Research Article

Error Bound of Periodic Signals in the Hölder Metric

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We obtain two theorems to determine the error bound between input periodic signals and processed output signals, whenever signals belong to $H_\omega$-space and as a processor we have taken $(C,1)(E,1)$-mean and generalized an early result of Lal and Yadav in (2001).

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1. Introduction

2. Definitions and notations

Let the transforms

\[ A_n = \sum_{k=1}^{n} a_{nk}s_k, \]  
\[ B_n = \sum_{k=1}^{n} b_{nk}s_k, \]

be two regular methods of summability. Then, the A transform of the B transform of a sequence \( \{s_n\} \) is given by

\[ t_n = \sum_{p=1}^{n} a_{np}b_{p}s_k, \]

the sequence \( \{s_n\} \) is said to be summable \( t_n \) to the sum \( s \), if

\[ \lim_{n \to \infty} t_n = s. \]

Let \( s(t) \in C_{2\pi} \) be a 2\( \pi \)-periodic analog signal whose Fourier trigonometric expansion be given by

\[ s(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nt + b_n \sin nt \right) \equiv \sum_{n=0}^{\infty} A_n(t), \]

and let \( \{s_n(t)\} \) be the sequence of partial sums of (2.5).

Let the \((E, 1)\) and \((C, 1)\) transforms for the sequence \( \{s_n\} \) be defined by

\[ E_n^1 = \frac{1}{2^n} n \sum_{k=0}^{n} \binom{n}{k} s_k(t), \]
\[ \sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k(t), \]

respectively.

The product \((C, 1)(E, 1)\)-transform is expressed as the \((C, 1)\)-transform of \((E, 1)\)-transform of \( \{s_n\} \) and is given by sequence-to-sequence transformation (see, e.g., [9]):

\[ t_n(s; t) = \frac{1}{n+1} \sum_{k=0}^{n} E_k^1. \]

The sequence \( \{s_n\} \) is said to be summable \((C, 1)(E, 1)\) to the sum \( s \), if

\[ \lim_{n \to \infty} t_n(s; t) = s. \]
2.1. Regularity condition of \((C, 1)(E, 1)\)-method

\[
t_n(s; t) = \frac{1}{n + 1} \sum_{k=0}^{n} E_k = \frac{1}{n + 1} \sum_{k=0}^{n} \left\{ \frac{1}{2^k} \sum_{v=0}^{k} \binom{k}{v} s_k \right\} = \sum_{k=0}^{\infty} C_{n,k} s_k,
\]

where

\[
C_{n,k} = \begin{cases} 
\frac{1}{n + 1} 2^{-k} \sum_{v=0}^{k} \binom{k}{v}, & k \leq n \\
0, & k > n.
\end{cases}
\]

Now,

(i) \(\sum_{k=0}^{\infty} |C_{n,k}| = \sum_{k=0}^{n} \left\{ (1/2) (1/2)^{k} \sum_{v=0}^{k} \binom{k}{v} \right\} = 1,

(ii) \(C_{n,k} = (1/2) (1/2)^{k} (1/2)^{k} \to 0\), as \(n \to \infty\), for fixed \(k\),

(iii) \(\sum_{k=0}^{\infty} C_{n,k} = 1\),

thus, \((C, 1)(E, 1)\)-method is regular.

Singh [4] defined the space \(H_\omega\) by

\[
H_\omega = \{ s(t) \in C_{2\pi} : |s(t_1) - s(t_2)| \leq K\omega(|t_1 - t_2|) \},
\]

and the norm \(\| \cdot \|_{\omega}\) by

\[
\|s\|_{\omega} = \|s\|_c + \sup_{t_1,t_2} \left\{ \Delta^\omega s(t_1, t_2) \right\},
\]

where

\[
\|s\|_c = \sup_{0 \leq t \leq 2\pi} |s(t)|,
\]

\[
\Delta^\omega s(t_1, t_2) = \left| s(t_1) - s(t_2) \right| / \omega(|t_1 - t_2|),
\]

and choosing \(\Delta^\omega s(t_1, t_2) = 0\), \(\omega(t)\) and \(\omega^*(t)\) being increasing signals of \(t\). If \(\omega(|t_1 - t_2|) \leq A|t_1 - t_2|^\alpha\) and \(\omega^*||t_1 - t_2| \leq K|t_1 - t_2|^{\beta}, 0 \leq \beta < \alpha \leq 1\), \(A\) and \(K\) being positive constants, then the space

\[
H_\alpha = \{ s(t) \in C_{2\pi} : |s(t_1) - s(t_2)| \leq K|t_1 - t_2|^\alpha, 0 < \alpha \leq 1 \}
\]

is Banach space [2] and the metric induced by the norm \(\| \cdot \|_\alpha\) on \(H_\alpha\) is said to be Hölder metric.

We write

\[
\phi_1(t) = s(t_1 + t) + s(t_1 - t) - 2s(t_1),
\]

\[
K_n(t) = \sin(n + 1) \frac{n}{2} \sum_{k=0}^{n} \binom{n}{k} \sin(k + \frac{1}{2})t.
\]
3. Known result

Lal and Yadav [10] established the following theorem to estimate the error between the input signal \(s(t)\) and the signal obtained after passing through the \((C,1)(E,1)\)-transform.

**Theorem A.** If a function \(s : R \rightarrow R\) is \(2\pi\)-periodic and belonging to class \(\text{Lip } \alpha, 0 < \alpha \leq 1\), then the degree of approximation by \((C,1)(E,1)\) means of its Fourier series is given by

\[
\|t_n(s; t_1) - s(t_1)\|_{\infty} = \begin{cases} O(n^{-\alpha}), & 0 < \alpha < 1 \\ O\left(\frac{\log n}{n}\right), & \alpha = 1. \end{cases} \tag{3.1}
\]

4. Main result

The object of this paper is to generalize the above result under much more general assumptions. We will measure the error between the input signal \(s(t)\) and the processed output signal \(t_n(s; t) = (1/(n + 1))\sum_{k=1}^{n} E_k(t)\), by establishing the following theorems.

**Theorem 4.1.** Let \(\omega(t)\) defined in (2.12) be such that

\[
\int_{t}^{\pi} \frac{\omega(u)}{u^{\beta}} du = O\{H(t)\}, \quad H(t) \geq 0, \quad (4.1)
\]

\[
\int_{0}^{t} H(u) du = O\{tH(t)\}, \quad \text{as } t \to 0^+, \quad (4.2)
\]

then, for \(0 \leq \beta < \eta \leq 1\) and \(s \in H_\omega\), we have

\[
\|t_n(s; t_1) - s\|_{\omega^\eta} = O\left\{\left((n + 1)^{-1}H\left(\frac{\pi}{n + 1}\right)\right)^{1-\beta/\eta}\right\}. \tag{4.3}
\]

**Theorem 4.2.** Let \(\omega(t)\) defined in (2.12) and for \(0 \leq \beta < \eta \leq 1\) and \(s \in H_\omega\), we have

\[
\|t_n(s; t_1) - s\|_{\omega^\eta} = O\left\{\left(\omega\left(\frac{\pi}{n + 1}\right)\right)^{1-\beta/\eta} + \left((n + 1)^{-1}\sum_{k=1}^{n+1} \omega\left(\frac{1}{k + 1}\right)\right)^{1-\beta/\eta}\right\}. \tag{4.4}
\]

5. Lemmas

We will use following lemmas.

**Lemma 5.1.** Let \(\phi_1(t)\) be defined in (2.16), then for \(s \in H_\omega\), we have

\[
|\phi_1(t) - \phi_2(t)| \leq 4K\omega(|t_1 - t_2|), \quad (5.1)
\]

\[
|\phi_1(t) - \phi_2(t)| \leq 4K\omega(|t|). \quad (5.2)
\]

It is easy to verify.
Lemma 5.2. Let \( K_n(t) \) be defined in (2.17), then

\[
K_n(t) \leq C \left( \frac{2^{n+1}}{t} \right) \cos^n \left( \frac{t}{2} \right) \sin(n + 1) \left( \frac{t}{2} \right),
\]

(5.3)

where “\( C \)” is an absolute constant, not necessarily the same at each occurrence.

Proof.

\[
K_n(t) = \frac{1}{\sin(t/2)} \text{I.P.} \left\{ \sum_{k=0}^{n} \binom{n}{k} e^{i(k+1)\ell t} \right\}
\]

\[
= \frac{1}{\sin(t/2)} \text{I.P.} \left\{ e^{it/2} (1 + e^{it})^n \right\}
\]

\[
= \frac{1}{\sin(t/2)} \text{I.P.} \left\{ 2^n \cos^n \left( \frac{t}{2} \right) e^{i(n+1)\ell t/2} \right\}
\]

\[
\leq C \left( \frac{2^{n+1}}{t} \right) \cos^n \left( \frac{t}{2} \right) \sin(n + 1) \left( \frac{t}{2} \right).
\]

\[
\square
\]

Lemma 5.3.

\[
\sum_{k=0}^{n} \left( \frac{1}{t} \right) \cos^k \left( \frac{t}{2} \right) \sin(k + 1) \left( \frac{t}{2} \right) \leq \left( \frac{C}{t^2} \right) \left( 1 - \cos(n + 1) \left( \frac{t}{2} \right) \cos^{n+1} \left( \frac{t}{2} \right) \right).
\]

(5.5)

Proof.

\[
\sum_{k=0}^{n} \left( \frac{1}{t} \right) \cos^k \left( \frac{t}{2} \right) \sin(k + 1) \left( \frac{t}{2} \right)
\]

\[
= \sum_{k=0}^{n} \left( \frac{1}{t} \right) \text{I.P.} \left\{ e^{i(k+1)\ell/2} \cos^k \left( \frac{t}{2} \right) \right\}
\]

\[
= \left( \frac{1}{t} \right) \text{I.P.} \left\{ e^{it/2} \left( \frac{1 - e^{i(n+1)\ell/2} \cos^{n+1} (t/2)}{1 - e^{it/2} \cos(t/2)} \right) \right\}
\]

\[
\leq \left( \frac{C}{t^2} \right) \text{I.P.} \left\{ i - i \cos(n + 1) \left( \frac{t}{2} \right) \cos^{n+1} \left( \frac{t}{2} \right) + \sin(n + 1) \left( \frac{t}{2} \right) \cos^{n+1} \left( \frac{t}{2} \right) \right\}
\]

\[
= \left( \frac{C}{t^2} \right) \left( 1 - \cos(n + 1) \left( \frac{t}{2} \right) \cos^{n+1} \left( \frac{t}{2} \right) \right).
\]

\[
\square
\]

Lemma 5.4 (see [9]). For \( 0 \leq t \leq 1/n + 1 \), then

\[
1 - \cos(n + 1) \left( \frac{t}{2} \right) \cos^{n+1} \left( \frac{t}{2} \right) = O \left( (n + 1)^2 t^2 \right).
\]

(5.7)

Lemma 5.5 (see [6]). If \( \omega(t) \) satisfies conditions (4.1) and (4.2), then

\[
\int_0^u t^{-1} \omega(t) dt = O(uH(u)), \quad u \rightarrow 0^+.
\]

(5.8)
6. Proof of Theorem 4.1

Proof of Theorem 4.1. Following Zygmund [11], we have

\[ s_n(t_1) - s = \frac{1}{2\pi} \int_0^\pi \frac{\phi_n(t)}{\sin(t/2)} \sin\left(n + \frac{1}{2}\right)t\ dt. \]  
(6.1)

From (2.6) and (2.16), we have

\[ E_n^1(t_1) - s = \frac{2^{-n}}{2\pi} \int_0^\pi \phi_n(t)K_n(t)dt. \]  
(6.2)

Using Lemma 5.2, we have

\[ E_n^1(t_1) - s \leq C \frac{2^{-(n+1)}}{\pi} \int_0^\pi \frac{\phi_n(t)}{t} 2^{n+1}\cos^n\left(\frac{t}{2}\right) \sin(n + 1)\left(\frac{t}{2}\right)dt. \]  
(6.3)

Now from (2.8), the (C,1)-transform of (E,1)-transform is given by

\[ |t_n(s; t_1) - s| \leq \frac{C}{n+1} \int_0^\pi \left|\frac{\phi_n(t)}{t}\right| \sum_{k=0}^{n} \cos^k\left(\frac{t}{2}\right) \sin(n + 1)\left(\frac{t}{2}\right)dt. \]  
(6.4)

Setting

\[ E_n(t_1) = |t_n(s; t_1) - s(t_1)| \leq \frac{C}{n+1} \int_0^\pi \left|\frac{\phi_n(t)}{t}\right| \left|\sum_{k=0}^{n} \cos^k\left(\frac{t}{2}\right) \sin(n + 1)\left(\frac{t}{2}\right)\right|dt, \]

\[ E_n(t_1, t_2) = |E_n(t_1) - E_n(t_2)| \leq \frac{C}{n+1} \int_0^\pi \left|\frac{\phi_n(t)}{t} - \frac{\phi_n(t)}{t}\right| \left|\sum_{k=0}^{n} \cos^k\left(\frac{t}{2}\right) \sin(n + 1)\left(\frac{t}{2}\right)\right|dt \]

\[ = O\left(\frac{1}{n+1}\right)\left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^\pi\right) = I_1 + I_2, \]  
(6.5)

now using (4.1), (4.2), (5.2), and Lemma 5.5, we get

\[ I_1 = O(1) \frac{1}{n+1} \int_0^{\pi/n+1} t^{-2}\omega(t)dt = O\left( (n + 1)^{-1}H\left(\frac{\pi}{n+1}\right) \right). \]  
(6.6)

Again using (5.2), (4.1), and Lemma 5.3, we have

\[ I_2 = O(1) \frac{1}{n+1} \int_{\pi/n+1}^\pi t^{-2}\omega(t)\left|1 - \cos(n + 1)\left(\frac{t}{2}\right)\cos^{n+1}\left(\frac{t}{2}\right)\right|dt \]

\[ = O(1) \frac{1}{n+1} \int_{\pi/n+1}^\pi t^{-2}\omega(t)dt \]  
(6.7)

\[ = O\left( (n + 1)^{-1}H\left(\frac{\pi}{n+1}\right) \right). \]
Proof of Theorem 4.2.\footnote{Footnote for Theorem 4.2.}

Now from (5.1), Lemmas 5.3 and 5.4, we have

\[
I_1 = O(1) \frac{1}{n+1} \int_0^{\pi/n} \frac{\omega(|t_1 - t_2|)}{2^2} \left| 1 - \cos(n + 1) \left( \frac{t}{2} \right) \cos^{n+1} \left( \frac{t}{2} \right) \right| dt
\]

\[
= O(1) \frac{\omega(|t_1 - t_2|)}{n+1} \int_0^{\pi/n} t^2(n+1)^{\frac{1}{2}} dt
\]

\[
= O\{\omega(|t_1 - t_2|)\},
\]

\[
I_2 = O(1) \frac{\omega(|t_1 - t_2|)}{n+1} \int_{\pi/(n+1)}^{\pi} t^2 dt
\]

\[
= O\{\omega(|t_1 - t_2|)\}.
\]

Now noting that

\[
I_r = I_r^{\frac{1}{\beta/\eta}} r^{\frac{1}{\beta/\eta}}, \quad r = 1, 2,
\]

we have, from (6.6) and (6.8),

\[
I_1 = O\left\{ (\omega(|t_1 - t_2|))^{\beta/\eta} (n+1)^{-1} H\left( \frac{\pi}{n+1} \right) \right\}^{1-\beta/\eta},
\]

and from (6.7) and (6.9), we have

\[
I_2 = O\left\{ (\omega(|t_1 - t_2|))^{\beta/\eta} (n+1)^{-1} H\left( \frac{\pi}{n+1} \right) \right\}^{1-\beta/\eta}.
\]

Thus, from (2.13), (6.11) and (6.12), we have

\[
\sup_{t_1,t_2} \Delta^\omega \left| E_n(t_1,t_2) \right| = \sup_{t_1,t_2} \frac{|E_n(t_1) - E_n(t_2)|}{\omega^*(|t_1 - t_2|)}
\]

\[
= O\left\{ (\omega(|t_1 - t_2|))^{\beta/\eta} (\omega^*(|t_1 - t_2|))^{-1} (n+1)^{-1} H\left( \frac{\pi}{n+1} \right) \right\}^{1-\beta/\eta}.
\]

It is to be noted from (6.6) and (6.7),

\[
\|E_n(t_1)\|_\infty = \max_{0 \leq t_1 \leq 2\pi} |t_n(s; t_1) - s| = O\left\{ (n+1)^{-1} H\left( \frac{\pi}{n+1} \right) \right\}.
\]

Combining (6.13) and (6.14), we get

\[
\|t_n(s; t_1) - s\|_{\omega^*} = O\left\{ (n+1)^{-1} H\left( \frac{\pi}{n+1} \right) \right\}^{1-\beta/\eta}.
\]

This completes the proof of Theorem 4.1.

\[
\square
\]

Proof of Theorem 4.2.\footnote{Footnote for Theorem 4.2.} Follows analogously as the proof of Theorem 4.1 with slight changes, so we omit details.\[
\square
\]
7. Applications

The following results can easily be derived from the Theorem 4.1. If we put \( \omega^*(|t_1 - t_2|) \leq K|t_1 - t_2|^\beta \), \( \omega(|t_1 - t_2|) \leq A|t_1 - t_2|^\alpha \) and replace \( \eta \) by \( \alpha \) and set

\[
H(u) = \begin{cases} 
  u^{\alpha-1}, & 0 < \alpha < 1 \\
  \log \left( \frac{1}{u} \right), & \alpha = 1 
\end{cases} 
\] (7.1)

then we get Corollary 7.1.

**Corollary 7.1.** If \( s \in H_\alpha, 0 \leq \beta < \alpha \leq 1 \), then

\[
\|t_n(s; t_1) - s\|_\beta = \begin{cases} 
  O(n+1)^{\beta-\alpha}, & 0 < \alpha < 1 \\
  O\left( \frac{\log(n+1)}{(n+1)} \right)^{1-\beta}, & \alpha = 1 
\end{cases} \] (7.2)

If we put \( \beta = 0 \), then from above corollary, we have Corollary 7.2.

**Corollary 7.2.** If \( s \in \text{Lip} \alpha, 0 < \alpha \leq 1 \), then

\[
\|t_n(s; t_1) - s\| = \begin{cases} 
  O(n^{-\alpha}), & 0 < \alpha < 1 \\
  O\left( \frac{\log n}{n} \right), & \alpha = 1 
\end{cases} \] (7.3)

Hence Theorem 3 is particular case of Theorem 4.1.

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**References**


