Research Article

Note on Product Summability of an Infinite Series

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New results concerning product summability of an infinite series are given. Some special cases are also deduced.

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1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums s_n . Let u_n^α denote the n th Cesaro mean of order $\alpha > -1$ of the sequence (s_n) . The series $\sum a_n$ is summable $|C, \alpha|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k < \infty \quad (1.1)$$

(Flett [1]). For $\alpha = 1$, $|C, \alpha|_k$ reduces to $|C, 1|_k$ summability.

Let (p_n) be a sequence of positive real constants such that $P_n = p_0 + \cdots + p_n \rightarrow \infty$ as $n \rightarrow \infty$ ($P_{-1} = p_{-1} = 0$). The (N, p) transform ϕ_n of (s_n) generated by (p_n) is defined by

$$\phi_n = \frac{1}{P_n} \sum_{v=0}^n p_{n-v} s_v. \quad (1.2)$$

The sequence-to-sequence transformation

$$\Phi_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v \quad (1.3)$$

defines the sequence (Φ_n) of $(\overline{N}, p_n)$ transform of (s_n) generated by (p_n) . The series $\sum a_n$ is summable $|R, p_n|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |\Phi_n - \Phi_{n-1}|^k < \infty. \quad (1.4)$$

In the special case when $p_n = 1$ for all n (resp., $k = 1$), $|R, p_n|_k$ summability reduces to $|C, 1|_k$ (resp., $|R, p_n|$) summability.

The series $\sum a_n$ is said to be summable $|(N, p)(N, q)|$, when the (N, p) transform of the (N, q) transform of (s_n) is a sequence of bounded variation (see Das [2]).

We give the following new definition.

Let (T_n) define the sequence of the $(\bar{N}, q_n)$ transform of the $(\bar{N}, p_n)$ transform of (s_n) generated by the sequences (q_n) and (p_n) , respectively. The series $\sum a_n$ is said to be summable $|(R, q_n)(R, p_n)|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty. \quad (1.5)$$

We may assume through the paper that $Q_n = q_0 + \dots + q_n \rightarrow \infty$, as $n \rightarrow \infty$; $R_n = r_0 + \dots + r_n \rightarrow \infty$, as $n \rightarrow \infty$.

2. New results

We state and prove the following.

Theorem 2.1. Let $k \geq 1$, (λ_n) be a sequence of constants. Define

$$f_v = \sum_{r=v}^n \frac{q_r}{P_r}, \quad F_v = \sum_{r=v}^n p_r f_r. \quad (2.1)$$

Let

$$p_n Q_n = O(P_n), \quad (2.2)$$

$$\sum_{n=v+1}^{\infty} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}} = O\left(\frac{(v q_v)^{k-1}}{Q_v^k}\right). \quad (2.3)$$

Then, sufficient conditions for the implication

$$\sum a_n \text{ is summable } |R, r_n|_k \implies \sum a_n \lambda_n \text{ is summable } |(R, q_n)(R, p_n)|_k \quad (2.4)$$

are

$$|\lambda_v| F_v = O(Q_v), \quad (2.5)$$

$$|\lambda_n| = O(Q_n), \quad (2.6)$$

$$p_v R_v |\lambda_v| = O(Q_v), \quad (2.7)$$

$$p_v q_v R_v |\lambda_v| = O(Q_v Q_{v-1} r_v), \quad (2.8)$$

$$p_n q_n R_n |\lambda_n| = O(P_n Q_n r_n), \quad (2.9)$$

$$R_{v-1} |\Delta \lambda_v| F_{v+1} = O(Q_v r_v), \quad (2.10)$$

$$R_{v-1} |\Delta \lambda_v| = O(Q_v r_v). \quad (2.11)$$

Proof. Let (S_n) be the sequence of partial sums of $\sum a_n \lambda_n$. Let v_n, V_n be the $(\overline{N}, r_n), (\overline{N}, q_n)(\overline{N}, p_n)$ transforms of the sequences $(s_n), (S_n)$, respectively. We write $t_n = v_n - v_{n-1}$, $T_n = V_n - V_{n-1}$. Therefore,

$$t_n = \frac{r_n}{R_n R_{n-1}} \sum_{v=1}^n R_{v-1} a_v, \quad (2.12)$$

$$\begin{aligned} V_n &= \frac{1}{Q_n} \sum_{r=0}^n q_r \frac{1}{P_r} \sum_{v=0}^r p_v S_v \\ &= \frac{1}{Q_n} \sum_{v=0}^n p_v S_v \sum_{r=v}^n \frac{q_r}{P_r} \\ &= \frac{1}{Q_n} \sum_{v=0}^n p_v S_v f_v. \end{aligned} \quad (2.13)$$

Also,

$$\begin{aligned} T_n &= V_n - V_{n-1} \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{r=0}^n p_r S_r f_r + \frac{p_n S_n f_n}{Q_{n-1}} \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{r=0}^v p_r f_r \sum_{v=0}^r a_v \lambda_v + \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=0}^n a_v \lambda_v \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=0}^n a_v \lambda_v \sum_{r=v}^n p_r f_r + \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=0}^n a_v \lambda_v \\ &= \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^n R_{v-1} a_v \frac{\lambda_v}{R_{v-1}} \sum_{r=v}^n p_r f_r + \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=1}^n R_{v-1} a_v \frac{\lambda_v}{R_{v-1}} \\ &= \frac{q_n}{Q_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(\sum_{r=1}^v R_{r-1} a_r \right) \Delta_v \left(\frac{\lambda_v}{R_{v-1}} \sum_{r=v}^n p_r f_r \right) + \left(\sum_{v=1}^n R_{v-1} a_v \right) \frac{\lambda_n p_n f_n}{R_{n-1}} \right) \\ &\quad + \frac{p_n q_n}{P_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(\sum_{r=1}^v R_{r-1} a_r \right) \Delta \left(\frac{\lambda_v}{R_{v-1}} \right) + \left(\sum_{v=1}^n R_{v-1} a_v \right) \frac{\lambda_n}{R_{n-1}} \right) \\ &= \frac{q_n}{Q_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(t_v \lambda_v F_v + \frac{R_{v-1}}{r_v} p_v t_v \lambda_v f_v + \frac{R_{v-1}}{r_v} t_v \Delta \lambda_v F_{v+1} \right) \right) + \frac{p_n q_n R_n}{Q_n Q_{n-1} r_n} t_n \lambda_n f_n \\ &\quad + \frac{p_n q_n}{P_n Q_{n-1}} \left(\sum_{v=1}^{n-1} \left(t_v \lambda_v + \frac{R_{v-1}}{r_v} t_v \Delta \lambda_v \right) \right) + \frac{p_n q_n R_n}{P_n Q_{n-1} r_n} t_n \lambda_n, \\ &= \sum_{j=1}^7 T_{nj}. \end{aligned} \quad (2.14)$$

In order to complete the proof, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} |T_{nj}|^k < \infty, \quad j = 1, 2, 3, 4, 5, 6, 7. \quad (2.15)$$

Applying Holder's inequality,

$$\begin{aligned}
 \sum_{n=2}^{m+1} n^{k-1} |T_{n1}|^k &= \sum_{n=2}^{m+1} n^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} t_v \lambda_v F_v \right|^k \\
 &\leq \sum_{n=2}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}^k} \sum_{v=1}^{n-1} \frac{1}{q_v^{k-1}} |t_v|^k |\lambda_v|^k F_v^k \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^{k-1} \\
 &= O(1) \sum_{v=1}^m \frac{1}{q_v^{k-1}} |t_v|^k |\lambda_v|^k F_v^k \sum_{n=v+1}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}^k} \\
 &= O(1) \sum_{v=1}^m v^{k-1} |t_v|^k \frac{|\lambda_v|^k F_v^k}{Q_v^k} \\
 &= O(1),
 \end{aligned}$$

$$\begin{aligned}
 \sum_{n=2}^{m+1} n^{k-1} |T_{n2}|^k &= \sum_{n=2}^{m+1} n^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{R_{v-1} p_v}{r_v} t_v \lambda_v f_v \right|^k \\
 &\leq \sum_{n=2}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}^k} \sum_{v=1}^{n-1} \frac{R_v^k p_v^k}{q_v^{k-1}} |t_v|^k |\lambda_v|^k f_v^k \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^{k-1} \\
 &= O(1) \sum_{v=1}^m \frac{R_v^k p_v^k}{q_v^{k-1}} |t_v|^k |\lambda_v|^k f_v^k \sum_{n=v+1}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}^k} \\
 &= O(1) \sum_{v=1}^m v^{k-1} |t_v|^k \frac{R_v^k p_v^k}{Q_v^k} |\lambda_v|^k f_v^k \\
 &= O(1) \sum_{v=1}^m v^{k-1} |t_v|^k \\
 &= O(1),
 \end{aligned}$$

$$\begin{aligned}
 \sum_{n=2}^{m+1} n^{k-1} |T_{n3}|^k &= \sum_{n=2}^{m+1} n^{k-1} \left| \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{R_{v-1}}{r_v} t_v \Delta \lambda_v F_{v+1} \right|^k \\
 &\leq \sum_{n=1}^{m+1} n^{k-1} \frac{q_n^k}{Q_n^k Q_{n-1}^k} \sum_{v=1}^{n-1} \frac{R_{v-1}^k}{q_v^{k-1} r_v^k} |t_v|^k |\Delta \lambda_v|^k F_{v+1}^k \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^k \\
 &= O(1) \sum_{v=1}^m \frac{R_{v-1}^k}{q_v^{k-1} r_v^k} |t_v|^k |\Delta \lambda_v|^k F_{v+1}^k \sum_{n=v+1}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}^k} \\
 &= O(1) \sum_{v=1}^m v^{k-1} |t_v|^k \frac{R_{v-1}^k}{Q_v^k r_v^k} |\Delta \lambda_v|^k F_{v+1}^k \\
 &= O(1),
 \end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^m n^{k-1} |T_{n4}|^k &= \sum_{n=1}^m n^{k-1} \left| \frac{p_n q_n R_n}{Q_n Q_{n-1} r_n} t_n \lambda_n f_n \right|^k \\
&= O(1) \sum_{n=1}^m n^{k-1} |t_n|^k |\lambda_n|^k \frac{p_n^k q_n^k R_n^k}{Q_n^k Q_{n-1}^k r_n^k} \\
&= O(1), \\
\sum_{n=2}^{m+1} n^{k-1} |T_{n5}|^k &= \sum_{n=2}^{m+1} n^{k-1} \left| \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=1}^{n-1} t_v \lambda_v \right|^k \\
&\leq \sum_{n=1}^{m+1} n^{k-1} \frac{p_n^k q_n^k}{P_n^k Q_{n-1}^k} \sum_{v=1}^{n-1} |t_v|^k |\lambda_v|^k \frac{1}{q_v^{k-1}} \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^{k-1} \\
&= O(1) \sum_{v=1}^m |t_v|^k |\lambda_v|^k \frac{1}{q_v^{k-1}} \sum_{n=v+1}^{m+1} \frac{n^{k-1} p_n^k q_n^k}{P_n^k Q_{n-1}^k} \\
&= O(1) \sum_{v=1}^m |t_v|^k |\lambda_v|^k \frac{1}{q_v^{k-1}} \sum_{n=v+1}^{m+1} \frac{n^{k-1} q_n^k}{Q_n^k Q_{n-1}^k} \\
&= O(1) \sum_{v=1}^m v^{k-1} |t_v|^k |\lambda_v|^k \frac{1}{Q_v^k} \\
&= O(1), \\
\sum_{n=2}^{m+1} n^{k-1} |T_{n6}|^k &= \sum_{n=2}^{m+1} n^{k-1} \left| \frac{p_n q_n}{P_n Q_{n-1}} \sum_{v=1}^{n-1} \frac{R_{v-1}}{r_v} t_v \Delta \lambda_v \right|^k \\
&\leq \sum_{n=2}^{m+1} n^{k-1} \frac{p_n^k q_n^k}{P_n^k Q_{n-1}^k} \sum_{v=1}^{n-1} \frac{R_{v-1}^k}{q_v^{k-1} r_v^k} |t_v|^k |\Delta \lambda_v|^k \left(\sum_{v=1}^{n-1} \frac{q_v}{Q_{n-1}} \right)^{k-1} \\
&= O(1) \sum_{v=1}^m \frac{R_{v-1}^k}{q_v^{k-1} r_v^k} |t_v|^k |\Delta \lambda_v|^k \sum_{n=v+1}^{m+1} n^{k-1} \frac{p_n^k q_n^k}{P_n^k Q_{n-1}^k} \\
&= O(1) \sum_{v=1}^m v^{k-1} |t_v|^k |\Delta \lambda_v|^k \frac{R_{v-1}^k}{Q_v^k r_v^k} \\
&= O(1).
\end{aligned} \tag{2.16}$$

Finally,

$$\begin{aligned}
\sum_{n=1}^m n^{k-1} |T_{n7}|^k &= \sum_{n=1}^m n^{k-1} \left| \frac{p_n q_n R_n}{P_n Q_{n-1} r_n} t_n \lambda_n \right|^k \\
&= O(1) \sum_{n=1}^m n^{k-1} |t_n|^k |\lambda_n|^k \left(\frac{p_n q_n R_n}{P_n Q_{n-1} r_n} \right)^k \\
&= O(1).
\end{aligned} \tag{2.17}$$

This completes the proof of the theorem. $\square$

Theorem 2.2. Let (2.3) be satisfied and

$$P_v = O(p_v Q_v), \quad (2.18)$$

$$Q_n = O(nq_n). \quad (2.19)$$

Then, necessary conditions for the implication (2.4) to be satisfied are

$$|\lambda_v| = O\left(\frac{Q_v Q_{v-1} r_v}{(1+F_v) q_v R_v}\right), \quad |\lambda_v| = O\left(\frac{v^{1-1/k} r_v Q_v}{p_v f_v R_v}\right), \quad |\Delta \lambda_v| = O\left(\frac{v^{1-1/k} r_v Q_v}{(1+F_{v+1}) R_v}\right). \quad (2.20)$$

Proof. For $k \geq 1$ define

$$\begin{aligned} A^* &= \left\{ (a_j): \sum a_j \text{ is summable } |R, r_n|_k \right\}, \\ B^* &= \left\{ (b_j): \sum b_j \lambda_j \text{ is summable } |(R, q_n)(R, p_n)|_k \right\}. \end{aligned} \quad (2.21)$$

From (2.14), we have

$$T_n = \sum_{v=1}^n \left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}} \right) a_v \lambda_v. \quad (2.22)$$

With t_n and T_n as defined by (2.12) and (2.22), the spaces A^* and B^* are BK-spaces with norms defined by

$$\begin{aligned} \|c\|_1 &= \left\{ |t_0|^k + \sum_{n=1}^{\infty} n^{k-1} |t_n|^k \right\}^{1/k}, \\ \|c\|_2 &= \left\{ |T_0|^k + \sum_{n=1}^{\infty} n^{k-1} |T_n|^k \right\}^{1/k}, \end{aligned} \quad (2.23)$$

respectively. By the hypothesis of the theorem,

$$\|c\|_1 < \infty \implies \|c\|_2 < \infty. \quad (2.24)$$

The inclusion map $i: A^* \rightarrow B^*$ defined by $i(a) = a$ is continuous since A^* and B^* are BK-spaces. By the closed graph theorem, there exists a constant $K > 0$ such that

$$\|c\|_2 \leq K \|c\|_1. \quad (2.25)$$

Let e_n denote the n th coordinate vector. From (2.12) and (2.22) with (a_n) defined by $a_n = e_n - e_{n+1}$, $n = v$, $a_n = 0$, otherwise, we have

$$\begin{aligned} t_n &= \begin{cases} 0, & n < v, \\ \frac{r_v}{R_v}, & n = v, \\ -\frac{r_n r_v}{R_n R_{n-1}}, & n > v, \end{cases} \\ T_n &= \begin{cases} 0, & n < v, \\ \left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}} \right) \lambda_v, & n = v, \\ \Delta_v \left(\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}} \right) \lambda_v \right), & n > v. \end{cases} \end{aligned} \quad (2.26)$$

From (2.23), we have

$$\begin{aligned} \|c\|_1 &= \left\{ v^{k-1} \left(\frac{q_v}{Q_v} \right)^k + \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{q_n q_v}{Q_n Q_{n-1}} \right)^k \right\}^{1/k}, \\ \|c\|_2 &= \left\{ v^{k-1} \left| \left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}} \right) \lambda_v \right|^k + \sum_{n=v+1}^{\infty} n^{k-1} \left| \Delta_v \left(\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}} \right) \lambda_v \right) \right|^k \right\}^{1/k}. \end{aligned} \quad (2.27)$$

Applying (2.25), we obtain

$$\begin{aligned} &v^{k-1} \left| \left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}} \right) \lambda_v \right|^k + \sum_{n=v+1}^{\infty} n^{k-1} \left| \Delta_v \left(\left(\frac{q_n F_v}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}} \right) \lambda_v \right) \right|^k \\ &= O(1) \left(v^{k-1} \left(\frac{r_v}{R_v} \right)^k + \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{r_n r_v}{R_n R_{n-1}} \right)^k \right). \end{aligned} \quad (2.28)$$

As the right-hand side of (2.28), by (2.3), is

$$\begin{aligned} &= O(1) \left(v^{k-1} \left(\frac{r_v}{R_v} \right)^k + \frac{r_v^k}{R_v^{k-1}} \sum_{n=v+1}^{\infty} \frac{n^{k-1} r_n^k}{R_n^k R_{n-1}} \right) \\ &= O(1) \left(v^{k-1} \left(\frac{r_v}{R_v} \right)^k + \left(\frac{r_v}{R_v} \right)^{k-1} v^{k-1} \left(\frac{r_v}{R_v} \right)^k \right) \\ &= O \left(v^{k-1} \left(\frac{r_v}{R_v} \right)^k \right), \end{aligned} \quad (2.29)$$

and the fact that each term of the left-hand side of (2.28) is $O(v^{k-1}(r_v/R_v)^k)$, we obtain

$$v^{k-1} \left(\frac{q_v F_v}{Q_v Q_{v-1}} + \frac{p_v q_v}{P_v Q_{v-1}} \right)^k |\lambda_v|^k = O \left(v^{k-1} \left(\frac{r_v}{R_v} \right)^k \right), \quad (2.30)$$

which implies by (2.18)

$$\left(\frac{q_v}{Q_v Q_{v-1}} \right)^k (1 + F_v)^k |\lambda_v|^k = O \left(\frac{r_v}{R_v} \right)^k, \quad (2.31)$$

that is,

$$|\lambda_v| = O \left(\frac{Q_v Q_{v-1} r_v}{(1 + F_v) q_v R_v} \right). \quad (2.32)$$

Also, we have, by (2.28),

$$\sum_{n=v+1}^{\infty} n^{k-1} \left| \left(\frac{q_n p_v f_v}{Q_n Q_{n-1}} \right) \lambda_v + \left(\frac{q_n F_{v+1}}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}} \right) \Delta \lambda_v \right|^k = O \left(v^{k-1} \left(\frac{r_v}{R_v} \right)^k \right). \quad (2.33)$$

The above, via the linear independence of λ_v and $\Delta\lambda_v$, implies

$$\begin{aligned} \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{q_n F_{v+1}}{Q_n Q_{n-1}} + \frac{p_n q_n}{P_n Q_{n-1}} \right)^k |\Delta\lambda_v|^k &= O \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k \right) \\ |\Delta\lambda_v|^k \left(1 + F_{v+1} \right)^k \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^k &= O \left(v^{k-1} \left(\frac{q_v}{Q_v} \right)^k \right) \end{aligned} \quad (2.34)$$

by (2.18). As by (2.19), via the mean value theorem,

$$\frac{1}{Q_v^k} = \sum_{n=v+1}^{\infty} \Delta \left(\frac{1}{Q_{n-1}^k} \right) = O(1) \sum_{n=v+1}^{\infty} \frac{|\Delta Q_{n-1}^k|}{Q_n^k Q_{n-1}^k} = O(1) \sum_{n=v+1}^{\infty} \frac{Q_{n-1}^{k-1} q_n}{Q_n^k Q_{n-1}^k} = O(1) \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^k. \quad (2.35)$$

Then,

$$|\Delta\lambda_v|^k (1 + F_{v+1})^k \frac{1}{Q_v^k} = O \left(v^{k-1} \left(\frac{r_v}{R_v} \right)^k \right), \quad (2.36)$$

which implies

$$\Delta\lambda_v = O \left(\frac{v^{1-1/k} r_v Q_v}{(1 + F_{v+1}) R_v} \right). \quad (2.37)$$

Also, by (2.28),

$$\begin{aligned} \sum_{n=v+1}^{\infty} n^{k-1} \left| \frac{q_n p_v f_v}{Q_n Q_{n-1}} \lambda_v \right|^k &= O \left(v^{k-1} \left(\frac{r_v}{R_v} \right)^k \right), \\ p_v^k f_v^k |\lambda_v|^k \sum_{n=v+1}^{\infty} n^{k-1} \left(\frac{q_n}{Q_n Q_{n-1}} \right)^k &= O \left(v^{k-1} \left(\frac{r_v}{R_v} \right)^k \right), \\ p_v^k f_v^k |\lambda_v|^k \frac{1}{Q_v^k} &= O \left(v^{k-1} \left(\frac{r_v}{R_v} \right)^k \right), \end{aligned} \quad (2.38)$$

which implies

$$\lambda_v = O \left(\frac{v^{1-1/k} r_v Q_v}{p_v f_v R_v} \right). \quad (2.39)$$

□

3. Applications

Corollary 3.1. *Let $k \geq 1$. Define*

$$f_v = \sum_{r=v}^n \frac{q_r}{r}, \quad F_v = \sum_{r=v}^n f_r. \quad (3.1)$$

Let

$$n = O(Q_n). \quad (3.2)$$

Then, sufficient conditions for the implication

$$\sum a_n \text{ is summable } |C, 1|_k \implies \sum a_n \lambda_n \text{ is summable } |(R, q_n)(C, 1)|_k \quad (3.3)$$

are (2.5), (2.6), and the following:

$$\begin{aligned} v|\lambda_v| &= O(Q_v), \\ vq_v|\lambda_v| &= O(Q_v Q_{v-1}), \\ nq_n|\lambda_n| &= O(nQ_n), \\ v|\Delta\lambda_v|_{F_{v+1}} &= O(Q_v), \\ |\Delta\lambda_v| &= O(q_v), \\ v|\Delta\lambda_v| &= O(Q_v). \end{aligned} \quad (3.4)$$

Proof. The proof follows from Theorem 2.1 by putting $p_n = r_n = 1$ for all n . $\square$

Corollary 3.2. Let $k \geq 1$. Define

$$f_v = \sum_{r=v}^n \frac{1}{P_r}, \quad F_v = \sum_{r=v}^n p_r f_r. \quad (3.5)$$

Let (2.2) be satisfied. Then, sufficient conditions for the implication

$$\sum a_n \text{ is summable } |C, 1|_k \implies \sum a_n \lambda_n \text{ is summable } |(C, 1)(R, p_n)|_k \quad (3.6)$$

are

$$\begin{aligned} |\lambda_v|_{F_v} &= O(v), \\ |\lambda_n| &= O(n), \\ p_v|\lambda_v| &= O(1), \\ |\Delta\lambda_v|_{F_{v+1}} &= O(1), \\ |\Delta\lambda_v| &= O(1). \end{aligned} \quad (3.7)$$

Proof. The proof follows from Theorem 2.1, by putting $q_n = r_n = 1$, for all n , noticing that (2.3) is satisfied as

$$\sum_{n=v+1}^{\infty} \frac{1}{n(n-1)} = \sum_{n=v+1}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = \frac{1}{v}. \quad (3.8)$$

$\square$

Corollary 3.3. Let f_v, F_v be as defined in (3.1). Let (2.3) and (3.2) be satisfied. Then, sufficient conditions for the implication

$$\sum a_n \text{ is summable } |R, r_n|_k \implies \sum a_n \lambda_n \text{ is summable } |(R, q_n)(C, 1)|_k \quad (3.9)$$

are (2.5), (2.6), (2.10), (2.11), and the following:

$$\begin{aligned} R_v |\lambda_v| &= O(Q_v), \\ q_v R_v |\lambda_v| &= O(Q_v Q_{v-1} r_v), \\ q_n R_n |\lambda_n| &= O(n Q_n r_n). \end{aligned} \quad (3.10)$$

Proof. The proof follows from Theorem 2.1, by putting $p_n = 1$ for all n . $\square$

Corollary 3.4. Let f_v, F_v be as defined in (3.1). Let (2.3), (2.19) be satisfied and

$$v = O(Q_v). \quad (3.11)$$

Then, necessary conditions for the implication (3.3) are

$$\lambda_v = O\left(\frac{Q_v Q_{v-1}}{(1 + F_v) v q_v}\right), \quad \lambda_v = O\left(\frac{Q_v}{v^{1/k} f_v}\right), \quad \Delta \lambda_v = O\left(\frac{Q_v}{v^{1/k} (1 + F_{v+1})}\right). \quad (3.12)$$

Proof. The proof follows from Theorem 2.2 by putting $p_n = r_n = 1$ for all n . $\square$

Corollary 3.5. Let f_v, F_v be as defined in (3.5). Let

$$P_v = O(v p_v). \quad (3.13)$$

Then, necessary conditions for the implication (3.5) to be satisfied are

$$\lambda_v = O\left(\frac{v}{1 + F_v}\right), \quad \lambda_v = O\left(\frac{v^{1-1/k}}{p_v f_v}\right), \quad \Delta \lambda_v = O\left(\frac{v^{1-1/k}}{1 + F_{v+1}}\right). \quad (3.14)$$

Proof. The proof follows from Theorem 2.2, by putting $q_n = r_n = 1$, keeping in mind that (2.3) is satisfied as in the case of (3.8). $\square$

Corollary 3.6. Let f_v, F_v be as defined in (3.1). Let (2.3), (2.19), and (3.2) be all satisfied. Then, necessary conditions for the implication (3.9) to be satisfied are

$$\lambda_v = O\left(\frac{Q_v Q_{v-1} r_v}{(1 + F_v) q_v R_v}\right), \quad \lambda_v = O\left(\frac{v^{1-1/k} r_v Q_v}{f_v R_v}\right), \quad \Delta \lambda_v = O\left(\frac{v^{1-1/k} r_v Q_v}{(1 + F_{v+1}) R_v}\right). \quad (3.15)$$

Proof. The proof follows from Theorem 2.2, by putting $p_n = 1$ for all n . $\square$

References

- [1] T. M. Flett, "On an extension of absolute summability and some theorems of Littlewood and Paley," *Proceedings of the London Mathematical Society*, vol. 7, no. 1, pp. 113–141, 1957.
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