Research Article

AGQP-Injective Modules

Zhanmin Zhu¹ and Xiaoxiang Zhang²

¹ Department of Mathematics, Jiaxing University, Jiaxing, Zhejiang 314001, China

² Department of Mathematics, Southeast University, Nanjing 210096, China

Correspondence should be addressed to Zhanmin Zhu, zhanmin_zhu@hotmail.com

Received 23 December 2007; Revised 20 April 2008; Accepted 20 June 2008

Recommended by Robert Lowen

Let R be a ring and let M be a right R -module with $S = \text{End}(M_R)$. M is called *almost general quasi-principally injective* (or *AGQP-injective* for short) if, for any $0 \neq s \in S$, there exist a positive integer n and a left ideal X_{s^n} of S such that $s^n \neq 0$ and $\text{I}_S(\text{Ker}(s^n)) = Ss^n \oplus X_{s^n}$. Some characterizations and properties of AGQP-injective modules are given, and some properties of AGQP-injective modules with additional conditions are studied.

Copyright © 2008 Z. Zhu and X. Zhang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Throughout R is an associative ring with identity, and all modules are unitary. Recall that a ring R is called *right principally injective* [1] (or *right P-injective* for short) if, every homomorphism from a principal right ideal of R to R can be extended to an endomorphism of R , or equivalently, $\text{I}_r(a) = Ra$ for all $a \in R$. The concept of right P-injective rings has been generalized by many authors. For example, in [2, 3], right P-injective rings are generalized in two directions, respectively. Following [2], a ring R is called *right GP-injective* if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R -homomorphism from $a^n R$ to R can be extended to an endomorphism of R . Note that GP-injective rings are also called *YJ-injective* in [4]. From [5], we know that GP-injective rings need not to be P-injective. Following [3], a right R -module M_R with $S = \text{End}(M_R)$ is called *quasiprincipally injective* (or *QP-injective* for short) if, every homomorphism from an M -cyclic submodule of M to M can be extended to an endomorphism of M , or equivalently, $\text{I}_S(\text{Ker}(s)) = Ss$ for all $s \in S$. In 1998, Page and Zhou [6] generalized the concept of GP-injective rings to that of AGP-injective rings. According to [6], a ring R is called *right AGP-injective* if, for any $0 \neq a \in R$, there exist a positive integer n and a left ideal X_{a^n} such that $a^n \neq 0$ and $\text{I}_r(a^n) = Ra^n \oplus X_{a^n}$. In [7], the first author introduced the notion of GQP-injective modules which can be regarded as the generalization of GP-injective rings and QP-injective modules. According to [7], a right R -module M with $S = \text{End}(M_R)$ is called *GQP-injective* if, for any

$0 \neq s \in S$, there exists a positive integer n such that $s^n \neq 0$ and any right R -homomorphism from $s^n(M)$ to M can be extended to an endomorphism of M , or equivalently, for any $0 \neq s \in S$, there exists a positive integer n such that $s^n \neq 0$ and $\mathbf{l}_S(\text{Ker}(s^n)) = Ss^n$. The nice structure of AGP-injective rings and GQP-injective modules draws our attention to define almost GQP-injective modules, in a similar way to AGP-injective rings, and to investigate their properties.

2. Results

Definition 2.1. Let M_R be a right R -module with $S = \text{End}(M_R)$. Then, M is said to be almost general quasiprincipally injective (briefly, AGQP-injective) if, for any $0 \neq s \in S$, there exist a positive integer n and a left ideal X_{s^n} of S such that $s^n \neq 0$ and $\mathbf{l}_S(\text{Ker}(s^n)) = Ss^n \oplus X_{s^n}$.

Clearly, a ring R is right AGP-injective if and only if R_R is AGQP-injective, GQP-injective modules are AGQP-injective.

Our next result gives the relationship between the AGQP-injectivity of a module and the AGP-injectivity of its endomorphism ring.

Theorem 2.2. *Let M_R be a right R -module with $S = \text{End}(M_R)$. Then,*

- (1) *if S is right AGP-injective, then M_R is AGQP-injective;*
- (2) *if M_R is AGQP-injective and M generates $\text{Ker}(s)$ for each $s \in S$, then S is right AGP-injective.*

Proof. (1) Suppose that S is right AGP-injective then for any $0 \neq s \in S$, there exist a positive integer n and a left ideal I_{s^n} of S such that $s^n \neq 0$ and $\mathbf{l}_S \mathbf{r}_S(s^n) = Ss^n \oplus I_{s^n}$. If $a \in \mathbf{l}_S(\text{Ker}(s^n))$ and $b \in \mathbf{r}_S(s^n)$, then $s^n b = 0$, that is, $b(M) \subseteq \text{Ker}(s^n)$. Hence, $(ab)M = 0$, that is, $ab = 0$. This shows that $\mathbf{l}_S(\text{Ker}(s^n)) \subseteq \mathbf{l}_S \mathbf{r}_S(s^n)$. Therefore, we have $Ss^n \subseteq \mathbf{l}_S(\text{Ker}(s^n)) \subseteq Ss^n \oplus I_{s^n}$, which guarantees that

$$\mathbf{l}_S(\text{Ker}(s^n)) = Ss^n \oplus (\mathbf{l}_S(\text{Ker}(s^n)) \cap I_{s^n}). \quad (2.1)$$

Thus, (1) is proved.

(2) Suppose that M_R is AGQP-injective then for any $0 \neq s \in S$, there exist a positive integer n and a left ideal X_{s^n} of S such that $s^n \neq 0$ and $\mathbf{l}_S(\text{Ker}(s^n)) = Ss^n \oplus X_{s^n}$. Assume that $a \in \mathbf{l}_S \mathbf{r}_S(s^n)$ and $\text{Ker}(s^n) = \sum_{t \in T} t(M)$ for some subset T of S . It is easy to see that $at = 0$ for each $t \in T$, so we have $ax = 0$ for each $x \in \text{Ker}(s^n)$. This implies that $\mathbf{l}_S \mathbf{r}_S(s^n) \subseteq \mathbf{l}_S(\text{Ker}(s^n))$, from which we have

$$Ss^n \subseteq \mathbf{l}_S \mathbf{r}_S(s^n) \subseteq \mathbf{l}_S(\text{Ker}(s^n)) = Ss^n \oplus X_{s^n}, \quad (2.2)$$

and hence

$$\mathbf{l}_S \mathbf{r}_S(s^n) = Ss^n \oplus (\mathbf{l}_S \mathbf{r}_S(s^n) \cap X_{s^n}). \quad (2.3)$$

Therefore, S is right AGP-injective. □

Recall that a module N is called M -cyclic [3], if it is a homomorphic image of M . Let $S = \text{End}(M_R)$, following [8], we write $W(S) = \{s \in S \mid \text{Ker}(s) \subseteq^{\text{ess}} M\}$.

Theorem 2.3. *Let M_R be an AGQP-injective module with $S = \text{End}(M_R)$. Then,*

- (1) $W(S) \subseteq J(S)$,
- (2) *if every nonzero submodule of M contains a nonzero M -cyclic submodule, then $W(S) = J(S)$.*

Proof. (1) Let $s \in W(S)$. Then, for each $t \in S$, $ts \in W(S)$ and so $1 - ts \neq 0$. Since M_R is AGQP-injective, there exist a positive integer n and a left ideal $X_{(1-ts)^n}$ such that $(1 - ts)^n \neq 0$ and $I_S(\text{Ker}(1 - ts)^n) = S(1 - ts)^n \oplus X_{(1-ts)^n}$. Note that $(1 - ts)^n = 1 - u$ for some $u \in W(S)$. Since $\text{Ker}(u) \cap \text{Ker}(1 - u) = 0$, we have $\text{Ker}(1 - u) = 0$, and then $S = S(1 - u) \oplus X_{1-u}$. So $1 = e + x$ for some $e \in S(1 - u)$ and $x \in X_{1-u}$, it follows that $e^2 = e$ and $S(1 - u) = Se \oplus S(1 - e) \cap S(1 - u) = Se$. Therefore, $1 - u = ve$ for some $v \in S$, since $\text{Ker}(u)$ is essential in M_R , if $e \neq 1$, then there exists a nonzero element $(1 - e)m \in (1 - e)M \cap \text{Ker}(u)$, and hence $(1 - u)(1 - e)m = (1 - e)m$. But $(1 - u)(1 - e)m = ve(1 - e)m = 0$, a contradiction. So $e = 1$, and hence $1 - u$ is left invertible, which implies $s \in J(S)$.

(2) We need only to prove that $J(S) \subseteq W(S)$. Let $s \in J(S)$. If $s \notin W(S)$, then there exists $0 \neq t \in S$ such that $\text{Ker}(s) \cap t(M) = 0$ by hypothesis. Clearly, $st \neq 0$ and $\text{Ker}(st) = \text{Ker}(t)$. Since M_R is AGQP-injective, there exist a positive integer n and a left ideal $X_{(st)^n}$ such that $(st)^n \neq 0$ and

$$I_S(\text{Ker}(st)^n) = S(st)^n \oplus X_{(st)^n}. \quad (2.4)$$

If $m \in \text{Ker}(st)^n$, then $(st)^{n-1}m \in \text{Ker}(st) = \text{Ker}(t)$, and so $m \in \text{Ker}(t(st)^{n-1})$. This shows that $\text{Ker}(st)^n = \text{Ker}(t(st)^{n-1})$. Hence, $t(st)^{n-1} \in S(st)^n \oplus X_{(st)^n}$. Write $t(st)^{n-1} = u(st)^n + v$, where $u \in S$, $v \in X_{(st)^n}$. Then $(1 - us)t(st)^{n-1} = v$, which gives that $(st)^n = s(1 - us)^{-1}v \in S(st)^n \cap X_{(st)^n} = 0$, a contradiction. $\square$

Corollary 2.4 (see [6, Corollary 2.3]). *If R is a right AGP-injective ring, then $J(R) = Z(R_R)$.*

Following [9], for a set $X \subseteq \text{Hom}(N_R, M_R)$, the submodule

$$\text{Ker } X = \cap \{\text{Ker } g \mid g \in X\} \quad (2.5)$$

of N is called an M -annihilator submodule of N . By [7, Lemma 9] and Theorem 2.3, we have the following corollary.

Corollary 2.5. *Let M_R be an AGQP-injective module with $S = \text{End}(M_R)$. If every nonzero submodule of M contains a nonzero M -cyclic submodule, and $M/\text{Soc}(M)$ satisfies ACC on M -annihilator submodules, then $J(S)$ is nilpotent.*

Recall that a module M_R is said to be a GC2 module [10] if every submodule $N \leq M$ with $N \cong M$ is a direct summand of M . For convenience, we write $N \mid M$ to denote that N is a direct summand of M .

Theorem 2.6. *Let M_R be an AGQP-injective module. Then,*

- (1) *if M_1 and M_2 are submodules of M such that $M_1 \subseteq M_2$ and $M_1 \cong M_2 \mid M$, then $M_1 \mid M$. In particular M is a GC2 module;*
- (2) *if M_1 and M_2 are simple submodules of M such that $M_1 \cong M_2 \mid M$, then $M_1 \mid M$.*

Proof. (1) Let $S = \text{End}(M_R)$. It is trivial in case $M_1 = 0$. Now suppose that $M_1 \neq 0$ and $M_2 \stackrel{f}{\cong} M_1$. Then $M_1 = aM$ and $M_2 = eM$, where $e^2 = e \in S$ and $a = fe$. Since M_R is AGQP-injective, there exist a positive integer n and a left ideal X_{a^n} such that $a^n \neq 0$ and $\text{I}_S(\text{Ker}(a^n)) = Sa^n \oplus X_{a^n}$. Let $a^0 = e$, then $f^{-1}(a^{i+1}M) = a^iM$ ($i = 0, 1, \dots, n-1$) since $M_1 \subseteq M_2 = eM$. So we have

$$a^iM \mid a^{i-1}M \iff f^{-1}(a^{i+1}M) \mid f^{-1}(a^iM) \iff a^{i+1}M \mid a^iM \quad (i = 1, \dots, n-1). \quad (2.6)$$

Consequently, $aM \mid eM \iff a^2M \mid aM \iff \dots \iff a^nM \mid a^{n-1}M$. Thus, to show $aM \mid M$, it suffices to show that $a^nM \mid M$. Note that $a|_{eM} : eM \rightarrow eM$ is monic and $a^n(m) = a^n(em)$ for every $m \in M$, $eM \stackrel{a^n}{\cong} a^nM$ and hence $\text{Ker}(a^n) = \text{Ker}(e)$. It follows that $e \in \text{I}_S(\text{Ker}(e)) = \text{I}_S(\text{Ker}(a^n)) = Sa^n \oplus X_{a^n}$. Now, let $e = ba^n + x$ with $b \in S$ and $x \in X_{a^n}$, then $a^n = a^n e = a^n ba^n + a^n x = a^n ba^n$. Finally, let $g = a^n b$, then $g^2 = g$ and $a^nM = gM$ as required.

(2) Let $M_2 = e_1M$, where $e_1^2 = e_1 \in S$, and let $M_2 \stackrel{f_1}{\cong} M_1$. Then $M_1 = a_1M$, where $a_1 = f_1 e_1$. Since M_R is AGQP-injective, there exist a positive integer n_1 and a left ideal $X_{a_1^{n_1}}$ such that $a_1^{n_1} \neq 0$ and $\text{I}_S(\text{Ker}(a_1^{n_1})) = Sa_1^{n_1} \oplus X_{a_1^{n_1}}$. Note that $0 \neq a_1^{n_1}M \subseteq a_1M$, and a_1M is simple. We have $a_1^{n_1}M = a_1M$. Clearly, $\text{Ker}(e_1) = \text{Ker}(a_1)$ because f_1 is a monomorphism. Since a_1M is simple, $\text{Ker}(a_1)$ is a maximal submodule of M . But $\text{Ker}(a_1) \subseteq \text{Ker}(a_1^{n_1}) \neq M$, so $\text{Ker}(a_1) = \text{Ker}(a_1^{n_1})$ and then $\text{Ker}(e_1) = \text{Ker}(a_1^{n_1})$. It follows that $e_1 \in \text{I}_S(\text{Ker}(e_1)) = \text{I}_S(\text{Ker}(a_1^{n_1})) = Sa_1^{n_1} \oplus X_{a_1^{n_1}}$. Now, let $e_1 = b_1 a_1^{n_1} + y$ with $b_1 \in S$ and $y \in X_{a_1^{n_1}}$, then $a_1^{n_1} = a_1^{n_1} e_1 = a_1^{n_1} b_1 a_1^{n_1} + a_1^{n_1} y = a_1^{n_1} b_1 a_1^{n_1}$. Finally, let $g_1 = a_1^{n_1} b_1$, then $g_1^2 = g_1$ and $M_1 = a_1M = a_1^{n_1}M = g_1M$ as required. $\square$

Recall that a module M is said to be *weakly injective* [11] if, for any finitely generated submodule $N \leq E(M)$, there exists $X \leq E(M)$ such that $N \subseteq X \cong M$.

Corollary 2.7. *Let M be a finitely generated module. Then, M is injective if and only if M is weakly injective and AGQP-injective. In particular, a ring R is right self-injective if and only if R_R is weakly injective and AGP-injective.*

Proof. We need only to prove the sufficiency. Let $x \in E(M)$. Then, there exists $X \subseteq E(M)$ such that $M + xR \subseteq X \cong M$. Hence, X is AGQP-injective and $M \mid X$ follows from Theorem 2.6(1). But M is essential in $E(M)$, so $M = X$ and hence $x \in M$. $\square$

Corollary 2.8. *Let M_R be an AGQP-injective module with $S = \text{End}(M_R)$.*

- (1) *If M_R is of finite Goldie dimension, then S is semilocal.*
- (2) *If M_R is a noetherian self-generator, then S is semiprimary.*

Proof. (1) Since M_R is AGQP-injective, it satisfies the GC2-condition by Theorem 2.6(1) and then (1) follows immediately by [12, Lemma 1.1].

(2) By (1) and Corollary 2.5. $\square$

Recall that if M and U are two right R -modules, then U is called M -projective in case for each epimorphism $g : M_R \rightarrow N_R$ and each homomorphism $\gamma : U_R \rightarrow N_R$, there is an R -homomorphism $\bar{\gamma} : U_R \rightarrow M_R$ such that $\gamma = g\bar{\gamma}$. A module M_R is called *quasiprojective* if it is M -projective.

Let R be a ring. Recall that an element $a \in R$ is called π -regular if there exists a positive integer m such that $a^m = a^m b a^m$ [13] for some $b \in R$. An element $x \in R$ is called *generalized π -regular* if there exists a positive integer n such that $x^n = x^n y x$ for some $y \in R$. A ring R is called π -regular (resp., *generalized π -regular*) if every element in R is π -regular (resp., *generalized π -regular*). If A is a subset of R , then we say that A is *regular* if every element in A is regular.

Proposition 2.9. *Let M_R be quasiprojective with $S = \text{End}(M_R)$. Then, S is regular if and only if M_R is AGQP-injective and $s(M)$ is M -projective for every $s \in S$.*

Proof. Assume that S is regular. Then, every right ideal of S is a direct summand of S_S , and so every homomorphism from a principal right ideal of S to S can be extended to an endomorphism of S . Hence, S is right P-injective and then right AGP-injective. By Theorem 2.2, M_R is AGQP-injective. The regularity of S also implies that $s(M)$ is a direct summand of M by [14, Theorem 37.7]. But M is quasiprojective, so $s(M)$ is M -projective for every $s \in S$.

Conversely, suppose M_R is AGQP-injective and $s(M)$ is M -projective for every $s \in S$. Then for any $0 \neq a \in S$, by the AGQP-injectivity of M_R , there exist a positive integer n and a left ideal X_{a^n} of S such that $a^n \neq 0$ and $\text{I}_S(\text{Ker}(a^n)) = Sa^n \oplus X_{a^n}$. Since $a^n M$ is M -projective, $\text{Ker}(a^n) = eM$ for some $e^2 = e \in S$. Then, we have $S(1-e) = \text{I}_S(eM) = \text{I}_S(\text{Ker}(a^n)) = Sa^n \oplus X_{a^n}$, and so $1-e = ba^n + x$ for some $b \in S$ and $x \in X_{a^n}$. Thus, $a^n = a^n(1-e) = a^n ba^n + a^n x = a^n ba^n$. This proves that S is π -regular and hence generalized π -regular. Clearly, $N_1(S) = \{0 \neq a \in S \mid a^2 = 0\}$ is regular (in this case, n must be equal to 1). Therefore or, S is regular by [13, Theorem 2.2]. $\square$

Recall that a module M_R is called an *IN-module* [15] if $\text{I}_S(A \cap B) = \text{I}_S(A) + \text{I}_S(B)$ for any submodules A and B of M , where $S = \text{End}(M_R)$.

Proposition 2.10. *Let M_R be an AGQP-injective IN-module with $S = \text{End}(M_R)$. Then, S is regular if and only if $W(S) = 0$.*

Proof. By Theorem 2.3, we need only to prove the sufficiency. Let $0 \neq a \in S$. Since M_R is AGQP-injective, there exist a positive integer n and a left ideal X_{a^n} of S such that $a^n \neq 0$ and $\text{I}_S(\text{Ker}(a^n)) = Sa^n \oplus X_{a^n}$. Since $W(S) = 0$, $\text{Ker}(a^n)$ is not essential in M and then there exists a nonzero submodule K such that $\text{Ker}(a^n) \oplus K$ is essential in M . Moreover, we also have

$$\begin{aligned} \text{I}_S(\text{Ker}(a^n)) + \text{I}_S(K) &= \text{I}_S(\text{Ker}(a^n) \cap K) = S, \\ \text{I}_S(\text{Ker}(a^n)) \cap \text{I}_S(K) &\subseteq \text{I}_S(\text{Ker}(a^n) + K) = 0, \end{aligned} \tag{2.7}$$

because M_R is an IN-module and $W(S) = 0$. Thus,

$$S = \text{I}_S(\text{Ker}(a^n)) \oplus \text{I}_S(K) = Sa^n \oplus X_{a^n} \oplus \text{I}_S(K). \tag{2.8}$$

Let $1 = ba^n + x$ with $b \in S$, $x \in X_{a^n} \oplus I_S(K)$, then $a^n = a^nba^n$. It follows that S is regular by the last part of the proof of Proposition 2.9. $\square$

Lemma 2.11. *Let M_R be an AGQP-injective module in which every nonzero submodule contains a nonzero M -cyclic submodule and $S = \text{End}(M_R)$. If $s \notin W(S)$, then the inclusion $\text{Ker}(s) \subseteq \text{Ker}(s - sts)$ is strict for some $t \in S$.*

Proof. If $s \notin W(S)$, then $\text{Ker}(s) \cap K = 0$ for some nonzero submodule K of M , and so $\text{Ker}(s) \cap s'(M) = 0$ for some $0 \neq s' \in S$ by hypothesis. Clearly, $ss' \neq 0$. Since M_R is AGQP-injective, there exist a positive integer n and a left ideal $X_{(ss')^n}$ such that $(ss')^n \neq 0$ and $I_S(\text{Ker}(ss')^n) = S(ss')^n \oplus X_{(ss')^n}$. Thus,

$$s'(ss')^{n-1} \in I_S(\text{Ker}(s'(ss')^{n-1})) = I_S(\text{Ker}(ss')^n) = S(ss')^n \oplus X_{(ss')^n}. \quad (2.9)$$

Write $s'(ss')^{n-1} = t(ss')^n + x$, where $t \in S$ and $x \in X_{(ss')^n}$, then $(1 - ts)s'(ss')^{n-1} = x$ and hence

$$(1 - st)(ss')^n = (s - sts)s'(ss')^{n-1} = sx \in S(ss')^n \cap X_{(ss')^n}. \quad (2.10)$$

This means that $(s - sts)s'(ss')^{n-1} = 0$. It is obvious that $\text{Ker}(s) \subseteq \text{Ker}(s - sts)$. Note that $s'(ss')^{n-1}M$ is contained in $\text{Ker}(s - sts)$ but not contained in $\text{Ker}(s)$, the inclusion $\text{Ker}(s) \subseteq \text{Ker}(s - sts)$ is strict. $\square$

Theorem 2.12. *Let M_R be AGQP-injective with $S = \text{End}(M_R)$. If every nonzero submodule of M contains a nonzero M -cyclic submodule, then the following conditions are equivalent:*

- (1) S is right perfect;
- (2) for any sequence $\{s_1, s_2, \dots\} \subseteq S$, the chain $\text{Ker}(s_1) \subseteq \text{Ker}(s_2s_1) \subseteq \dots$ terminates.

Proof. By Theorem 2.3, Lemma 2.11, and [16, Lemma 2.8], one can complete the proof in a similar way to that of [16, Theorem 2.9]. $\square$

Acknowledgment

The authors are very grateful to the referees for their useful comments and suggestions.

References

- [1] W. K. Nicholson and M. F. Yousif, "Principally injective rings," *Journal of Algebra*, vol. 174, no. 1, pp. 77–93, 1995.
- [2] S. B. Nam, N. K. Kim, and J. Y. Kim, "On simple GP-injective modules," *Communications in Algebra*, vol. 23, no. 14, pp. 5437–5444, 1995.
- [3] N. V. Sanh, K. P. Shum, S. Dhompongsa, and S. Wongwai, "On quasi-principally injective modules," *Algebra Colloquium*, vol. 6, no. 3, pp. 269–276, 1999.
- [4] R. Yue Chi Ming, "On injectivity and p -injectivity," *Journal of Mathematics of Kyoto University*, vol. 27, no. 3, pp. 439–452, 1987.
- [5] J. Chen, Y. Zhou, and Z. Zhu, "GP-injective rings need not be P-injective," *Communications in Algebra*, vol. 33, no. 7, pp. 2395–2402, 2005.
- [6] S. S. Page and Y. Zhou, "Generalizations of principally injective rings," *Journal of Algebra*, vol. 206, no. 2, pp. 706–721, 1998.

- [7] Z. Zhu, "On general quasi-principally injective modules," *Southeast Asian Bulletin of Mathematics*, vol. 30, no. 2, pp. 391–397, 2006.
- [8] W. K. Nicholson, J. K. Park, and M. F. Yousif, "Principally quasi-injective modules," *Communications in Algebra*, vol. 27, no. 4, pp. 1683–1693, 1999.
- [9] N. V. Dung, D. V. Huynh, P. F. Smith, and R. Wisbauer, *Extending Modules*, vol. 313 of *Pitman Research Notes in Mathematics Series*, Longman Scientific & Technical, Harlow, UK, 1994.
- [10] M. F. Yousif and Y. Zhou, "Rings for which certain elements have the principal extension property," *Algebra Colloquium*, vol. 10, no. 4, pp. 501–512, 2003.
- [11] S. K. Jain and S. R. López-Permouth, "Rings whose cyclics are essentially embeddable in projective modules," *Journal of Algebra*, vol. 128, no. 1, pp. 257–269, 1990.
- [12] Y. Zhou, "Rings in which certain right ideals are direct summands of annihilators," *Journal of the Australian Mathematical Society*, vol. 73, no. 3, pp. 335–346, 2002.
- [13] J. Chen and N. Ding, "On regularity of rings," *Algebra Colloquium*, vol. 8, no. 3, pp. 267–274, 2001.
- [14] R. Wisbauer, *Foundations of Module and Ring Theory*, vol. 3 of *Algebra, Logic and Applications*, Gordon and Breach Science, Philadelphia, Pa, USA, German edition, 1991.
- [15] R. Wisbauer, M. F. Yousif, and Y. Zhou, "Ikeda-Nakayama modules," *Contributions to Algebra and Geometry*, vol. 43, no. 1, pp. 111–119, 2002.
- [16] Z. Zhu, Z. Xia, and Z. Tan, "Generalizations of principally quasi-injective modules and quasiprincipally injective modules," *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, no. 12, pp. 1853–1860, 2005.