Research Article

Characterization for the Convergence of Krasnoselskij Iteration for Non-Lipschitzian Operators

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We establish the convergence of Krasnoselskij iteration for various classes of non-Lipschitzian operators.

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1. Introduction

Let $X$ be a real Banach space; $B$ a nonempty, convex subset of $X$; and $T : B \to B$ an operator. Let $x_0 \in B$. The following iteration is known as Krasnoselskij iteration (see [1]):

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n.$$  \hspace{1cm} (1.1)

The map $J : X \to 2^{X^*}$ given by $Jx := \{ f \in X^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\| \}$, for all $x \in X$, is called the normalized duality mapping. It is easy to see that we have

$$\langle y, j(x) \rangle \leq \|x\|\|y\|, \quad \forall x, y \in X, \: \forall j(x) \in J(x).$$  \hspace{1cm} (1.2)

Denote

$$\Psi := \{ \varphi : [0, +\infty) \to [0, +\infty) \text{ is a strictly increasing map with } \varphi(0) = 0 \}.$$  \hspace{1cm} (1.3)

**Definition 1.1.** Let $X$ be a real Banach space, and let $B$ be a nonempty subset of $X$. A map $T : B \to B$ is called uniformly pseudocontractive if there exists a map $\varphi \in \Psi$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x - y\|^2 - \varphi(\|x - y\|), \quad \forall x, y \in B.$$  \hspace{1cm} (1.4)
A map $S : X \to X$ is called uniformly accretive if there exists a map $\varphi \in \Psi$ and $j(x-y) \in \mathcal{J}(x-y)$ such that
\begin{equation}
\langle Sx - Sy, j(x-y) \rangle \geq \varphi(\|x-y\|), \quad \forall x, y \in X.
\end{equation}

Taking $\varphi(a) := \varphi(a) \cdot a$, for all $a \in [0, +\infty)$, $(\varphi \in \Psi)$, reduces to the usual definitions of $\varphi$-strongly pseudocontractive and $\varphi$-strongly accretive. Taking $\varphi(a) := \gamma \cdot a^2$, $\gamma \in (0, 1)$, for all $a \in [0, +\infty)$, $(\varphi \in \Psi)$, we get the usual definitions of strongly pseudocontractive and strongly accretive. Therefore, the class of strongly pseudocontractive maps is included strictly in the class of $\varphi$-strongly pseudocontractive maps. The example from [2] shows that this inclusion is proper. Remark, further, that the class of $\varphi$-strongly pseudocontractive maps is also included strictly in the class of uniformly pseudocontractive maps (see also [3]).

We will give a characterization for the convergence of (1.1) when applied to uniformly pseudocontractive operators. For this purpose, we need the following lemma similar to [4, Lemma 1]. Next, $\mathbb{N}$ denotes the set of all natural numbers.

**Lemma 1.2.** Let $\{a_n\}$ be a positive bounded sequence and assume that there exists $n_0 \in \mathbb{N}$ such that
\begin{equation}
a_{n+1} \leq (1 - \lambda)a_n + \lambda a_{n+1} - \lambda \frac{\varphi(a_{n+1})}{a_{n+1}} + \lambda \epsilon_n, \quad \forall n \geq n_0,
\end{equation}
where $\lambda \in (0, 1)$, $\epsilon_n \geq 0$, for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \epsilon_n = 0$. Then $\lim_{n \to \infty} a_n = 0$.

**Proof.** There exists an $M > 0$ such that $a_n \leq M$, for all $n \in \mathbb{N}$. Denote $a := \lim \inf a_n$. We will prove that $a = 0$. Suppose on the contrary that $a > 0$. Then there exists an $N_1 \in \mathbb{N}$ such that
\begin{equation}
a_n \geq \frac{a}{2}, \quad \forall n \geq N_1.
\end{equation}

From $\lim_{n \to \infty} \epsilon_n = 0$, we know that there exists an $N_2 \in \mathbb{N}$ such that
\begin{equation}
\epsilon_n \leq \frac{\varphi(a/2)}{2M}, \quad \forall n \geq N_2.
\end{equation}

Set $N_0 := \max\{N_1, N_2\}$. Using the fact that $-(1/M) \geq -(1/a_{n+1})$, we get the following:
\begin{equation}
a_{n+1} \leq (1 - \lambda)a_n + \lambda a_{n+1} - \lambda \frac{\varphi(a_{n+1})}{a_{n+1}} + \lambda \epsilon_n
\leq (1 - \lambda)a_n + \lambda a_{n+1} - \lambda \frac{\varphi(a/2)}{M} + \lambda \frac{\varphi(a/2)}{2M}
\leq (1 - \lambda)a_n + \lambda a_{n+1} - \lambda \frac{\varphi(a/2)}{2M},
\end{equation}
which implies that $(1 - \lambda)a_{n+1} \leq (1 - \lambda)a_n - \lambda((\varphi(a/2))/2M)$, or
\begin{equation}
a_{n+1} \leq a_n - \frac{\lambda}{1 - \lambda} \frac{\varphi(a/2)}{2M} \leq a_n - \frac{\lambda \varphi(a/2)}{2M},
\end{equation}

$\Box$
Theorem 2.1. Let X be a real Banach space, B a nonempty, closed, convex, bounded subset of X. Let T : B → B be a uniformly pseudocontractive and uniformly continuous operator with F(T) ≠ ∅. Then for x₀ ∈ B, the Krasnoselskij iteration (1.1) converges to the fixed point of T if and only if limₙ→∞‖xₙ₊₁ − xₙ‖ = 0.

Proof. Since T is a self-map of B, which is bounded and convex, then, from (1.1), each xₙ ∈ B, so {xₙ} is bounded for each n ∈ ℕ. Uniqueness of the fixed point follows from (1.4). If {xₙ} converges to the fixed point of T, that is, limₙ→∞xₙ = x*, then, obviously, limₙ→∞‖xₙ₊₁ − xₙ‖ = 0. Conversely, we will prove that if limₙ→∞‖xₙ₊₁ − xₙ‖ = 0, then limₙ→∞xₙ = x*. Suppose that
\[ x_n = x^* \] for some \( n \in \mathbb{N} \). Then from (1.1), it follows that \( x_m = x^* \) for each \( m > n \), and the theorem is proved. Now suppose that \( x_n \neq x^* \) for each \( n \in \mathbb{N} \). Using (1.1) and (1.2),

\[
\begin{align*}
\|x_{n+1} - x^*\|^2 &= \langle x_{n+1} - x^*, f(x_{n+1} - x^*) \rangle \\
&= \langle (1 - \lambda)(x_n - x^*) + \lambda(Tx_n - Tx^*), f(x_{n+1} - x^*) \rangle \\
&= (1 - \lambda)\langle (x_n - x^*), f(x_{n+1} - x^*) \rangle + \lambda \langle Tx_n - Tx^*, f(x_{n+1} - x^*) \rangle \\
&\leq (1 - \lambda)\|x_n - x^*\|\|x_{n+1} - x^*\| + \lambda \langle Tx_n - Tx^*, f(x_{n+1} - x^*) \rangle + \lambda \langle Tx_n - Tx_{n+1}, f(x_{n+1} - x^*) \rangle \\
&\leq (1 - \lambda)\|x_n - x^*\|\|x_{n+1} - x^*\| + \lambda\|x_{n+1} - x^*\|^2 - \lambda \varphi(\|x_{n+1} - x^*\|) + \lambda \|Tx_n - Tx_{n+1}\|\|x_{n+1} - x^*\| \\
&\leq \|x_{n+1} - x^*\| \left( (1 - \lambda)\|x_n - x^*\| + \lambda\|x_{n+1} - x^*\| - \lambda \frac{\varphi(\|x_{n+1} - x^*\|)}{\|x_{n+1} - x^*\|} + \lambda \|Tx_n - Tx_{n+1}\| \right).
\end{align*}
\]  

(2.1)

Hence

\[
\|x_{n+1} - x^*\| \leq (1 - \lambda)\|x_n - x^*\| + \lambda\|x_{n+1} - x^*\| - \lambda \frac{\varphi(\|x_{n+1} - x^*\|)}{\|x_{n+1} - x^*\|} + \lambda \|Tx_n - Tx_{n+1}\|.
\]  

(2.2)

Since \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \) and \( T \) is uniformly continuous, it follows that

\[
\lim_{n \to \infty} \|Tx_n - Tx_{n+1}\| = 0.
\]  

(2.3)

Set \( a_n = \|x_n - x^*\|, \ v_n = \|Tx_n - Tx_{n+1}\| \) and use Lemma 1.2 to obtain the conclusion.

**Remark 2.2.** (1) If \( B \) is not bounded, then Theorem 2.1 holds under the assumption that \( \{x_n\} \) is bounded.

(2) If \( \{x_n\} \) is bounded.

(3) If \( T \) is strongly pseudocontractive, then automatically \( F(T) \neq \emptyset \).

**3. Further results**

Let \( I \) denote the identity map. A map \( T : B \to B \) is called pseudocontractive if there exists \( j(x - y) \in J(x - y) \) such that \( \langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 \).

**Remark 3.1.** The operator \( T \) is a (uniformly, strongly) pseudocontractive map if and only if \( (I - T) \) is a (uniformly, strongly) accretive map.

**Remark 3.2.** (1) Let \( T, S : X \to X \), and let \( f \in X \) be given. A fixed point for the map \( Tx = f + (I - S)x \), for all \( x \in X \), is a solution for \( Sx = f \).

(2) Let \( f \in X \) be a given point. If \( S \) is an accretive map, then \( T = f - S \) is a strongly pseudocontractive map.
Consider Krasnoselskij iteration with $Tx = f + (I - S)x$

$$x_{n+1} = (1 - \lambda)x_n + \lambda(f + (I - S)x_n). \quad (3.1)$$

Remarks 3.1 and 3.2 and Theorem 2.1 lead to the following result.

**Corollary 3.3.** Let $X$ be a real Banach space and let $S : X \rightarrow X$ be a uniformly accretive and uniformly continuous operator, with $(I - S)(X)$ bounded. Suppose that $Sx = f$ has a solution. Then for any $x_0 \in X$, the Krasnoselskij iteration (3.1) converges to the solution of $Sx = f$ if and only if $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

Let $S$ be an accretive operator. The operator $Tx = f - Sx$ is strongly pseudocontractive for a given $f \in X$. A solution for $Tx = x$ becomes a solution for $x + Sx = f$. Consider Krasnoselskij iteration with $Tx := f - Sx$,

$$x_{n+1} = (1 - \lambda)x_n + \lambda(f - Sx_n). \quad (3.2)$$

Again, using Remarks 3.1 and 3.2 and Theorem 2.1, we obtain the following result.

**Corollary 3.4.** Let $X$ be a real Banach space and let $S : X \rightarrow X$ be an accretive and uniformly continuous operator, with $(I - S)(X)$ bounded. Suppose that $x + Sx = f$ has a solution. Then for $x_0 \in X$, the Krasnoselskij iteration (3.2) converges to the solution of $x + Sx = f$ if and only if $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$.

**Remark 3.5.** If (1.4) holds for all $x \in B$ and $y := x^* \in F(T)$, then such a map is called uniformly hemicontractive. It is trivial to see that our results hold for the uniformly hemicontractive maps.

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**References**


