Review Article

On Some Subclasses of Harmonic Functions Defined by Fractional Calculus

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The purpose of this paper is to study subclasses of normalized harmonic functions with positive real part using fractional derivative. Sharp estimates for coefficients and distortion theorems are given.

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1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain $C$ if both $u$ and $v$ are real harmonic in $C$. In any simply connected domain $D \subseteq C$, we can write $f = h + \overline{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the coanalytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and orientation-preserving in $D$ is that $|g'(z)| < |h'(z)|$ in $D$, see [1].

Denote by $H$ the class of functions $f = h + \overline{g}$ which are harmonic univalent and orientation-preserving in the open unit disk $U = \{z : |z| < 1\}$ so that $f = h + \overline{g}$ is normalized by $f(0) = h(0) = f_2(0) - 1 = 0$. Therefore, for $f = h + \overline{g} \in H$, we can express $h$ and $g$ by the following power series expansion:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1.$$  \hspace{1cm} (1.1)

Observe that $H$ reduces $S$, the class of normalized univalent analytic functions, if the coanalytic part of $f$ is zero.

For $f = h + \overline{g}$ given by (1.1) and $n > -1$, Murugusundaramoorthy [2] defined the Ruscheweyh derivative of the harmonic function $f = h + \overline{g}$ in $H$ by

$$D^n f(z) = D^n h(z) + D^n \overline{g}(z),$$  \hspace{1cm} (1.2)
where the Ruscheweh derivative of a power series \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is given by

\[
D^n f(z) = \frac{z}{(1 - z)^{n+1}} \ast f.
\]  

(1.3)

The operator \( \ast \) stands for the Hadamard product or convolution of two power series

\[
f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n
\]  

(1.4)

defined by

\[
(f \ast g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n.
\]  

(1.5)

In [3], Owa introduced the following definition.

**Definition 1.1.** Let the function \( f(z) \) be analytic in a simply connected domain of the \( z \)-plane containing the origin and let \( 0 \leq \lambda < 1 \). The fractional derivative of \( f \) of order \( \lambda \) is defined by

\[
D^\lambda_z f(z) := \frac{1}{\Gamma(1 - \lambda)} \frac{d}{dz} \int_0^1 \frac{f(\zeta)}{(z - \zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1),
\]  

(1.6)

where the multiplicity of \( (z - \zeta)^{-\lambda} \) is removed by requiring \( \log(z - \zeta) \) to be real when \( z - \zeta > 0 \).

In [4], Owa gave the relation between the fractional derivative and Ruscheweyh operator for the function \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) as

\[
D^\lambda f(z) := \frac{1}{\Gamma(1 + \lambda)} D^\lambda z [z^{\lambda-1} f(z)], \quad 0 < \lambda < 1,
\]

(1.7)

\[
D^0 f(z) = \lim_{\lambda \to 1} D^\lambda f(z),
\]

\[
D^1 f(z) = \lim_{\lambda \to 1} D^\lambda f(z).
\]

Using (1.2) and the relation between the fractional derivative and Ruscheweyh operator, we define the fractional derivative of order \( \lambda, 0 \leq \lambda < 1 \), for the harmonic function \( f = h + g \) as

\[
D^\lambda_z [z^{\lambda-1} f(z)] = D^\lambda_z [z^{\lambda-1} h(z)] + D^\lambda_z [z^{\lambda-1} g(z)], \quad 0 < \lambda < 1,
\]

\[
D^0 f(z) = \lim_{\lambda \to 0} D^\lambda f(z),
\]

\[
D^1 f(z) = \lim_{\lambda \to 1} D^\lambda f(z).
\]  

(1.8)
Since $D^λf = D^λh + \overline{D^λg}$, it was proved in [1] that the harmonic function $D^λf$ is starlike of order 1/2 if and only if the analytic function $D^λh - D^λg$ is starlike of order 1/2, and it was shown in [4, Theorem 3] that $D^λh - D^λg$ is starlike of order 1/2 if and only if $\Re\{D^{λ+1}_z[z^λ(h - g)] / D^λ_z[z^λ-1(h - g)]\} > (1 + λ)/2$ for $0 < λ < 1$. Since $\Re\{D^{λ+1}_z[z^λ(h - g)] / D^λ_z[z^λ-1(h - g)]\} = \Re(\Gamma(2 + λ)D^{λ+1}(h - g) / \Gamma(1 + λ)D^λ(h - g))$, then $D^λh - D^λg$ is starlike of order $(1 + λ)/2\Gamma(2 + λ)$, hence $D^λf = D^λh + D^λg$ is starlike of order $(1 + λ)\Gamma(1 + λ)/2\Gamma(2 + λ)$. This means

$$\Re\frac{DD^{λ+1}_z[f]}{D^λz[f]} > \frac{(1 + λ)\Gamma(1 + λ)}{2\Gamma(2 + λ)} \Rightarrow \Re\frac{D^{λ+1}_z[z]f}{D^λ_z[z]^{λ-1}} > \frac{(1 + λ)}{2}.$$  

(1.9)

Recently, Owa and Srivastava [5] studied the linear operator $Ω^λ$ defined by operator

$$Ω^λf(z) := \Gamma(2 - λ)z^λD^λ_zf(z) \quad (0 ≤ λ < 1),$$  

(1.10)

where $f$ is normalized and analytic function on $U$.

It is easily seen that

$$Ω^0 f = f, \quad Ω^1 f = zf'.$$  

(1.11)

Analogously, we studied the linear operator $Ω^λ$ defined on the harmonic function $f = h + \overline{g}$ by

$$Ω^λh(z) := \Gamma(2 - λ)z^λD^λ_zh(z) = \sum_{n=0}^{∞} \frac{Γ(n + 2)Γ(2 - λ)}{Γ(n + 2 - λ)} a_{n+1} z^{n+1}, \quad a_1 = 1,$$

$$Ω^λg(z) := \Gamma(2 - λ)z^λD^λ_zg(z) = \sum_{n=0}^{∞} \frac{Γ(n + 2)Γ(2 - λ)}{Γ(n + 2 - λ)} b_{n+1} z^{n+1}, \quad b_1 = 0.$$  

(1.13)

We will define subclasses of normalized harmonic functions obtained by the Hadamard product and using the fractional derivative.

2. Main results

Let $h$ and $g$ be analytic in $U$. Let $P_H$ stand for harmonic functions $f = h + \overline{g}$ so that $\Re f > 0$, $z \in U$ and $f(0) = 1$.

If the function $f_z + \overline{f_z} = h' + \overline{g'}$ belongs to $P_H$ for the analytic and normalized functions

$$h(z) = z + \sum_{n=2}^{∞} a_n z^n, \quad g(z) = \sum_{n=2}^{∞} b_n z^n,$$  

(2.1)

then the class of functions $f = h + \overline{g}$ is denoted by $P^0_H$ [6].
The function

\[ t_\alpha(z) = z + \frac{1}{1 + \alpha} z^2 + \cdots + \frac{1}{1 + (n-1)\alpha} z^n + \cdots \quad (2.2) \]

is analytic on \( \mathcal{U} \) when \( \alpha \) is a complex number different from \(-1, -(1/2), -(1/3), \ldots \). For \( \Omega^1 f \in \tilde{P}_H^{1.0} \), we denote by \( \tilde{P}_H^{1,0}(\alpha) \) the class of functions defined by

\[ \Omega^1 F = \Omega^1 f(t_\alpha + \overline{t_\alpha}). \quad (2.3) \]

Therefore,

\[
\begin{align*}
\Omega^1 F &= \Omega^1 H + \overline{\Omega^1 g} \\
&= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)[1+(n-1)\alpha]} a_n z^n + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)[1+(n-1)\alpha]} b_n z^n \\
&= z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} B_n z^n, \quad z \in \mathcal{U}
\end{align*}
\]

is in \( \tilde{P}_H^{1,0}(\alpha) \). Conversely, if \( \Omega^1 F \) is in the form (2.4), with \( a_n, b_n \) being the coefficients of \( \Omega^1 f \in \tilde{P}_H^{0.0} \), then \( \Omega^1 F = \tilde{P}_H^{1,0}(\alpha) \).

Note that \( \tilde{P}_H^{0.0}(\alpha) = \tilde{P}_H^{0.0}(\alpha) \) [7] and \( \tilde{P}_H^{0.0}(0) \equiv \tilde{P}_H^0 \).

**Theorem 2.1.** If \( \Omega^1 F \in \tilde{P}_H^{1.0}(\alpha) \), then there exists \( \Omega^1 f \in \tilde{P}_H^{0.0} \) so that

\[ a \left[ z(\Omega^1 F)_z(z) + \overline{\Xi(\Omega^1 F)_{\overline{z}}(z)} \right] + (1 - \alpha) \Omega^1 F(z) = \Omega^1 f(z). \quad (2.5) \]

Conversely, for any function \( f \) such that \( \Omega^1 f \in \tilde{P}_H^{0.0} \), there exists \( \Omega^1 F \in \tilde{P}_H^{1.0}(\alpha) \) satisfying (2.5).

**Proof.** Let \( \Omega^1 F \in \tilde{P}_H^{1.0}(\alpha) \). If \( \Omega^1 f \in \tilde{P}_H^{0.0} \), then since

\[ azt_\alpha(z) + (1 - \alpha)t_\alpha(z) = t_0(z) \quad (2.6) \]

as \( \Omega^1 F = \Omega^1 f(t_\alpha + \overline{t_\alpha}) \), we obtain that

\[ \Omega^1 f(z) = a \left[ \Omega^1 f(z) \ast (zt_\alpha + \overline{zt_\alpha}) \right] + (1 - \alpha) \left[ \Omega^1 f(z) \ast (t_\alpha + \overline{t_\alpha}) \right]. \quad (2.7) \]

Therefore,

\[ \Omega^1 f(z) = a \left[ z(\Omega^1 F)_z(z) + \overline{\Xi(\Omega^1 F)_{\overline{z}}(z)} \right] + (1 - \alpha) \Omega^1 F(z). \quad (2.8) \]
Conversely, for $\Omega^k f \in \tilde{P}_H^0$, from (2.1), (2.2), and (2.5),

$$z + \sum_{n=2}^{\infty} \frac{\Gamma(n+2)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} b_n z^n$$

(2.9)

where

$$A_n = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(1+(n-1)\alpha)} a_n, \quad B_n = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)(1+(n-1)\alpha)} b_n. \quad (2.10)$$

Therefore,

$$\Omega^k F = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} B_n z^n = \Omega^k f \ast [t_\alpha(z) + \tilde{f}_\alpha(z)]. \quad (2.11)$$

\[ \square \]

**Theorem 2.2.** A function $\Omega^k F$ of the form (2.4) belongs to $\tilde{P}_H^{1,0}(\alpha)$, if and only if

$$\text{Re} \left\{ z(\Omega^k H(z))^n + \overline{\alpha}(\Omega^k G(z))^n + (\Omega^k H(z))' + (\Omega^k G(z))' \right\} > 0, \quad z \in U. \quad (2.12)$$

**Proof.** If $\Omega^k F = \Omega^k H + \overline{\Omega^k G} \in \tilde{P}_H^{1,0}(\alpha)$, then from Theorem 2.1,

$$a [z(\Omega^k H)' + z(\Omega^k G)'] + (1 - \alpha) [\Omega^k H + \overline{\Omega^k G}] = \Omega^k h + \overline{\Omega^k g} \in \tilde{P}_H^0 \quad (2.13)$$

and $(\Omega^k h)' + (\overline{\Omega^k g})' \in P_H$. Hence

$$0 < \text{Re} \left\{ (\Omega^k h)' + (\overline{\Omega^k g})' \right\}$$

$$\times \text{Re} \left\{ az(\Omega^k H)'' + a(\Omega^k H)' + (1 - a)(\Omega^k H)' + \overline{\alpha z(\Omega^k G)}'' + \overline{\alpha (\Omega^k G)'} + (1 - \overline{\alpha})(\Omega^k G)' \right\}$$

$$\times \text{Re} \left\{ z(a(\Omega^k H)'' + \overline{\alpha (\Omega^k G)'}) + (\Omega^k H)' + (\Omega^k G)' \right\}. \quad (2.14)$$

Conversely, if the function $\Omega^k F = \Omega^k H + \overline{\Omega^k G}$ of the form (2.4) satisfies (2.10), then by Theorem 2.1 $(\Omega^k h)' + (\overline{\Omega^k g})' \in P_H$ and the following function holds:

$$\Omega^k f = \Omega^k h + \overline{\Omega^k g} = a [z(\Omega^k H)' + z(\Omega^k G)'] + (1 - \alpha) [\Omega^k H + \overline{\Omega^k G}] \in \tilde{P}_H^0. \quad (2.15)$$

Then by Theorem 2.1, $\Omega^k F = \Omega^k H + \overline{\Omega^k G} \in \tilde{P}_H^{1,0}(\alpha). \quad \square$
Theorem 2.3. $\tilde{P}^{1,0}_H(\alpha)$ is convex and compact.

Proof. Let $\Omega^1 F_1 = \Omega^1 H_1 + \Omega^1 G_1$, $\Omega^1 F_2 = \Omega^1 H_2 + \Omega^1 G_2 \in \tilde{P}^{1,0}_H(\alpha)$ and let $\mu \in [0,1]$. Then

$$\text{Re}\{z[\alpha(\Omega^1 H_1(z))'' + (1 - \mu)(\Omega^1 H_2(z))''] + \overline{\alpha}(\Omega^1 G_1(z))'' + (1 - \mu)(\Omega^1 G_2(z))'']\}
+ \mu[(\Omega^1 H_1(z))' + (\Omega^1 G_1(z))'] + (1 - \mu)[(\Omega^1 H_2(z))' + (\Omega^1 G_2(z))']\}
= \mu \text{Re}\{z[\alpha(\Omega^1 H_1(z))' + \overline{\alpha}(\Omega^1 G_1(z))] + (\Omega^1 H_2(z))' + (\Omega^1 G_2(z))'\}
+ (1 - \mu)\text{Re}\{z[\alpha(\Omega^1 H_2(z))' + \overline{\alpha}(\Omega^1 G_2(z))] + (\Omega^1 H_2(z))' + (\Omega^1 G_2(z))'\}
> 0.$$ 

Hence from Theorem 2.2, $\mu \Omega^1 F_1 + (1 - \mu) \Omega^1 F_2 \in \tilde{P}^{1,0}_H(\alpha)$. Therefore, $\tilde{P}^{1,0}_H(\alpha)$ is convex.

Now, let $\Omega^1 F_n = \Omega^1 H_n + \Omega^1 G_n \in \tilde{P}^{1,0}_H(\alpha)$ and let $\Omega^1 F_n \to \Omega^1 F = \Omega^1 H + \Omega^1 G$. By Theorem 2.2,

$$\alpha[z(\Omega^1 H_n)' + \overline{z(\Omega^1 G_n)}] + (1 - \alpha)[\Omega^1 H_n + \Omega^1 G_n] \in \tilde{P}^0_H.$$ 

(2.17)

Since $\tilde{P}^0_H$ is compact, see [6],

$$\alpha[z(\Omega^1 H)' + \overline{z(\Omega^1 G)}] + (1 - \alpha)[\Omega^1 H + \Omega^1 G] \in \tilde{P}^0_H.$$ 

(2.18)

Hence by Theorem 2.1, $\Omega^1 F \in \tilde{P}^{1,0}_H(\alpha)$, therefore $\tilde{P}^{1,0}_H(\alpha)$ is compact.

Theorem 2.4. If $\Omega^1 F = \Omega^1 H + \Omega^1 G \in \tilde{P}^{1,0}_H(\alpha)$ and $|z| = r < 1$, then

$$-r + 2 \ln(1 + r) \leq \text{Re}\{\alpha[z(\Omega^1 H_n)' + \overline{z(\Omega^1 G_n)}] + (1 - \alpha)[\Omega^1 H_n + \Omega^1 G_n]\}
\leq -r - 2 \ln(1 - r).$$

(2.19)

Equality is obtained for the function (2.3) where

$$\Omega^1 f = 2z + \ln(1 - z) - 3z - 3 \ln(1 - z), \quad z \in \mathcal{U}.$$ 

(2.20)

Proof. From Theorem 2.1, if $\Omega^1 H + \Omega^1 G \in \tilde{P}^{1,0}_H(\alpha)$, then there exists $\Omega^1 f = \Omega^1 h + \Omega^1 g \in \tilde{P}^0_H$ so that

$$\alpha[z(\Omega^1 H)' + \overline{z(\Omega^1 G)}] + (1 - \alpha)[\Omega^1 H + \Omega^1 G] = \Omega^1 f.$$ 

(2.21)

Since by [5, Proposition 2.2]

$$-r + 2 \ln(1 + r) \leq \text{Re}(\Omega^1 f) \leq -r - 2 \ln(1 - r),$$

(2.22)

this completes the proof. □
Theorem 2.5. If $\Omega^1 F = \Omega^1 H + \Omega^1 G \in \tilde{P}_H^{1,0}(\alpha)$ and Re $\alpha > 0$, then there exists $\Omega^1 f \in \tilde{P}_H^0$ so that

$$\Omega^1 F = \frac{1}{\alpha} \int_0^1 \xi^{1/\alpha - 2} (\Omega^1 f)(z\xi) d\xi, \quad z \in U. \quad (2.23)$$

Proof. Since

$$t_\alpha(z) = \frac{1}{\alpha} \int_0^1 \xi^{1/\alpha - 1} \frac{z}{1 - z\xi} d\xi, \quad |\xi| \leq 1, \text{ Re } \alpha > 0, \quad (2.24)$$

and for $\Omega^1 f = \Omega^1 h + \Omega^1 g \in \tilde{P}_H^0$,

$$(\Omega^1 h) \ast \frac{z}{1 - z\xi} = \frac{(\Omega^1 h)(z\xi)}{\xi}, \quad (\Omega^1 g) \ast \frac{z}{1 - z\xi} = \frac{(\Omega^1 g)(z\xi)}{\xi}, \quad (2.25)$$

we have

$$(\Omega^1 H)(z) = (\Omega^1 h)(z) \ast t_\alpha = \frac{1}{\alpha} \int_0^1 \xi^{1/\alpha - 2} (\Omega^1 h)(z\xi) d\xi, \quad (2.26)$$

$$(\Omega^1 G)(z) = (\Omega^1 g)(z) \ast t_\alpha = \frac{1}{\alpha} \int_0^1 \xi^{1/\alpha - 2} (\Omega^1 g)(z\xi) d\xi.$$ 

Hence $\Omega^1 F$ is type (2.23). □

Theorem 2.6. If Re $\alpha > 0$, then $\tilde{P}_H^{1,0}(\alpha) \subset \tilde{P}_H^0$.

Proof. Let $\Omega^1 F \in \tilde{P}_H^{1,0}(\alpha)$ and Re $\alpha > 0$. Then there exists $\Omega^1 f \in \tilde{P}_H^0$ so that

$$\Omega^1 F = \Omega^1 H + \Omega^1 G = \Omega^1 f * (t_\alpha + \overline{t_\alpha}) = (\Omega^1 h * t_\alpha) + (\Omega^1 g * \overline{t_\alpha}). \quad (2.27)$$

Hence $0 < \text{Re} \{ (\Omega^1 h)' + (\Omega^1 g) \} = \text{Re} \{ (\Omega^1 h)' + (\Omega^1 g)' \}$ and since Re $\alpha > 0$, Re $\{ (\Omega^1 H)' + (\Omega^1 G) \} > 0$ and $\Omega^1 H(0) = 0$, $(\Omega^1 H)'(0) = 1$, $\Omega^1 G(0) = 0$, $(\Omega^1 G)'(0) = 0$. And hence $\Omega^1 F \in \tilde{P}_H^0$. □

Theorem 2.7. Let $\Omega^1 F = \Omega^1 H + \Omega^1 G \in \tilde{P}_H^{1,0}(\alpha)$. Then

(i)

$$\|A_n - B_n\| \leq \frac{2\Gamma(n+1)\Gamma(2-\lambda)}{n\Gamma(n+1-\lambda)(1+(n-1)\alpha)}, \quad n \geq 1, \quad (2.28)$$
If $\Omega^1 F$ is sense-preserving, then

$$\begin{align*}
|A_n| &\leq \frac{2n-1}{n} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)|1+(n-1)\alpha|}, \quad n = 1, 2, \ldots, \\
|B_n| &\leq \frac{2n-3}{n} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)|1+(n-1)\alpha|}, \quad n = 2, 3, \ldots.
\end{align*}$$  \hspace{1cm} (2.29)

Proof. By (2.10),

$$\|A_n| - |B_n\| = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)|1+(n-1)\alpha|} \|a_n| - |b_n\|. \hspace{1cm} (2.30)$$

Also, by [6, Theorem 2.3], we have

$$\|a_n| - |b_n\| \leq \frac{2}{n}. \hspace{1cm} (2.31)$$

The required results are obtained.

On the other hand, from (2.10), it is known [6, Corollary 2.5] that

$$|a_n| \leq \frac{2n-1}{n}, \quad |b_n| \leq \frac{2n-3}{n}. \hspace{1cm} (2.32)$$

Then we get the coefficient inequalities for $\tilde{P}^{1,0}_{0}(a)$.

Remark 2.8. Taking $\lambda = 0$ in Theorems 2.1–2.7, we get the similar results in [7].

Theorem 2.9. Let $\Omega^1 F = \Omega^1 H = \overline{\Omega^1 G} \in \tilde{P}^{1,0}_{H}(a)$ and sense-preserving in $\mathcal{U}$, then for $|z| = r < 1$,

$$\begin{align*}
|az(\Omega^1 H)' + (1 - a)\Omega^1 H| &\leq \frac{2r}{1-r} + \ln(1-r), \\
|\overline{\alpha}z(\Omega^1 G)' + (1 - \overline{\alpha})\Omega^1 G| &\leq \frac{3r - r^2}{1-r} + 3\ln(1-r).
\end{align*} \hspace{1cm} (2.33)$$

Proof. From Theorems 2.1 and 2.2, if $\Omega^1 F = \Omega^1 H + \overline{\Omega^1 G} \in \tilde{P}^{1,0}_{H}(a)$, then there exists $\Omega^1 f = \Omega^1 h + \overline{\Omega^1 g} \in \tilde{P}^{0}_{H}$ such that

$$\begin{align*}
ax(\Omega^1 H)' + (1 - a)\Omega^1 H &= \Omega^1 h, \\
\overline{ax}(\Omega^1 G)' + (1 - \overline{a})\Omega^1 G &= \Omega^1 g.
\end{align*} \hspace{1cm} (2.34)$$

By [6, Theorem 3.5], we obtain the results.
Remark 2.10. Taking \( \lambda = 0 \) and \( \alpha = 0 \) in Theorem 2.9, we get [6, Theorem 2.4].

3. Positive order

We say that the harmonic function \( f = h + \overline{g} \) of the form (2.1) is in the class \( P_H(\beta) \), \( 0 \leq \beta < 1 \) for \( |z| = r \) if \( \text{Re} \ f > \beta \) and \( f(0) = 1 \).

If the function \( f_z + \overline{f_z} = h' + \overline{g'} \) belongs to \( P_H(\beta) \) for the analytic and normalized functions \( h \) and \( g \) of the form (2.1), then the class of functions \( f = h + \overline{g} \) is denoted by \( \hat{P}_H^0(\beta) \).

Denote by \( \hat{P}_H^{1,0}(\beta, \alpha) \) the class of functions defined by (2.3) where \( \Omega^1 f \in \hat{P}_H^0(\beta) \).

Many of our results can be rewritten for functions in the class \( \hat{P}_H^{1,0}(\beta, \alpha) \). For instance, see the following theorems.

Theorem 3.1. If \( \Omega^1 F \in \hat{P}_H^{1,0}(\beta, \alpha) \), then there exists \( \Omega^1 f \in \hat{P}_H^0(\beta) \) so that

\[
\alpha [z(\Omega^1 F)_z(z) + \overline{z}(\Omega^1 F)_{\overline{z}}(z)] + (1 - \alpha)\Omega^1 F(z) = \Omega^1 f(z). \tag{3.1}
\]

Conversely, for any function \( f \) such that \( \Omega^1 f \in \hat{P}_H^0(\beta) \), there exists \( \Omega^1 F \in \hat{P}_H^{1,0}(\beta, \alpha) \) satisfying (3.1).

Theorem 3.2. A function \( \Omega^1 F \) belongs to \( \hat{P}_H^{1,0}(\beta, \alpha) \) if and only if

\[
\text{Re} \{z(\Omega^1 H(z))^n + \overline{z}(\Omega^1 G(z))^{n} + (\Omega^1 H(z))' + (\Omega^1 G(z))' \} > \beta, \quad z \in U. \tag{3.2}
\]

Theorem 3.3. If \( \Omega^1 F \in \hat{P}_H^{1,0}(\beta, \alpha) \) and \( \text{Re} \ \alpha > 0 \), then there exists \( \Omega^1 f \in \hat{P}_H^0(\beta) \) so that

\[
\Omega^1 F = \frac{1}{\alpha} \int_0^1 \xi^{1/2 - 2}(\Omega^1 f)(z \xi) d\xi, \quad z \in U. \tag{3.3}
\]

Theorem 3.4. If \( \text{Re} \ \alpha > 0 \), then \( \hat{P}_H^{1,0}(\beta, \alpha) \subset \hat{P}_H^0(\beta) \).

References

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