Research Article

Skew Polynomial Extensions over Zip Rings

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1. Introduction

Throughout this paper $R$ denotes an associative ring with identity and $\sigma : R \to R$ an automorphism of $R$, otherwise unless stated. We denote $R[[x; \sigma]]$ ($R[[x, x^{-1}; \sigma]]$) the skew series rings (skew Laurent series rings) whose elements are the series $\sum_{i=0}^{\infty} a_i x^i$ ($\sum_{i=-\infty}^{\infty} a_i x^i$), where the addition is defined as usual and the multiplication is defined by the rule, $xa = \sigma(a)x$ ($xa = \sigma(a)x$ and $x^{-1}a = \sigma^{-1}(a)x$), for any $a \in R$. Note that the skew polynomial rings of automorphism type $R[x; \sigma]$ (skew Laurent of polynomial $R[x, x^{-1}; \sigma]$) are subrings of $R[[x; \sigma]]$ ($R[[x, x^{-1}; \sigma]]$) whose elements are $\sum_{i=0}^{n} a_i x^i$ ($\sum_{i=-n}^{m} a_i x^i$) where the sum and multiplication are defined as before.

Rege and Chhawchharia in [1] introduced the notion of an Armendariz ring. A ring $R$ is called Armendariz if whenever polynomials $\sum_{i=0}^{n} a_i x^i$, $\sum_{j=0}^{m} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each $0 \leq i \leq n$ and $0 \leq j \leq m$. The name Armendariz ring was chosen because Armendariz [2] had shown that a reduced ring (i.e., ring without nonzero nilpotent elements) satisfies this condition. Some properties of Armendariz rings have been studied by Rege and Chhawchharia [1], Armendariz [2], Anderson and Camillo [3], and Kim and Lee [4].

Faith in [5] called a ring $R$ right zip if the right annihilator $r_R(X)$ of a subset $X$ of $R$ is zero, then $r_R(Y) = 0$ for a finite subset $Y \subseteq X$; equivalently, for a left ideal $L$ of $R$ with $r_R(L) = 0$, there exists a finitely generated left ideal $L_1 \subseteq L$ such that $r_R(L_1) = 0$. $R$ is zip if it is right and left zip. The concept of zip rings was initiated by Zelmanowitz [6] and appeared in various papers [5, 7–12], and references therein. Zelmanowitz stated that any ring satisfying
the descending chain condition on right annihilators is a right zip ring (although not so-called at that time), but the converse does not hold. Extensions of zip rings were studied by several authors. Beachy and Blair [7] showed that if $R$ is a commutative zip ring, then the polynomial ring $R[x]$ over $R$ is zip. The authors in [13] proved that $R$ is a right (left) zip ring if and only if $R[x]$ is a right (left) zip ring when $R$ is an Armendariz ring.

In this paper, we study skew polynomial extensions over zip rings by using skew versions of Armendariz rings and we generalized the results of [13]. Our skew versions of Armendariz rings follow the ideas of [14, Definition]. Moreover, we provide some examples to display some of the phenomena of Section 2.

2. Skew polynomial extensions over zip rings

Throughout this paper $\sigma$ is an automorphism of $R$ unless otherwise stated and $S$ will denote one of the following rings: $R[x;\sigma]$, $R[[x;\sigma]]$, $R[x,x^{-1}\sigma]$, and $R[[x,x^{-1};\sigma]]$. A left (right) annihilator of a subset $U$ of $R$ is defined by $l_R(U) = \{ a \in R : aU = 0 \}$ ($r_R(U) = \{ a \in R : Ua = 0 \}$). For a ring $R$, put $r_{Ann}(2^R) = \{ r_R(U) : U \subseteq R \}$ and $l_{Ann}(2^R) = \{ l_R(U) : U \subseteq R \}$.

We begin with the following lemma and use it without further mention.

Lemma 2.1. Let $S$ be one of the rings above and $U$ a subset of $R$. The following statements hold:

(i) $l_S(U) = SL_R(U)$,

(ii) $r_S(U) = r_R(U)S$.

Proof. (i) We only prove for the case $S = R[x;\sigma]$ because the other cases are similar. Let $f(x) = \sum_{i=0}^{n}a_ix^i \in R[x;\sigma]$ such that $f(x)U = 0$. Then $\sigma^{-i}(a_i)U = 0$ for all $0 \leq i \leq n$ and it follows that $\sigma^{-i}(a_i) \in l_R(U)$ for all $0 \leq i \leq n$. Hence $f(x) = \sum_{i=0}^{n}x^i\sigma^{-i}(a_i) \in R[x;\sigma]l_R(U)$. So $l_{R[x;\sigma]}(U) \subseteq R[x;\sigma]l_R(U)$. We clearly have that $R[x;\sigma]l_R(U) \subseteq l_{R[x;\sigma]}(U)$. Therefore, we have $l_{R[x;\sigma]}(U) = R[x;\sigma]l_R(U)$.

(ii) We only prove for the case $S = R[x;\sigma]$ because the other cases are similar. Let $f(x) = \sum_{i=0}^{n}a_ix^i \in R[x;\sigma]$ such that $Uf(x) = 0$. Then $Ua_i = 0$ for all $0 \leq i \leq n$ and it follows that $a_i \in r_R(U)$ for all $0 \leq i \leq n$. Hence $f(x) = \sum_{i=0}^{n}x^i(a_i) \in r_R(U)R[x;\sigma]$. So $r_{R[x;\sigma]}(U) \subseteq r_R(U)R[x;\sigma]$. We clearly have that $r_R(U)R[x;\sigma] \subseteq r_{R[x;\sigma]}(U)$. Therefore, we have $r_{R[x;\sigma]}(U) = r_R(U)R[x;\sigma]$. \hfill $\Box$

With the above lemma, we have maps $\phi : r_{Ann}(2^R) \rightarrow r_{Ann}(2^S)$ defined by $\phi(I) = IS$ for every $I \in r_{Ann}(2^R)$ and

$$\Psi : l_{Ann}(2^R) \rightarrow l_{Ann}(2^S)$$

(2.1)
defined by $\Psi(I) = SI$ for every $I \in l_{Ann}(2^R)$. Moreover, we have maps $\Phi : r_{Ann}(2^S) \rightarrow r_{Ann}(2^R)$ defined by $\Phi(J) = J \cap R$ for every $J \in r_{Ann}(2^S)$ and $\Gamma : l_{Ann}(2^S) \rightarrow l_{Ann}(2^R)$ defined by $\Gamma(J) = J \cap R$ for every $J \in l_{Ann}(2^S)$. Obviously, $\phi$ is injective and $\Phi$ is surjective. Clearly, $\phi$ is surjective if and only if $\Phi$ is injective, and in this case $\phi$ and $\Phi$ are the inverses of each other. Note that $\Psi$ and $\Gamma$ satisfy the same relations as above. The first item of the definition below appears in [14, Definition].

Definition 2.2. (i) Suppose that $\sigma$ is an endomorphism of $R$. A ring $R$ satisfies $SA1'$ if for $f(x) = \sum_{i=0}^{n}a_ix^i$ and $g(x) = \sum_{j=0}^{m}b_jx^j \in R[x;\sigma]$, $f(x)g(x) = 0$ implies that $a_i\sigma^j(b_j) = 0$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$.
(ii) Suppose that $\sigma$ is an endomorphism of $R$. A ring $R$ satisfies SA2′ if for $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x;\sigma]]$, $f(x)g(x) = 0$ implies that $a_i \sigma^j (b_j) = 0$ for all $i \geq 0$, $j \geq 0$.

(iii) Suppose that $\sigma$ is an automorphism of $R$. A ring $R$ satisfies SA3′ if for $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x, x^{-1}; \sigma]$, $f(x)g(x) = 0$ implies that $a_i \sigma^j (b_j) = 0$ for all $s \leq i \leq q$ and $t \leq j \leq n$.

(iv) Suppose that $\sigma$ is an automorphism of $R$. A ring $R$ satisfies SA4′ if for $f(x) = \sum_{i=0}^{\infty} a_i x^i$ and $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x, x^{-1}; \sigma]]$, $f(x)g(x) = 0$ implies that $a_i \sigma^j (b_j) = 0$ for all $i \geq 0$ and $j \geq t$.

Note that if $R$ satisfies one of the conditions above, then all subrings $S$ of $R$ such that $\sigma(S) \subseteq S$ satisfies the same property. The following implications are easy to verify: SA4′ $\Rightarrow$ SA3′ and SA2′ $\Rightarrow$ SA1′. Following [15, Example 2.1] when $\sigma = id_R$, the last implication is not reversible.

**Lemma 2.3.** Let $\sigma$ be an automorphism of $R$. Then

(i) $R$ satisfies SA1′ if and only if $R$ satisfies SA3′;

(ii) $R$ satisfies SA2′ if and only if $R$ satisfies SA4′.

**Proof.** Let $f(x), g(x) \in R[x, x^{-1}; \sigma]$ such that $f(x)g(x) = 0$, where $f(x) = \sum_{i=-p}^{q} a_i x^i$ and $g(x) = \sum_{j=-t}^{s} b_j x^j$. We clearly have $x^p f(x) \in R[x; \sigma]$ and $g(x)x^t \in R[x; \sigma]$, then $x^p f(x)g(x)x^t = 0$. By assumption, $\sigma^p (a_i) \sigma^t (b_j) = 0$ for all $-p \leq i \leq q$ and $-t \leq j \leq s$. Hence $a_i \sigma^j (b_j) = 0$ for all $-p \leq i \leq q$ and $-t \leq j \leq s$. Since $R[x; \sigma] \subseteq R[x, x^{-1}; \sigma]$, the converse follows.

The proof of the other statement is similar. □

The following definition appears in [16, Definition 2.1].

**Definition 2.4.** Let $R$ be a ring and $\sigma$ an endomorphism of $R$. Then $R$ is said $\sigma$-compatible like right $R$-module, if $ar = 0$ if and only if $a \sigma(r) = 0$ for any $a \in R$ and $r \in R$.

Let $R$ be a ring and $\alpha$ an endomorphism of $R$. Following [17], the endomorphism $\alpha$ is said $\alpha$-rigid if $r \alpha(r) = 0$, then $r = 0$. A ring $R$ is said a rigid ring if it exists a rigid endomorphism $\alpha$ of $R$.

**Proposition 2.5.** Let $\sigma$ be an endomorphism of $R$. If $R$ is a reduced ring and $\sigma$-compatible like right $R$-module, then $R$ is a $\sigma$-rigid ring and hence satisfies SA1′ and SA2′.

**Proof.** We only prove the case of SA2′ because the other are similar. We claim that $R[[x;\sigma]]$ is a reduced ring. In fact, let $f(x) = \sum_{i=0}^{\infty} a_i x^i$ such that $(f(x))^2 = 0$. We have that $a_i^2 = 0$. Since $R$ is reduced, then $a_0 = 0$. Next, we have $a_1 \sigma(a_1) = 0$, since $R$ is $\sigma$-compatible and reduced, then $a_1 = 0$. By induction, we get $f(x) = 0$. Hence $R[[x;\sigma]]$ is reduced. Using the same ideas of [14, Proposition 3], we have that $R$ is $\sigma$-rigid and using similar ideas of [14, Corollary 4], we obtain that $R$ satisfies SA2′. □

Without the assumption that $R$ is $\sigma$-compatible, Proposition 2.5 is not true. In fact, let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\sigma : R \rightarrow R$, defined by $\sigma((a, b)) = (b, a)$. By [14, Example 2], $R$ does not satisfy SA2′ because $R$ does not satisfy SA1′. Observe that $(1, 0)(0, 1) = (0, 0)$ but $(1, 0)\sigma(0, 1) \neq (0, 0)$ and so $R$ is not $\sigma$-compatible. We have the following natural questions.
Questions

(i) Let \( \sigma \) be an endomorphism of \( R \). Suppose that \( R \) satisfies \( SA2' \). Is \( R{\sigma}\)-compatible like right \( R\)-module?

(ii) Let \( \sigma \) be an endomorphism of \( R \). Suppose that \( R \) is \( \sigma \)-compatible like right \( R\)-module. Does \( R \) satisfy \( SA2' \)?

The question (i) is false. Let \( R_0 \) be any domain and \( R = R_0[x] \). Let \( \sigma : R \to R \) be defined by \( \sigma(t) = 0 \) and \( \sigma|_{R_0} = id_{R_0} \). By [16, Example 4.1], \( R \) is not \( \sigma \)-compatible and using the similar ideas of the proof of [14, Proposition 10], we have that \( R \) satisfies \( SA2' \) and consequently \( R \) satisfies \( SA1' \).

The question (ii) is false. Let \( R = K[x,y]/(x^2,y^2) \), where \( K \) is a field of characteristic 2, and consider \( T = M_2(R) \). In this case, take \( \sigma = id_T \). By [18, Example 3.6], \( S \) does not satisfy \( SA2' \) because \( T \) does not satisfy \( SA1' \). Moreover, \( T \) is \( \sigma \)-compatible like right \( T\)-module.

In [19] the authors introduced the following version of skew Armendariz rings.

(i) Suppose that \( \sigma \) is an endomorphism of \( R \). Let \( f(x) = \sum_{i=0}^{n} a_i x^i \), \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\sigma] \) such that \( f(x)g(x) = 0 \) implies \( a_i b_j = 0 \) for all \( 0 \leq i \leq n \) and \( 0 \leq j \leq m \).

(ii) Suppose that \( \sigma \) is an endomorphism of \( R \). Let \( f(x) = \sum_{i=0}^{n} a_i x^i \), \( g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\sigma] \) such that \( f(x)g(x) = 0 \) implies \( a_i b_j = 0 \) for all \( i \geq 0 \) and \( j \geq 0 \).

Note that the item (i) above in [20, Definition 1.1] the authors called it by \( \sigma \)-Armendariz, the item (ii) above is similar with [20, Definition 1.1] and we call it here by \( \sigma \)-power Armendariz.

In the next proposition, we give a relationship between the definition above and the skew versions of Armendariz rings used in this paper. Using [21, Lemma 2.1] and [20, Theorem 1.8], the proof of next proposition is easy to verify.

**Proposition 2.6.** Let \( \sigma \) be an endomorphism of \( R \) and suppose that \( R \) is \( \sigma \)-compatible like right \( R\)-module. Then

(i) \( R \) satisfies \( SA1' \) if and only if \( R \) is \( \sigma \)-Armendariz;

(ii) \( R \) satisfies \( SA2' \) if and only if \( R \) is \( \sigma \)-power Armendariz.

The proposition above without the compatibility assumption is not true according to [20, Example 1.9] and the authors in [22, Theorem 2.2] obtained an approach of the result above without the compatibility assumption.

The following proposition is a generalization of [18, Proposition 3.4] and partially generalizes [15, Proposition 2.6].

**Lemma 2.7.** Let \( S \) be any of the rings \( R[x;\sigma] \) and \( R[[x;\sigma]] \). The following conditions are equivalent:

(i) \( R \) satisfies \( SA2' \) (SA1');

(ii) \( \phi : r\text{Ann}_R(2^R) \to r\text{Ann}_S(2^S) \) defined by \( \phi(J) = JS \) is bijective;

(iii) \( \Psi : l\text{Ann}_R(2^R) \to l\text{Ann}_S(2^S) \) defined by \( \Psi(J) = SJ \) is bijective.

**Proof.** We only prove the proposition in the case of \( SA2' \) because the equivalence of (i) and (ii) when \( R \) satisfies \( SA1' \) was proved in [23, Proposition 3.2]. The equivalence between (i) and (iii) when \( R \) satisfies \( SA1' \) has similar proof.

(i) \( \Rightarrow \) (ii). It is only necessary to show that \( \phi \) is surjective. For an element \( f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x;\sigma]] \), define \( C_{f(x)} = \{ \sigma^{-i}(a_i), i \geq 0 \} \), and for a subset \( T \) of \( R[[x;\sigma]] \), we denote the set
We show that \( r_R[[x;\sigma]](f(x)) = r_R[[x;\sigma]](C_f(x)) \). In fact, given \( g(x) = \sum_{i=0}^{\infty} b_i x^i \) in \( r_R[[x;\sigma]](f(x)) \), we have \( f(x)g(x) = 0 \). Since \( R \) satisfies SA2', then \( a_i\sigma^j(b_j) = 0 \) for all \( i \geq 0 \) and \( j \geq 0 \). Hence \( g(x) \in r_R[[x;\sigma]](C_f(x)) \).

On the other hand, let \( h(x) = \sum_{i=0}^{\infty} c_i x^i \) be an element in \( R[[x;\sigma]] \) such that \( C_f(x)h(x) = 0 \). It is clear that \( a_i\sigma^j(c_k) = 0 \) for all \( i \geq 0 \) and \( k \geq 0 \). So \( f(x)h(x) = 0 \). Since \( R \) satisfies SA2' then \( r_R[[x;\sigma]](T) = r_R[[x;\sigma]](\bigcup_{f(x)\in T} C_f(x)) \). Thus

\[
  r_R[[x;\sigma]](T) = \bigcap_{f(x)\in T} r_R[[x;\sigma]](f(x)) = \bigcap_{f(x)\in T} r_R[[x;\sigma]](C_f(x))
  = \left( \bigcap_{f(x)\in T} r_R(C_f(x)) \right) R[[x;\sigma]] = r_R(C_T) R[[x;\sigma]].
\] (2.2)

Therefore, \( \phi \) is surjective.

(ii)\( \to \) (i). Let \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) and \( g(x) = \sum_{i=0}^{\infty} b_i x^i \) be elements in \( R[[x;\sigma]] \) such that \( f(x)g(x) = 0 \). By assumption, \( r_R[[x;\sigma]](f(x)) = BR[[x;\sigma]] \), for some right ideal \( B \) of \( R \). Hence \( g(x) \in BR[[x;\sigma]] \) and we have that \( b_j \in B \subseteq r_R[[x;\sigma]](f(x)) \) for all \( j \geq 0 \). So \( a_i\sigma^j(b_j) = 0 \) for all \( i \geq 0 \) and \( j \geq 0 \).

(iii)\( \to \) (i). Let \( f(x) = \sum_{i=0}^{\infty} a_i x^i \) and \( g(x) = \sum_{i=0}^{\infty} b_i x^i \) be elements in \( R[[x;\sigma]] \) such that \( f(x)g(x) = 0 \). By assumption, \( I_{r_R[[x;\sigma]]}(g(x)) = R[[x;\sigma]]B \) for some left ideal \( B \) of \( R \). We can write \( f(x) = \sum_{i=0}^{\infty} x^i \sigma^{-i}(a_i) \in R[[x;\sigma]]B \). By the equality of the polynomials with the coefficients on the right side, we have that \( \sigma^{-i}(a_i) \in B \subseteq I_{r_R[[x;\sigma]]}(f(x)) \) for all \( i \geq 0 \). So \( a_i\sigma^j(b_j) = 0 \) for all \( i \geq 0 \) and \( j \geq 0 \).

(i)\( \to \) (iii). It is only necessary to show that \( \Psi \) is surjective. Let \( f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x;\sigma]] \).

Define \( C_f(x) = \{ a_i \mid i \geq 0 \} \), and for a subset \( T \) of \( R[[x;\sigma]] \), we denote the set \( \bigcup_{f(x)\in T} C_f(x) \) by \( C_T \).

We show that

\[
  I_{r_R[[x;\sigma]]}(f(x)) = I_{r_R[[x;\sigma]]}(C_f(x)).
\] (2.3)

In fact, given \( g(x) = \sum_{i=0}^{\infty} b_i x^i \in I_{r_R[[x;\sigma]]}(f(x)) \), we have \( g(x)f(x) = 0 \). Since \( R \) satisfies SA2', then \( b_j\sigma^j(a_i) = 0 \) for all \( i \geq 0 \) and \( j \geq 0 \). Hence \( g(x) = \sum_{j=0}^{\infty} x^j \sigma^{-j}(b_i) \in I_{r_R[[x;\sigma]]}(C_f(x)) \).

On the other hand, let \( g(x) \in R[[x;\sigma]] \) such that \( g(x)C_f(x) = 0 \). Thus \( g(x)a_i = 0 \) for all \( i \geq 0 \). So \( g(x) \sum_{i=0}^{\infty} a_i x^i = g(x)f(x) = 0 \), and we have that \( g(x) \in I_{r_R[[x;\sigma]]}(f(x)) \).

We easily have that for each subset \( T \) of \( R[[x;\sigma]] \),

\[
  I_{r_R[[x;\sigma]]}(T) = I_{r_R[[x;\sigma]]}\left( \bigcup_{f(x)\in T} C_f(x) \right). \] (2.4)

We claim that \( I_{r_R[[x;\sigma]]}(C_f(x)) = R[[x;\sigma]]I_R(C_f(x)) \). In fact, let \( g(x) = \sum_{j=0}^{\infty} b_j x^j \) such that \( g(x)C_f(x) = 0 \). Then we have that \( 0 = g(x)a_i = \sum_{j=0}^{\infty} b_j x^j a_i = \sum_{j=0}^{\infty} x^j \sigma^{-j}(b_j) a_i \). Thus \( \sigma^{-j}(b_j) \in I_R(C_f(x)) \), and it follows that

\[
  \sum_{j=0}^{\infty} x^j \sigma^{-j}(b_j) \in R[[x;\sigma]]I_R(C_f(x)). \] (2.5)

The other inclusion is trivial. So

\[
  I_{r_R[[x;\sigma]]}(T) = \bigcap_{f(x)\in T} I_{r_R[[x;\sigma]]}(C_f(x)) = \bigcap_{f(x)\in T} I_{r_R[[x;\sigma]]}(C_f(x))
  = R[[x;\sigma]]\left( \bigcap_{f(x)\in T} I_R(C_f(x)) \right) = R[[x;\sigma]]I_R(C_T). \] (2.6)

Therefore, \( \Psi \) is surjective.

\[ \square \]
Now we are able to prove the main results of this paper.

**Theorem 2.8.** Let $\sigma$ be an automorphism of $R$.

(i) Suppose that $R$ satisfies SA1'. The following conditions are equivalent:

(a) $R$ is a right (left) zip ring;
(b) $R[x;\sigma]$ is a right (left) zip ring;
(c) $R[x,x^{-1};\sigma]$ is a right (left) zip ring.

(ii) Suppose that $R$ satisfies SA2'. The following conditions are equivalent:

(a) $R$ is right (left) zip ring;
(b) $R[[x;\sigma]]$ is right (left) zip ring;
(c) $R[[x,x^{-1};\sigma]]$ is right (left) zip ring.

**Proof.** (i) We will show the right case because the left case is similar.

Suppose that $R[x;\sigma]$ is right zip. Let $X$ be a subset of $R$ such that $r_R(X) = 0$, and $f(x) = \sum_{i=0}^{n} ax^i \in R[x;\sigma]$ such that $xf(x) = 0$. Thus $a_i \in r_R(X) = 0$ and it follows that $f(x) = 0$. By assumption, there exists $X_1 = \{x_0, \ldots, x_n\}$ such that $r_R[x;\sigma](X_1) = 0$. Hence $r_R(X_1) = r_R[x;\sigma](X_1) \cap R = (0)$.

Conversely, let $Y \subseteq R[x;\sigma]$ such that $r_R[x;\sigma](Y) = 0$. By Lemma 2.7, $r_R[x;\sigma](Y) = r_R(T)R[x;\sigma]$, where $T = C_Y = \cup_{f(x)\in Y}C_f(x)$ such that $C_f(x) = \{\sigma^{-i}(a_i) : 0 \leq i \leq n\}$ with $f(x) = \sum_{i=0}^{n} a_i x^i \in Y$. We have that $r_R(T) = 0$ and, by assumption, there exists $T_1 = \{\sigma^{-i}(a_i), \ldots, \sigma^{-i}(a_i)\}$ such that $r_R(T_1) = 0$. For each $\sigma^{-i}(a_i) \in T_i$, there exists $g_{a_i}(x) \in Y$ such that some of the coefficients of $g_{a_i}(x)$ are $a_i$ for each $1 \leq j \leq n$. Let $Y_0$ be a minimal subset of $Y$ such that $g_{a_i}(x) \in Y_0$ for each $1 \leq j \leq n$. Then $Y_0$ is nonempty finite subset of $Y$. Set $T_0 = \cup_{f(x)\in Y_0}(C_f(x))$ and we have that $T_1 \subseteq T_0$. Hence $r_R(T_0) \subseteq r_R(T_1) = 0$. By Lemma 2.7, $r_R[x;\sigma](Y_0) = r_R(T_0)R[x;\sigma]$ and it follows that $r_R[x;\sigma](Y_0) = 0$.

The proofs of (a)⇔(c) and of item (ii) follow similarly. $\square$

Let $\sigma$ be an endomorphism of $R$ and $\delta : R \rightarrow R$ an additive map of $R$. The application $\delta$ is said to be a $\sigma$-derivation if $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$. The Ore extension $R[x;\sigma,\delta]$ is the set of polynomials $\sum_{i=0}^{n} a_i x^i$ with the usual sum, and the multiplication rule is $x a = \sigma(a)x + \delta(a)$.

Following [16], $R$ is said to be $(\sigma, \delta)$-compatible, where $\sigma$ is an endomorphism of $R$ and $\delta$ is a $\sigma$-derivation of $R$ if $ab = 0 \iff a\sigma(b) = 0$ and $ab = 0$ implies that $a\delta(b) = 0$.

In the next result we obtain a necessary and sufficient condition for $R[x;\sigma,\delta]$ to be left zip, when $\sigma$ is an endomorphism of $R$ using the skew version of Armendariz rings of [19].

**Theorem 2.9.** Let $\sigma$ be an endomorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Suppose that if $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x;\sigma,\delta]$, then $a_i b_j = 0$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$. Then $R$ is left zip if and only if $R[x;\sigma,\delta]$ is left zip.

**Proof.** Let $X$ be any subset of $R[x;\sigma,\delta]$ and $C_X = \cup_{f(x)\in X}C_f(x)$, where $C_f(x) = \{a_i : 0 \leq i \leq n\}$ with $f(x) = \sum_{i=0}^{n} a_i x^i$. Suppose that $I_R[x;\sigma,\delta](X) = 0$. We clearly have $I_R(C_X) = 0$. By assumption, there exists $\{b_0, \ldots, b_t\} \subseteq C_X$ such that $I_R(Y) = 0$. Let $f_b(x) \in X$ be an element of $X$ with some of its coefficients are equal to $b_i$ for all $1 \leq i \leq t$. Take $X_0$ be a minimal subset of $X$ with this property. We clearly have that $X_0$ is a finite set. We claim that $I_R[x;\sigma,\delta](X_0) = 0$. In fact, we
easily have $l_R(C_{X_0}) = 0$, where $C_{X_0} = \cup_{f(x) \in X_0} C_{f(x)}$ with $C_{f(x)}$ being defined as before. Next, let $g(x) = \sum_{i=0}^m b_jx^j$ such that $g(x)X_0 = 0.$ Hence for any $f(x) = \sum_{i=0}^n a_ix^i \in X_0,$ $g(x)f(x) = 0,$ and we have, by assumption, $b_ja_i = 0$ for all $0 \leq j \leq m$ and $0 \leq i \leq n.$ Thus $b_jC_{X_0} = 0$ for all $0 \leq j \leq m$ and it follows that $g(x) = 0.$ So $l_{R[x;\sigma]}(X_0) = 0.$

Using the methods of Theorem 2.8, the converse follows.

**Remark 2.10.** Let $R$ be a ring and $\sigma$ an endomorphism of $R.$ Suppose that $R$ is $\sigma$-power Armendariz and left zip. Using similar methods of [20, Theorem 1.8], $R$ satisfies SA2' and with similar ideas of Theorem 2.9, we have that $R$ is a left zip ring if and only if $R[[x;\sigma]]$ is a left zip ring.

### 3. Examples

In this section, we present some examples of rings that satisfy SA1' and SA2', and they are zip rings. Moreover, an example of a $\sigma$-rigid ring that is a zip ring is given.

**Example 3.1.** Let $F$ be any field and $\sigma : F \rightarrow F$ any automorphism of $F.$ Following [14, page 113], we consider the ring $T(F,F)$ with automorphism $\tilde{\sigma}(a,b) = (\sigma(a),\sigma(b))$ and we denote it by $\sigma.$ Note that

$$T(F,F) = \left\{ \left( \begin{array}{cc} a & b \\ 0 & a \end{array} \right) : a,b \in F \right\}. \quad (3.1)$$

By [14, Proposition 15], $T(F,F)$ satisfies SA1', and using similar methods, we can prove that $T(F,F)$ satisfies SA2'. We claim that $T(F,F)$ is a zip ring. In fact, the unique one-sided ideals of $T(F,F)$ are $\{ (0,0) \},$

$$I = \left\{ \left( \begin{array}{c} 0 \\ b \\ 0 \end{array} \right) : b \in F \right\}, \quad (3.2)$$

and $T(F,F).$ Note that $r_{T(F,F)}(I) \neq 0$ and $l_{T(F,F)}(I) \neq 0.$ So we easily have that $T(F,F)$ is a zip ring.

**Example 3.2.** Let $F$ be any field and $\sigma$ a monomorphism of $F,$ and let

$$R = \left\{ \left( \begin{array}{ccc} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{array} \right) : a,b,c \in F \right\} \quad (3.3)$$

with usual addition and multiplication of matrix. Note that the monomorphism $\sigma$ is naturally extended to $R,$ and $R$ has the following one-sided ideals:

$$I_1 = \left\{ \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{array} \right) : a \in F \right\}, \quad I_2 = \left\{ \left( \begin{array}{ccc} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) : c \in F \right\}, \quad (3.4)$$

$R$ and the zero ideal. We easily have $r_R(I_2) \neq 0,$ $l_R(I_2) \neq 0,$ $r_R(I_1) \neq 0,$ and $l_R(I_1) \neq 0.$ Now we clearly have that $R$ is a zip ring and by [14, Proposition 17], $R$ satisfies SA1', and with similar methods of [14, Proposition 17], we can prove that $R$ satisfies SA2'.
Example 3.3. Let $D$ be any domain with identity, $R = D[x]$, $\sigma$ an endomorphism of $R$ defined by $\sigma(f(x)) = f(0)$. Since $R$ is a domain, then $R$ is right and left zip. Moreover, using similar methods of [14, Example 5], we have that $R$ satisfies $SA1'$ and $SA2'$.

Example 3.4. Let $D$ and $D_1$ be any domains, $\sigma$ an monomorphism of $D$, and $\tau$ an monomorphism of $D_1$. Set $R = D \times D_1$ with usual addition and multiplication, and we define an endomorphism $\gamma$ of $R$ by $\gamma(a, b) = (\sigma(a), \tau(b))$. We easily have that $\gamma$ is a monomorphism of $R$. Since $D$ is $\sigma$-rigid and $D_1$ is $\tau$-rigid, we easily obtain that $R$ is $\gamma$-rigid. We claim that $R$ is left and right zip. In fact, let $I$ be any left ideal of $R$. It is well known that $I = A \times B$, where $A$ is a left ideal of $D$ and $B$ is a left ideal of $D_1$. Suppose that $r_R(I) = 0$. Then $A \neq 0$ and $B \neq 0$. It is not difficult to show that $r_D(A) = 0$ and $r_{D_1}(B) = 0$. Since $D$ and $D_1$ are left zip, then there exists a left finitely generated ideal $L$ of $D$ contained in $A$ such that $r_D(L) = 0$ and a left finitely generated ideal $L_1$ of $D_1$ contained in $B$ such that $r_{D_1}(L_1) = 0$. Thus $r_R(L \times L_1) = 0$ and $L \times L_1$ is a left finitely generated ideal of $R$ contained in $A \times B$. Hence $R$ is left zip. Using similar methods, we have that $R$ is right zip.

Example 3.5. Let $F$ be a field, $\sigma$ an automorphism of $F$,

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c \in F \right\},$$

and $D$ a domain with automorphism $\tau$. Set $T = R \times D$ and we define an endomorphism $\gamma$ of $T$ by $\gamma(a, b, c) = (\sigma(a), \tau(b), \tau(c))$. It is clear that $\gamma$ is an automorphism of $T$ and it is not difficult to show that $T$ satisfies $SA1'$ and $SA2'$ because $R$ and $D$ satisfy $SA1'$ by [14, Proposition 17] and [14, Proposition 10], respectively, and using similar methods of [14, Proposition 17] and [14, Proposition 10], $R$ and $D$ satisfy $SA2'$, respectively.

Using similar methods of Example 3.4, we have that $T$ is right and left zip and note that $T$ is not $\gamma$-rigid, since $T$ is not a reduced ring.

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