in A \times B\).

In the real case, the functions \(\varphi, \varphi\) have only the values \(\pm 1\), then we can choose measurable subsets \(A_0 \subseteq A\) and \(B_0 \subseteq B\) such that \(\mu(A_0)\mu(B_0) > 0\), where \(\varphi, \varphi\) are constants on \(A_0, B_0\), respectively. If \(t = \pm 1\) is the product of these constants, then we have clearly \(th(x, y) = \|h\|_\infty\) for \([\mu \otimes \mu]\)-almost every \((x, y) \in A_0 \times B_0\), so we get that (3) \(\Rightarrow\) (1), as required.

In the special case \(h = \chi_E\), the characteristic function of a measurable set \(E \in \mathcal{A} \times \mathcal{A}\), we have the following result.

**Corollary 2.4.** Let \((\Omega, \mathcal{A}, \mu)\) be a finite measure space, let \(E \in \mathcal{A} \times \mathcal{A}\) be a measurable set with \((\mu \otimes \mu)(E) > 0\), and consider the following bilinear form \(\tilde{\chi}_E\) corresponding to the characteristic function of \(E\). The following statements are equivalent:

1. \(\tilde{\chi}_E \in \mathcal{A} \mathcal{L}^2(L_1(\mu))\);
2. \(\tilde{\chi}_E \in \mathcal{A} \mathcal{L}^2(L_1(\mu))\);
3. There exist subsets \(A, B \in \mathcal{A}\) with \(\mu(A)\mu(B) > 0\) such that \([\mu \otimes \mu]((A \times B) \cap E) = \mu(A)\mu(B)\).

Note that we can say that the measurable rectangle \(A \times B\) is contained in the set \(E\).

**Proof.** (1) \(\Rightarrow\) (2). This is trivial.

(2) \(\Rightarrow\) (3). Let \(h \in L_\infty(\mu \otimes \mu)\) be such that \(\|\tilde{\chi}_E - h\|_\infty < 1/2\), and \(\tilde{h} \in \mathcal{A} \mathcal{L}^2(L_1(\mu))\), then it is clear that \(\|h\|_\infty > 1/2\). From the implication (3) \(\Rightarrow\) (2) of Proposition 2.3, we have two measurable sets \(A, B \in \mathcal{A}\) with \(\mu(A)\mu(B) > 0\), and measurable functions \(\varphi, \varphi\) on \(\Omega\) with \(|\varphi(x)| = |\varphi(y)| = 1\), such that

\[ \varphi(x)\varphi(y)h(x, y) = \|h\|_\infty, \]  

(2.12)

then

\[ |h(x, y)| = \|h\|_\infty > \frac{1}{2}, \]  

(2.13)

for \([\mu \otimes \mu]\)-almost every \((x, y) \in A \times B\). Hence

\[ |\chi_E(x, y)| \geq |h(x, y)| - |h(x, y) - \chi_E(x, y)| > \frac{1}{2} - \|h - \chi_E\|_\infty > 0. \]  

(2.14)

for \([\mu \otimes \mu]\)-almost every \((x, y) \in A \times B\), from which we get that \(\chi_E = 1\), for \([\mu \otimes \mu]\)-almost every \((x, y) \in A \times B\), which means that (3) holds.

(3) \(\Rightarrow\) (1). If \(A, B\) are the sets that satisfy the conditions of the statement (3), then we may clearly see that the function \(\chi_E = 1 = \|\chi_E\|_\infty\), for \([\mu \otimes \mu]\)-almost every \((x, y) \in A \times B\), then the function \(f = \chi_E\) verifies the statement (1) of Proposition 2.3 including the case \(t = 1\).
Remark 2.5. Let us point out the following consequence of the representation theory for $L_1$-spaces. Indeed, if $\nu$ is a finite measure, we may write
\[
L_1(\nu) = \left( \oplus_{i \in I} X_i \right)_{\ell_1},
\]
where each space $X_i$ is either 1-dimensional or of the form $L_1([0,1]^{\Lambda})$ and $\Lambda$ is a finite or infinite set. For each coordinate interval, we consider the Lebesgue measure on the Borel subsets of $[0,1]$ and $[0,1]^\Lambda$ provided with the product measure on the Borel $\sigma$-algebra (see [13]).

We are now ready to provide the main result.

Theorem 2.6. Given an arbitrary measure $\mu$, the following statements are equivalents.

1. $\mu$ is purely atomic.
2. $\mathcal{A} \mathcal{L}^N(L_1(\mu))$ is dense in $\mathcal{L}^N(L_1(\mu))$ for any number $N$.
3. $\mathcal{A} \mathcal{L}^N(L_1(\mu))$ is dense in $\mathcal{L}^N(L_1(\mu))$ for any number $N \geq 2$.
4. $\mathcal{A} \mathcal{L}^2(L_1(\mu))$ is dense in $\mathcal{L}^2(L_1(\mu))$.

Proof. (1) $\Rightarrow$ (2). If $\mu$ is purely atomic, then $L_1(\mu)$ has the Radon-Nikodym property, and (2) follows from [2, Theorem 1].

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (4). This follows from [8, Proposition 2.1].

(4) $\Rightarrow$ (1). Given an arbitrary nonempty set $\Lambda$, consider the product $[0,1]^\Lambda$ of so many copies of $[0,1]$ as indicated by $\Lambda$ with product measure. We have clearly $\mu = \nu \otimes m$, where $\nu$ is an arbitrary nonzero finite measure and $m$ denotes the Lebesgue measure on $[0,1]$. Then it follows from Lemma 2.1 that $\mathcal{A} \mathcal{L}^2(L_1([0,1]^\Lambda))$ is not dense in $\mathcal{L}^2(L_1([0,1]^\Lambda))$. Indeed, if $\mu$ is a finite measure satisfying statement (4) of the above theorem, then by Remark 2.5, $L_1(\mu) \cong (\oplus_{i \in I} X_i)_{\ell_1}$ for each $i \in I$, where $X_i$ is 1-dimensional or of the form $L_1([0,1]^{\Lambda_i})$ for appropriate nonempty set $\Lambda_i$ (see [13, Theorem 14]). It follows then from Lemma 2.2 that $\mathcal{A} \mathcal{L}^2(X_i)$ is dense in $\mathcal{L}^2(X_i)$ for all $i \in I$. But in view of Remark 2.5, none of the spaces $X_i$ are of the form $L_1([0,1]^{\Lambda_i})$. Then all $X_i$ are 1-dimensional, and then $L_1(\mu) \cong \ell_1(I)$, which means that $\mu$ is purely atomic. Finally, if $\mu$ is not necessarily a finite measure satisfying (4) of our theorem, we recall that $L_1(\mu) \cong (\oplus_{i \in I} L_1(\mu_i))_{\ell_1}$, where $\mu_i$ is a finite measure for all $i \in I$. So by Lemma 2.2, we get that $\mathcal{A} \mathcal{L}^2(L_1(\mu_i))$ is dense in $\mathcal{L}^2(L_1(\mu_i))$, and this proves that $\mu_i$ is purely atomic for each $i \in I$, which clearly means that $\mu$ is purely atomic.

Remark 2.7. Let us mention the relation between the $\mathcal{L}^2(X)$, the space of all continuous bilinear forms on the Banach space $X$, and $L(X,X^*)$, the space of all bounded linear operators from $X$ into $X^*$, to see that just consider the canonical identification of $\mathcal{L}^2(X)$ with $L(X,X^*)$. The operator $T \in L(X,X^*)$ corresponding to a bilinear form $\varphi \in \mathcal{L}^2(X)$ is given by
\[
[T(x)](y) = \varphi(x,y) \quad (x, y \in X).
\]

The bilinear form $\varphi$ attains its norm if and only if the operator $T$ attains its norm at a point $x \in B_X$, that is, $T(x)$ also attains its norm as a functional on $X$, therefore, $T \in NA(X,X^*)$ whenever $\varphi \in \mathcal{A} \mathcal{L}^N(X)$, but the converse is not true (see [4, 14, 15]). Connecting our main result in this paper with Theorem 1.1, we get a new example of a Banach space $X$ such that the set of norm attaining bounded linear operators from $X$ into $X^*$ is dense in the space of all bounded linear operators from $X$ into $X^*$, but $\mathcal{A} \mathcal{L}^2(X)$ is not dense in $\mathcal{L}^2(X)$. 
Therefore, the following result is inevitable.

**Corollary 2.8.** If \( \mu \) is a localizable and not purely atomic measure, then the set of norm attaining bounded linear operators from \( L_1(\mu) \) into \( L_\infty (\mu) \) is dense in the space \( L(L_1(\mu), L_\infty (\mu)) \) but \( \mathcal{A}L^2(L_1(\mu)) \) is not dense in \( L^2(L_1(\mu)) \).

**References**


