Research Article

A Note on Locally Inverse Semigroup Algebras

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Received 9 December 2007; Accepted 29 January 2008

Let $R$ be a commutative ring and $S$ a finite locally inverse semigroup. It is proved that the semigroup algebra $R[S]$ is isomorphic to the direct product of Munn algebras $\mathcal{M}(R[G_j], m_j, n_j; P_j)$ with $J \in S/J$, where $m_j$ is the number of $R$-classes in $J$, $n_j$ the number of $L$-classes in $J$, and $G_j$ a maximum subgroup of $J$. As applications, we obtain the sufficient and necessary conditions for the semigroup algebra of a finite locally inverse semigroup to be semisimple.

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1. Main results

A regular semigroup $S$ is called a locally inverse semigroup if for all idempotent $e \in S$, the local submonoid $eSe$ is an inverse semigroup under the multiplication of $S$. Inverse semigroups are locally inverse semigroups. Inverse semigroup algebras are a class of semigroup algebras which is widely investigated. One of fundamentally important results is that a finite inverse semigroup algebra is the direct product of full matrix algebras over group algebras of the maximum subgroups of this finite inverse semigroup. Consider that all local submonoids of a locally inverse semigroup are inverse semigroups, it is a very natural problem whether a finite locally inverse semigroup algebra has a similar representation to inverse semigroup algebras. This is the main topic of this note.

Let $A$ be an $R$-algebra. Let $m$ and $n$ be positive integers, and let $P$ be a fixed $n \times m$ matrix over $A$. Let $\mathcal{M} := \mathcal{M}(A; m, n; P)$ be the vector space of all $m \times n$ matrices over $A$. Define a product $\circ$ in $\mathcal{M}$ by

$$A \circ B = APB \quad (A, B \in \mathcal{M}), \quad (1.1)$$

where $APB$ is the usual matrix product of $A$, $P$, and $B$. Then $\mathcal{M}$ is an algebra over $R$. Following [1], we call $\mathcal{M}$ the Munn $m \times n$ matrix algebra over $A$ with sandwich matrix $P$. 


By a semisimple semigroup, we mean a semigroup each of whose principal factor is either a completely 0-simple semigroup or a completely simple semigroup. It is well known that a finite regular semigroup is semisimple. The Rees theorem tells us that any completely 0-simple semigroup (completely simple semigroup) is isomorphic to some Rees matrix semigroup \( M(G, I, \Lambda; P) \) (\( M(G, I, \Lambda; P) \)), and vice versa (for Rees matrix semigroups, refer to [1]).

In what follows, by the phrase “Let \( S = \bigcup_{J \in S/\mathcal{J}} M^0(G_J; I_J, \Lambda_J; P_J) \) be a finite regular semigroup,” we mean that \( S \) is a finite regular semigroup in which the principal factor of \( S \) determined by the \( \mathcal{J} \)-class \( J \) is isomorphic to the Rees matrix semigroup \( M^0(G_J; I_J, \Lambda_J; P_J) \) or \( M(G_J; I_J, \Lambda_J; P_J) \) for any \( J \in S/\mathcal{J} \).

The following is the main result of this paper.

**Theorem 1.1.** Let \( S = \bigcup_{J \in S/\mathcal{J}} M^0(G_J; I_J, \Lambda_J; P_J) \) be a finite locally inverse semigroup. Then the semigroup algebra \( R[S] \) is isomorphic to the direct product of \( M(R[G_J]; I_J, \Lambda_J; P_J) \) with \( J \in S/\mathcal{J} \).

Based on Theorem 1.1 and [1, Lemma 5.17, page 162, and Lemma 5.18, page 163], the following corollary is straightforward.

**Corollary 1.2.** Let \( S = \bigcup_{J \in S/\mathcal{J}} M^0(G_J; I_J, \Lambda_J; P_J) \) be a finite locally inverse semigroup. Then the semigroup algebra \( R[S] \) has an identity if and only if \( |I_J| = |\Lambda_J| \) and \( P_J \) is invertible in the full matrix algebra \( M_{|I_J|}(R[G_J]) \) for all \( J \in S/\mathcal{J} \).

Reference [1, Lemma 5.18, page 163] told us that \( M(R[G_J], m_J, n_J; P_J) \) is isomorphic to the full matrix algebra \( M_{n_J}(R[G_J]) \) if \( M(R[G_J], m_J, n_J; P_J) \) has an identity. Now, we have the following.

**Corollary 1.3.** Let \( S = \bigcup_{J \in S/\mathcal{J}} M^0(G_J; I_J, \Lambda_J; P_J) \) be a finite locally inverse semigroup. If \( R[S] \) has an identity, then \( R[S] \) is isomorphic to the direct product of the full matrix algebras \( M_{|I_J|}(R[G_J]) \) with \( J \in S/\mathcal{J} \).

The following corollary is a consequence of Corollary 1.3.

**Corollary 1.4.** Let \( S = \bigcup_{J \in S/\mathcal{J}} M^0(G_J; I_J, \Lambda_J; P_J) \) be a finite locally inverse semigroup. Then the semigroup algebra \( R[S] \) is semisimple if and only if for all \( J \in S/\mathcal{J} \),

1. \( |I_J| = |\Lambda_J| \);
2. \( P_J \) is invertible in the full matrix algebra \( M_{|I_J|}(R[G_J]) \);
3. \( R[G_J] \) is semisimple.

## 2. Proof of Theorem 1.1

For our purpose, we have the Möbius inversion theorem [2].

**Lemma 2.1.** Let \((P, \leq)\) be a locally finite partially ordered set (i.e., intervals are finite) in which each principal ideal has a maximum and \( G \) be an Abelian group. Suppose that \( f : P \rightarrow G \) is a function and define \( g : P \rightarrow G \) by \( g(x) = \sum_{y \leq x} f(y) \). Then \( f(x) = \sum_{y \leq x} g(y)\mu(x, y) \), where \( \mu \) is a Möbius function.

Now assume that \( S \) is a regular semigroup and \( a, b \in S \). Define

\[
a \leq b \iff \text{there exist } e, \ f \in E(S) \text{ such that } a = eb = bf.
\]
Then \( \leq \) is a partial order on \( S \). Following [3], we call \( \leq \) the natural partial order on \( S \). Equivalently, \( a \leq b \) if and only if for every (for some) \( f \in E(R_0) \) (\( f \in E(L_b) \)), there exists \( e \in E(R_a) \) (\( e \in E(L_a) \)) such that \( e \leq f \) and \( a = eb \) (\( a = be \)). Moreover, Nambooripad [3, 4] proved that \( S \) is a locally inverse semigroup if and only if the natural partial order \( \leq \) is compatible with respect to the multiplication of \( S \).

**Lemma 2.2.** Let \( S \) be a locally inverse semigroup and \( a, b \in S \). Then for any \( u \leq ab \), there exist \( x \leq a \) and \( y \leq b \) such that \( u = xy \), \( x \in R_a \) and \( y \in L_b \).

**Proof.** For any \( e \in E(R_a) \), we have \( ea = a \) and \( eab = ab \). Let \( z \) be an inverse of \( ab \). Clearly, \( abz \in E(R_{ab}) \). Note that \( eabz = abz \). It is easy to check that \( abze \in E(S) \), \( abze \leq e \), and \( abze \in E(S) \). Hence \( abze \in E(S) \) such that \( u = gab \) and \( g \leq abze \). Thus \( ga \leq a \). On the other hand, since \( R \) is a left congruence and since \( abze \in E(S) \), we have \( u = gab \) and \( g \leq abze \). These imply that \( uRga \). Dually, we have \( h \in E(S) \) such that \( u = abh \) and \( uEbh \). Since \( u = gab = abh = uh = (ga)(bh) \), we know that \( ga \) and \( bh \) are the required elements \( x \) and \( y \).

Define a multiplication \( \otimes \) on \( S^0 = S \cup \{0\} \) by

\[
x \otimes y = \begin{cases} xy & \text{if } x \neq 0, y \neq 0, \text{ and } xy \in J_x; \\ 0 & \text{otherwise}, \end{cases}
\]

where \( xy \) is the product of \( x \) and \( y \) in \( S \). By the arguments of [4, page 9], \( (S^0, \otimes) \) is a semigroup. We denote by \( S^0 \) the semigroup \((S^0, \otimes)\). For any \( J \in S/\mathcal{J} \), we denote \( J^0 = J \cup \{0\} \). It is easy to check that \((J^0, \otimes)\) is a subsemigroup of \( S^0 \), which is isomorphic to the principal factor of \( S \) determined by \( J \). We will denote the semigroup \((J^0, \otimes)\) by \( J^0 \). By the definition of \( \otimes \), it is easy to see that in the semigroup \( S^0 \),

\[
\begin{align*}
(i) & \quad J_x^0 \otimes J_y^0 \subseteq J_x^0 \\
(ii) & \quad J_x^0 \otimes J_y^0 = 0 \text{ for all } x, y \in S \text{ such that } x \notin J_y.
\end{align*}
\]

Thus \( R_0[S^0] \) is the direct sum of the contracted semigroup algebras \( R_0[J^0] \) with \( J \in S/\mathcal{J} \). Note that \( J^0 \) is isomorphic to some principal factor of \( S \). We observe that \( J^0 \) is a completely 0-simple semigroup since \( S \) is a semisimple semigroup, and thus \( J^0 \) is isomorphic to some Rees matrix semigroup \( \mathcal{M}^0(G, I, J, \Lambda; P) \). By a result of [1], \( R_0[\mathcal{M}^0(G, I, J, \Lambda; P)] \) is isomorphic to \( \mathcal{M}(R[G], [I], [J]; [P]) \). Consequently, to verify Theorem 1.1, we need only to prove that \( R[S] \) is isomorphic to \( R_0[S^0] \).

For the convenience of description, we introduce the semigroup \( \overline{S} \). Put \( \overline{S} = \{x \mid x \in S\} \cup \{0\} \). Define a multiplication on \( \overline{S} \) as follows:

\[
\overline{x} \ast \overline{y} = \overline{xy},
\]

where \( \overline{0} \) is the zero element of \( \overline{S} \). It is easy to see that \( \overline{S} \) is isomorphic to \( S^0 \). Hence the contracted semigroup algebra \( R_0[\overline{S}] \) is isomorphic to the contracted semigroup algebra \( R_0[S^0] \). For \( J \in S/\mathcal{J} \), we denote \( \overline{J} = \{x \mid x \in J\} \cup \{0\} \). It is easy to check that \( (\overline{J}, \ast) \) is a subsemigroup of \( \overline{S} \) isomorphic to the semigroup \( J^0 \). So, for any \( J, K \in S/\mathcal{J} \), we have

\[
\overline{J} \ast \overline{K} \begin{cases} \subseteq \overline{J} & \text{if } K = J, \\ = 0 & \text{otherwise}. \end{cases}
\]
For Theorem 1.1, it remains to prove the following lemma.

**Lemma 2.3.** \( R[S] \cong R_0[\mathcal{S}] \).

**Proof.** We consider the mapping \( \varphi : R[S] \to R_0[\mathcal{S}] \) given on the basis by \( \varphi(s) = \sum_{t \leq s} t(s \in S) \). Clearly, \( \varphi \) is well defined. Of course, \( \varphi \) and \( \bar{\psi} \) may be regarded as the mappings of the ordered set \( (S, \leq ) \) into the additive group of \( R_0[\mathcal{S}] \). Now, by applying the Möbius inversion theorem to the mappings \( \varphi \) and \( \bar{\psi} \), we have

\[
\bar{s} = \sum_{t \leq s} \varphi(t)\mu(t, s) = \varphi \left( \sum_{t \leq s} t\mu(t, s) \right),
\]

where \( \mu \) is the Möbius function for \( (S, \leq ) \). Hence \( \varphi \) is surjective.

We will prove that \( \varphi \) is injective. For \( \alpha_0 = \sum_{x \in S} p_{\alpha x}^0 \in R[S] \), we denote by \( \text{supp}(\alpha_0) \) the set \( \{ x \in S : p_{\alpha x}^0 \neq 0 \} \) and by \( M(\alpha_0) \) the set of maximal elements in the set \( \text{supp}(\alpha_0) \) with respect to the partial order \( \leq \). In recurrence, we define \( \alpha_n = \alpha_{n-1} - \sum_{x \in M(\alpha_{n-1})} p_{\alpha x}^{n-1}x \), where \( \alpha_n = \sum_{x \in \text{supp}(\alpha_0)} p_{\alpha x}^n x \). Let \( \beta_n = \sum_{x \in \text{supp}(\beta_0)} p_{\beta x}^n x \) with \( n = 0, 1, 2, \ldots \). If \( \varphi(\alpha_n) = \varphi(\beta_n) \), then by the definition of \( \varphi \), \( \sum_{x \in M(\alpha_n)} p_{\alpha x}^n x + \Gamma_n = \varphi(\alpha_n) = \varphi(\beta_n) = \sum_{y \in M(\beta_n)} p_{\beta y}^n y + \Gamma_n \), where \( \Gamma_n = \sum_{x \in M(\alpha_0)} \sum_{y \in S, y \leq x} p_{\alpha x}^n y \) and \( \Gamma_n = \sum_{x \in M(\beta_0)} \sum_{y \in S, y \leq x} p_{\beta y}^n y \) and hence \( \sum_{x \in M(\alpha_0)} p_{\alpha x}^n x = \sum_{x \in M(\beta_0)} p_{\beta x}^n x \), thus \( M(\alpha_n) = M(\beta_n) \) and \( p_{\alpha x}^n = p_{\beta x}^n \) for any \( x \in M(\alpha_n) \). This can imply the following.

**Fact 2.4.** If \( \varphi(\alpha_n) = \varphi(\beta_n) \), then \( M(\alpha_n) = M(\beta_n) \) and by the definition of \( \varphi \), \( \varphi(\alpha_{n+1}) = \varphi(\beta_{n+1}) \).

By the definition of \( \varphi \), the following facts are immediate.

**Fact 2.5.** \( \alpha_n = \beta_n \) if and only if \( M(\alpha_n) = M(\beta_n) \) and \( \alpha_{n+1} = \beta_{n+1} \).

**Fact 2.6.** If \( \varphi(\alpha_n) = \varphi(\beta_n) \) and \( M(\alpha_n) = M(\beta_n) \), \( M(\alpha_n) = M(\beta_n) \), then \( \alpha_n = \beta_n \).

Note that \( |\text{supp}(\alpha_0)| < \infty \) and \( \text{supp}(\alpha_{n+1}) \subseteq \text{supp}(\alpha_n) \). We thus have a smallest integer \( k \) such that \( M(\alpha_k) = \text{supp}(\alpha_k) \). Clearly, \( \alpha_{k+1} = 0 \). This means that \( k \) is the smallest integer \( t \) such that \( \alpha_{t+1} = 0 \). Similarly, there exists the smallest integer \( l \) such that \( \beta_{l+1} = 0 \) and \( M(\beta_l) = \text{supp}(\beta_l) \). Now, assume \( \varphi(\alpha_0) = \varphi(\beta_0) \). By using Fact 2.4 repeatedly,

\[
\varphi(\alpha_1) = \varphi(\beta_1), \quad \varphi(\alpha_2) = \varphi(\beta_2), \ldots, \quad \varphi(\alpha_{k+1}) = \varphi(\beta_{k+1}).
\]

But \( \varphi(\alpha_{k+1}) = 0 \), we have \( \varphi(\beta_{k+1}) = 0 \) and by the definition of \( \varphi \), \( \beta_{k+1} = 0 \). Thus \( k + 1 \geq l + 1 \) by the minimality of \( l \), and \( k \geq l \). Therefore \( k = l \). Since \( \varphi(\alpha_k) = \varphi(\beta_k) \), by Fact 2.6, we have \( \alpha_k = \beta_k \) since \( M(\alpha_k) = \text{supp}(\alpha_k) \) and \( M(\beta_k) = \text{supp}(\beta_k) \). Again by the hypothesis \( \varphi(\alpha_0) = \varphi(\beta_0) \), and by Fact 2.4, \( M(\alpha_0) = M(\beta_0) \); and by (2.6), \( M(\alpha_1) = M(\beta_1) \), \( M(\alpha_2) = M(\beta_2) \), \ldots, \( M(\alpha_k) = M(\beta_k) \). By Fact 2.5, \( M(\alpha_{k-1}) = M(\beta_{k-1}) \); and \( \alpha_k = \beta_k \) imply \( \alpha_{k-1} = \beta_{k-1} \); moreover, by using Fact 2.5 repeatedly, \( \alpha_{k-2} = \beta_{k-2}, \ldots, \alpha_1 = \beta_1 \) and \( \alpha_0 = \beta_0 \). We have now proved that \( \varphi \) is injective.

Finally, for any \( s, t \in S \), by (2.4), we have

\[
\bar{s}t = \begin{cases} 
0 & \text{if } s, t \in J, \\
\frac{st}{t} & \text{otherwise},
\end{cases}
\]
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and by Lemma 2.2,

\[ \varphi(s) \ast \varphi(t) = \left( \sum_{x \leq s} x \right) \ast \left( \sum_{y \leq t} y \right) = \sum_{x \in J_{st}, x \leq s} \sum_{y \in J_{st}, y \leq t} x \ast y. \]

Moreover, by Lemma 2.2, we have

\[ \varphi(st) = \sum_{u \leq sl} u = \sum_{x \in J_{st}, x \leq sl} \sum_{y \in J_{st}, y \leq tl} x \ast y = \sum_{x \leq s, x \in J_{st}} \sum_{y \leq t, y \in J_{st}} x \ast y = \varphi(s) \ast \varphi(t). \]

Thus \( \varphi \) is a homomorphism of \( R[S] \) into \( R_0[S] \). Consequently, \( \varphi \) is an isomorphism of \( R[S] \) onto \( R_0[S] \). \( \square \)

**Acknowledgment**

The research is supported by the NSF of Jiangxi Province, the SF of Education Department of Jiangxi Province, and the SF of Jiangxi Normal University.

**References**


