Research Article

Some Estimates of Certain Subnormal and Hyponormal Derivations

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We prove that if A and B^* are subnormal operators and X is a bounded linear operator such that $AX - XB$ is a Hilbert-Schmidt operator, then $f(A)X - Xf(B)$ is also a Hilbert-Schmidt operator and $\|f(A)X - Xf(B)\|_2 \leq L\|AX - XB\|_2$ for f belongs to a certain class of functions. Furthermore, we investigate the similar problem in the case that S, T are hyponormal operators and $X \in \mathcal{L}(\mathcal{H})$ is such that $SX - XT$ belongs to a norm ideal $(J, \|\cdot\|_J)$, and we prove that $f(S)X - Xf(T) \in J$ and $\|f(S)X - Xf(T)\|_J \leq C\|SX - XT\|_J$ for f being in a certain class of functions.

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1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and by $\mathcal{C}_2(\mathcal{H})$ the Hilbert-Schmidt class. For $T \in \mathcal{L}(\mathcal{H})$, $\sigma(T)$ denotes the spectrum of T , and for a compact subset $\Sigma \subset \mathbb{C}$, $\text{Lip}(\Sigma)$ denotes the set of Lipschitz functions on Σ . Furthermore, $\text{Rat}(\Sigma)$ denotes the algebra of rational functions with poles of Σ , and $R(\Sigma)$ denotes the closure of $\text{Rat}(\Sigma)$ in the supremum norm over Σ .

For operators $A, B \in \mathcal{L}(\mathcal{H})$, the mapping $\Delta_{A,B}(X) = AX - XB$ is called a (generalized) derivation. If A, B are normal (subnormal or co-subnormal, hyponormal or co-hyponormal) operators, then $\Delta_{A,B}$ will be called a normal (subnormal, hyponormal) derivation, respectively.

Next, we recall some theorems that involve normal derivations, and then we extend some of these theorems to the case in which A, B^* are subnormal operators and to the case in which $A = S, B = T$ are hyponormal operators.

In [1], a generalization of Fuglede-Putnam theorem for normal operators was proved. For further results concerning normal derivations, the reader can see [2, 3].

Theorem 1.1 (see [1]). *If $A, B \in \mathcal{L}(\mathcal{H})$ are normal operators and $X \in \mathcal{L}(\mathcal{H})$ satisfies $AX - XB \in \mathcal{C}_2(\mathcal{H})$, then $A^*X - XB^* \in \mathcal{C}_2(\mathcal{H})$ and*

$$\|AX - XB\|_2 = \|A^*X - XB^*\|_2. \quad (1.1)$$

In [4], Furuta extended the above result to subnormal operators.

Theorem 1.2 (see [4]). *If $A, B^* \in \mathcal{L}(\mathcal{H})$ are subnormal operators and $X \in \mathcal{L}(\mathcal{H})$ satisfies $AX - XB \in \mathcal{C}_2(\mathcal{H})$, then $A^*X - XB^* \in \mathcal{C}_2(\mathcal{H})$ and*

$$\|AX - XB\|_2 \geq \|A^*X - XB^*\|_2. \quad (1.2)$$

In his paper [5], Kittaneh proved the following theorem using a famous result of Voiculescu [6] according to which every normal operator can be written as the sum of a diagonal operator and a Hilbert-Schmidt operator of an arbitrarily small Hilbert-Schmidt norm.

Theorem 1.3 (see [5]). *Let $A, B \in \mathcal{L}(\mathcal{H})$ be normal operators and $X \in \mathcal{L}(\mathcal{H})$ such that $AX - XB \in \mathcal{C}_2(\mathcal{H})$, and let $f \in \text{Lip}(\sigma(A) \cup \sigma(B))$. Then $f(A)X - Xf(B)$ is also a Hilbert-Schmidt operator and*

$$\|f(A)X - Xf(B)\|_2 \leq L\|AX - XB\|_2, \quad (1.3)$$

where L is the Lipschitz constant of the function f .

2. Subnormal derivations

In this section, we investigate the validity of this inequality in the case that A, B^* are subnormal operators, but with a drawback concerning the extent of the class of functions in which f can run.

The following lemma is elementary and can be easily established making use of the *minimal normal extension* of a subnormal operator. Its proof is left for the reader.

Lemma 2.1. *If $S_1, S_2 \in \mathcal{L}(\mathcal{H})$ are subnormal operators, then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and normal operators $N_1, N_2 \in \mathcal{L}(\mathcal{K})$ that are extensions of S_1, S_2 , respectively, and $\sigma(N_i) \subseteq \sigma(S_i)$, $i = 1, 2$*

For a subnormal operator $S \in \mathcal{L}(\mathcal{H})$ and a function $f \in R(\sigma(S))$, one can associate an operator $f(S) \in \mathcal{L}(\mathcal{H})$ as follows. Let $r_n \in \text{Rat}(\sigma(S))$, $n \in \mathbb{N}$, such that

$$\|f - r_n\|_{\sigma(S), \infty} \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (2.1)$$

and let $N_S \in \mathcal{L}(\mathcal{K})$, where $\mathcal{K} \supset \mathcal{H}$, be the minimal normal extension of S . Since $\sigma(N_S) \subseteq \sigma(S)$, we have

$$r_n(N_S) = \begin{pmatrix} r_n(S) & S'_{12} \\ 0 & S'_{22} \end{pmatrix}, \quad (2.2)$$

and $r_n(N_S) \rightarrow f(N_S)$ in the operator norm of $\mathcal{L}(\mathcal{K})$. Therefore, $r_n(S)$ converges in the operator norm of $\mathcal{L}(\mathcal{H})$ to an operator that will be denoted by $f(S)$. It is obvious that this operator does not depend on the sequence $\{r_n\}$. In a similar way, for $f \in R(\sigma(T))$, one can define $f(T)$, when $T^* \in \mathcal{L}(\mathcal{H})$ is a subnormal operator.

Theorem 2.2. Let $A, B^* \in \mathcal{L}(\mathcal{H})$ be subnormal operators and $X \in \mathcal{L}(\mathcal{H})$ such that $AX - XB \in \mathcal{C}_2(\mathcal{H})$, and let $\Sigma = \sigma(A) \cup \sigma(B)$ and $f \in \text{Lip}(\Sigma) \cap R(\Sigma)$. Then $f(A)X - Xf(B)$ is also a Hilbert-Schmidt operator and

$$\|f(A)X - Xf(B)\|_2 \leq L\|AX - XB\|_2, \quad (2.3)$$

where L is the Lipschitz constant of the function f .

Proof. For subnormal operators $A, B^* \in \mathcal{L}(\mathcal{H})$, according to Lemma 2.1, there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and there are some normal operators $N_A, N_{B^*} \in \mathcal{L}(\mathcal{K})$ such that relative to the decomposition of $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$, we have

$$N_A = \begin{pmatrix} A & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad N_{B^*} = \begin{pmatrix} B^* & B_{12} \\ 0 & B_{22} \end{pmatrix}, \quad (2.4)$$

and $\sigma(N_A) \subseteq \sigma(A)$, $\sigma(N_{B^*}) \subseteq \sigma(B^*)$.

If we put $\tilde{X} = X \oplus 0$ on $\mathcal{H} \oplus \mathcal{H}^\perp$, then we have $N_A \tilde{X} - \tilde{X} N_{B^*}^* = (AX - XB) \oplus 0$, and therefore $N_A \tilde{X} - \tilde{X} N_{B^*}^* \in \mathcal{C}_2(\mathcal{K})$.

For $r \in \text{Rat}(\Sigma)$, where $\Sigma = \sigma(A) \cup \sigma(B)$, a simple calculation shows that

$$r(N_A) = \begin{pmatrix} r(A) & A'_{12} \\ 0 & A'_{22} \end{pmatrix}, \quad r(N_{B^*}^*) = \begin{pmatrix} r(B) & 0 \\ B'_{21} & B'_{22} \end{pmatrix}. \quad (2.5)$$

Thus, if $f \in \text{Lip}(\Sigma) \cap R(\Sigma)$, using a limiting argument, one can see that $f(N_A)$ and $f(N_{B^*}^*)$ have similar matrix representation as in (2.5), but with f replacing r . According to Theorem 1.3,

$$\begin{aligned} f(N_A) \tilde{X} - \tilde{X} f(N_{B^*}^*) &\in \mathcal{C}_2(\mathcal{K}), \\ \|f(N_A) \tilde{X} - \tilde{X} f(N_{B^*}^*)\|_2 &\leq L \|N_A \tilde{X} - \tilde{X} N_{B^*}^*\|_2. \end{aligned} \quad (2.6)$$

Since $f(N_A) \tilde{X} - \tilde{X} f(N_{B^*}^*) = (f(A)X - Xf(B)) \oplus 0$, the proof is finished. $\square$

Corollary 2.3. Let $A, B^* \in \mathcal{L}(\mathcal{H})$ be subnormal operators and $X \in \mathcal{L}(\mathcal{H})$ such that $AX - XB \in \mathcal{C}_2(\mathcal{H})$, and let $\Sigma = \sigma(A) \cup \sigma(B)$ and $f \in \text{Lip}(\Sigma) \cap R(\Sigma)$. Then

$$\|f(A)^*X - Xf(B)^*\|_2 \leq \|f(A)X - Xf(B)\|_2, \quad (2.7)$$

and thus

$$\|f(A)^*X - Xf(B)^*\|_2 \leq L\|AX - XB\|_2, \quad (2.8)$$

where L is the Lipschitz constant of the function f .

Proof. The first inequality is a consequence of Theorem 1.2 after observing that $f(A)$ and $f(B)^*$ are subnormal operators. The second inequality follows from Theorem 2.2. $\square$

3. Hyponormal derivations

In this section, we approach the same problem, but in the case in which $A = S$, $B = T$ are hyponormal operators and the Hilbert-Schmidt class is replaced with an arbitrary norm ideal.

For a hyponormal operator $T \in \mathcal{L}(\mathcal{H})$, the analytic functional calculus can be extended to a class $A^\alpha(\sigma(T))$ of “pseudo-analytic” functions on $\sigma(T)$ that satisfy a certain growth condition at the boundary.

The extension of the analytic functional calculus for a hyponormal operator was introduced by Dyn’kin (cf. [7, 8]) and it also can be found in [9].

We briefly review the definition and the main tools that are necessary. Let Σ be a perfect compact set of the complex plane and let α be a positive noninteger with k its integer part, $[\alpha]$. The class $A^\alpha(\Sigma)$ is defined as the set of $(k+1)$ tuples of continuous functions on Σ , $(f_0, \dots, f_k) : \Sigma \rightarrow \mathbb{C}^{k+1}$ that are related by

$$f_j(z) = f_j(z_0) + \frac{f_{j+1}(z_0)}{1!}(z - z_0) + \dots + \frac{f_k(z_0)}{(k-j)!}(z - z_0)^{k-j} + R_j(z_0, z), \quad (3.1)$$

$$|R_j(z_0, z)| \leq C_j |z - z_0|^{\alpha-j}$$

for $j = 0, \dots, k$ and $z, z_0 \in \Sigma$. Since Σ is a perfect set,

$$f_j(z_0) = \lim_{z \rightarrow z_0} \frac{f_{j+1}(z) - f_{j+1}(z_0)}{z - z_0}, \quad j = 0, \dots, k-1, \quad (3.2)$$

and thus the $(k+1)$ tuple depends only on f_0 . The space $A^\alpha(\Sigma)$, endowed with the maximum of the smallest constants that satisfy (3.1) plus the supremum norm on Σ of f_0 , becomes a unital Banach algebra and is a closed subalgebra of $\text{Lip}(\alpha, \Sigma)$, the algebra of Lipschitz functions of order α .

Theorem 3.1 (see [8]). *Let Σ be a perfect compact set, $f \in C(\Sigma)$, and α a positive noninteger. The following are equivalent:*

- (a) $f \in A^\alpha(\Sigma)$;
- (b) f has an extension $F \in C^1(\mathbb{C} \setminus \Sigma)$ with $|\bar{\partial}F(z)| \leq C \cdot \text{dist}(z, \Sigma)^{\alpha-1}$, $z \notin \Sigma$;
- (c) there exists $\phi \in C_0(\mathbb{C})$ such that

$$f(z) = \int \frac{\phi(w)}{w - z} d\mu(w), \quad z \in \Sigma, \quad (3.3)$$

and $|\phi(w)| \leq C_0 \cdot \|f\|_{A^\alpha(\Sigma)} \cdot \text{dist}(w, \Sigma)^{\alpha-1}$, $w \in \mathbb{C}$, where μ is planar Lebesgue measure and C_0 is a constant that does not depend on f .

If $T \in \mathcal{L}(\mathcal{H})$ is a hyponormal operator, then $\|T\| = \|T\|_\sigma$, where $\|T\|_\sigma$ denotes the spectral radius of T , that is $\sup_{z \in \sigma(T)} |z|$. It is well known that if $z \notin \sigma(T)$, then $(z - T)^{-1}$ is also hyponormal and thus

$$\|(z - T)^{-1}\| = \frac{1}{\text{dist}(z, \sigma(T))}. \quad (3.4)$$

Thus, for a hyponormal operator T whose spectrum $\sigma(T)$ is a perfect set and for a function $f \in A^\alpha(\sigma(T))$ with $\alpha > 2$, one can associate an operator defined by

$$\int \phi(w)(w - T)^{-1} d\mu(w) \quad (3.5)$$

that will be denoted by $f(T)$. The above integral does not depend on ϕ , that is, the definition of $f(T)$ is not ambiguous, and the mapping $\phi \mapsto f(T)$ acting from $A^\alpha(\sigma(T))$ into $\mathcal{L}(\mathcal{H})$ is a continuous, unital morphism of Banach algebras, and which extends the Riesz-Dunford calculus.

Let $(J, \|\cdot\|_J)$ be a norm ideal, that is, a proper two-sided ideal J of $\mathcal{L}(\mathcal{H})$ with a norm $\|\cdot\|_J$ that satisfies the following: $(J, \|\cdot\|_J)$ is a Banach space and $\|AXB\|_J \leq \|A\| \|B\| \|X\|_J$, for all $X \in J$ and any $A, B \in \mathcal{L}(\mathcal{H})$. In particular, the Schatten-von Neumann p -classes, $\mathcal{C}_p(\mathcal{H})$, for $p \geq 1$, are instances of norm ideals.

Theorem 3.2. *Let $(J, \|\cdot\|_J)$ be a norm ideal, let $S, T \in \mathcal{L}(\mathcal{H})$ be hyponormal operators for which both $\sigma(S)$ and $\sigma(T)$ are perfect sets, let f belong to $A^\alpha(\Sigma)$ with $\alpha > 3$ and $\Sigma = \sigma(S) \cup \sigma(T)$, and let $X \in \mathcal{L}(\mathcal{H})$ such that $SX - XT \in J$. Then $f(S)X - Xf(T) \in J$ and*

$$\|f(S)X - Xf(T)\|_J \leq C_1 \cdot \|f\|_{A^\alpha(\Sigma)} \cdot \|SX - XT\|_J, \quad (3.6)$$

where C_1 is a constant that depends on Σ but it does not depend on f .

Proof. For $f \in A^\alpha(\Sigma)$, according to Theorem 3.1, there exists $\phi \in C_0(\mathbb{C})$ such that

$$f(z) = \int \frac{\phi(w)}{w - z} d\mu(w), \quad z \in \Sigma, \quad (3.7)$$

$$|\phi(w)| \leq C_0 \cdot \|f\|_{A^\alpha(\Sigma)} \cdot \text{dist}(w, \Sigma)^{\alpha-1}, \quad w \in \mathbb{C}.$$

Therefore,

$$f(S)X - Xf(T) = \int \phi(w) [(w - S)^{-1}X - X(w - T)^{-1}] d\mu(w). \quad (3.8)$$

The domain of integration is $\text{supp}(\phi)$, which is a compact set that has in common with Σ only possibly boundary points of Σ . For $w \in \text{supp}(\phi) \cap (\mathbb{C} \setminus \Sigma)$,

$$\begin{aligned} (w - S)^{-1}X - X(w - T)^{-1} &= (w - S)^{-1} [X(w - T) - (w - S)X] (w - T)^{-1} \\ &= (w - S)^{-1} [SX - XT] (w - T)^{-1} \in J, \end{aligned} \quad (3.9)$$

and, according to (3.4),

$$\begin{aligned} \|(w - S)^{-1}X - X(w - T)^{-1}\|_J &\leq \text{dist}(w, \sigma(S))^{-1} \cdot \text{dist}(w, \sigma(T))^{-1} \cdot \|SX - XT\|_J \\ &\leq C' \cdot \text{dist}(w, \Sigma)^{-2} \cdot \|SX - XT\|_J, \end{aligned} \quad (3.10)$$

where C' is a constant that depends on Σ . Therefore, the integrand in (3.8) belongs to the norm ideal J and

$$\|\phi(w)(w - S)^{-1}X - X(w - T)^{-1}\|_J \leq C_0 \cdot C' \cdot \|f\|_{A^\alpha(\Sigma)} \cdot \text{dist}(w, \Sigma)^{\alpha-3} \|SX - XT\|_J, \quad (3.11)$$

for $w \in \text{supp}(\phi) \cap (\mathbb{C} \setminus \Sigma)$. After integration one obtains the desired conclusion of the theorem. $\square$

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