Research Article

Subordination Properties for Certain Analytic Functions

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Received 13 September 2007; Accepted 7 November 2007

Recommended by Brigitte Forster-Heinlein

The purpose of the present paper is to derive a subordination result for functions in the class $H^*_n(\alpha, \lambda, b)$ of normalized analytic functions in the open unit disk $U$. A number of interesting applications of the subordination result are also considered.

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1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disc $\mathbb{D} = \{ z : |z| < 1 \}$. We also denote by $K$ the class of functions $f \in A$ that are convex in $\mathbb{D}$.

Given two functions $f, g \in A$, where $f$ is given by (1.1) and $g$ is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the Hadamard product (or convolution) $f*g$ is defined by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad (z \in \mathbb{D}).$$
By using the Hadamard product, Ruscheweyh [1] defined

\[ D^n f(z) = \frac{z}{(1 - z)^{\alpha+1}} f(z) \quad (\alpha \geq -1). \tag{1.4} \]

From the definition of (1.4), we observe that

\[ D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!} , \tag{1.5} \]

when \( n = \alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\} \). The symbol \( D^n f(z) (n \in \mathbb{N}_0) \) was called the \( n \)th-order Ruscheweyh derivative of \( f \) by Al-Amiri [2]. We also note that \( D^0 f(z) = f(z) \) and \( D^1 f(z) = zf'(z) \).

**Definition 1.1.** Suppose that \( f \in A \). Then the function \( f \) is said to be a member of the class \( H_n(a, \lambda, b) \) if it satisfies

\[
\left| \frac{\lambda(n+1)(D^{n+1}f(z)/z) + [1 - \lambda(n+1)] (D^n f(z)/z) - 1}{\lambda(n+1)(D^{n+1}f(z)/z) + [1 - \lambda(n+1)] (D^n f(z)/z) + 2b(1-\alpha) - 1} \right| < 1 \tag{1.6}
\]

\((z \in U; \ 0 \leq \alpha < 1; \ \lambda \geq 0; \ b \in \mathbb{C} \setminus \{0\}; \ n \in \mathbb{N}_0)\).

By specializing \( \alpha, \lambda, b, \) and \( n \), one can obtain various subclasses studied by many authors (see, e.g., [3–11]).

**Definition 1.2.** Let \( g \) be analytic and univalent in \( U \). If \( f \) is analytic in \( U \), \( f(0) = g(0) \), and \( f(U) \subset g(U) \), then one says that \( f \) is subordinate to \( g \) in \( U \), and we write \( f < g \) or \( f(z) < g(z) \). One also says that \( g \) is superordinate to \( f \) in \( U \).

**Definition 1.3.** An infinite sequence \( \{b_k\}_{k=1}^\infty \) of complex numbers will be called a subordinating factor sequence if for every univalent function \( f \) in \( K \), one has

\[
\sum_{k=1}^\infty b_k a_k z^k < f(z) \quad (z \in U; \ a_1 = 1). \tag{1.7}
\]

**Lemma 1.4** (see [12]). The sequence \( \{b_k\}_{k=1}^\infty \) is a subordinating factor sequence if and only if

\[
\text{Re} \left\{ 1 + 2 \sum_{k=1}^\infty b_k z^k \right\} > 0 \quad (z \in U). \tag{1.8}
\]

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class \( H_n(a, \lambda, b) \).
Lemma 1.5. If the function $f$ which is defined by (1.1) satisfies the following condition:

$$
\sum_{k=2}^{\infty} \left[ 1 + \lambda(k-1) \right] C(n,k) |a_k| (1-\alpha)b \leq (1-\alpha) |b| \quad (0 \leq \alpha < 1; \lambda \geq 0; \ b \in \mathbb{C} \setminus \{0\}; \ n \in \mathbb{N}_0),
$$

(1.9)

where

$$
C(n,k) = \prod_{j=2}^{k} \frac{(j+n-1)}{(k-1)!} \quad (k = 2, 3, \ldots),
$$

(1.10)

then $f \in H_n(\alpha, \lambda, b)$.

Proof. Suppose that the inequality (1.9) holds. Using the identity

$$
z(D^n f(z))' = (n+1)D^{n+1} f(z) - nD^n f(z),
$$

(1.11)

we have for $z \in \mathbb{U},$

$$
\left| (1-\lambda) \frac{D^n f(z)}{z} + \lambda \frac{(D^n f(z))'}{z} - 1 \right| - \left| 2b(1-\alpha) + (1-\lambda) \frac{D^n f(z)}{z} + \lambda \frac{(D^n f(z))'}{z} - 1 \right|
$$

$$
= \left| \sum_{k=2}^{\infty} \left[ 1 + \lambda(k-1) \right] C(n,k) |a_k| z^{k-1} \right| - \left| 2b(1-\alpha) + \sum_{k=2}^{\infty} \left[ 1 + \lambda(k-1) \right] C(n,k) a_k z^{k-1} \right|
$$

$$
\leq \sum_{k=2}^{\infty} \left[ 1 + \lambda(k-1) \right] C(n,k) |a_k| |z|^{k-1}
$$

$$
- \left\{ 2|b|(1-\alpha) - \sum_{k=2}^{\infty} \left[ 1 + \lambda(k-1) \right] C(n,k) |a_k| |z|^{k-1} \right\}
$$

$$
\leq 2 \left\{ \sum_{k=2}^{\infty} \left[ 1 + \lambda(k-1) \right] C(n,k) |a_k| - |b|(1-\alpha) \right\} \leq 0,
$$

(1.12)

which shows that $f$ belongs to $H_n(\alpha, \lambda, b)$. $\square$

Let $H_n^*(\alpha, \lambda, b)$ denote the class of functions $f$ in $A$ whose Taylor-Maclaurin coefficients $a_k$ satisfy the condition (1.9).

We note that

$$
H_n^*(\alpha, \lambda, b) \subseteq H_n(\alpha, \lambda, b).
$$

(1.13)

Example 1.6. (i) For $0 \leq \alpha < 1, \lambda > 0, \ b \in \mathbb{C} \setminus \{0\}, \text{and} \ n \in \mathbb{N}_0$, the following function defined by:

$$
f_0(z) = z + \frac{2b(1-\alpha)}{(n+1)(\lambda+1)} z^2 \mathcal{F}_2 \left( \frac{1}{2}, 1 + \frac{1}{\lambda}; 2 + \frac{1}{\lambda}, n+2; z \right) \quad (z \in \mathbb{U}),
$$

(1.14)

is in the class $H_n(\alpha, \lambda, b)$.

(ii) For $0 \leq \alpha < 1, \lambda > 0, \ b \in \mathbb{C} \setminus \{0\}, \text{and} \ n \in \mathbb{N}_0$, the following functions defined by:

$$
f_1(z) = z + \frac{(1-\alpha)b}{(\lambda+1)(n+1)} z^2 \quad (z \in \mathbb{U}),
$$

$$
f_2(z) = z + \frac{(1-\alpha)b}{(2\lambda+1)(n+1)(n+2)} z^3 \quad (z \in \mathbb{U}),
$$

$$
f_3(z) = z + \frac{1}{(\lambda+1)(n+1)} z^2 + \frac{2[(1-\alpha)b - 1]}{(2\lambda+1)(n+1)(n+2)} z^3 \quad (z \in \mathbb{U})
$$

(1.15)

are in the class $H_n^*(\alpha, \lambda, b)$.
In this paper, we obtain a sharp subordination result associated with the class $H_n^*(a, \lambda, b)$ by using the same techniques as in [13] (see also [14–16]). Some applications of the main result which give important results of analytic functions are also investigated.

2. Main theorem

Theorem 2.1. Let $f \in H_n^*(a, \lambda, b)$. Then

$$\frac{(\lambda + 1)(n + 1)}{2[(\lambda + 1)(n + 1) + |b|(1 - \alpha)]} (f*g)(z) < g(z) \quad (z \in \mathbb{U}) \quad (2.1)$$

for every function $g$ in $K$, and

$$\Re f(z) > -\frac{(\lambda + 1)(n + 1) + |b|(1 - \alpha)}{(\lambda + 1)(n + 1)} \quad (2.2)$$

The constant $(\lambda + 1)(n + 1)/2[(\lambda + 1)(n + 1) + |b|(1 - \alpha)]$ cannot be replaced by a larger one.

Proof. Let $f \in H_n^*(a, \lambda, b)$ and let

$$g(z) = z + \sum_{k=2}^{\infty} c_k z^k \quad (2.3)$$

be any function in the class $K$. Then we readily have

$$\frac{(\lambda + 1)(n + 1)}{2[(\lambda + 1)(n + 1) + |b|(1 - \alpha)]} (f*g)(z) = \frac{(\lambda + 1)(n + 1)}{2[(\lambda + 1)(n + 1) + |b|(1 - \alpha)]} \left( z + \sum_{k=2}^{\infty} a_k c_k z^k \right) \quad (2.4)$$

Thus, by Definition 1.2, the subordination result (2.1) will hold true if the sequence

$$\left\{ \frac{(\lambda + 1)(n + 1) a_k}{2[(\lambda + 1)(n + 1) + |b|(1 - \alpha)]} \right\}_{k=1}^{\infty} \quad (2.5)$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.4, this is equivalent to the following inequality:

$$\Re \left\{ 1 + \sum_{k=1}^{\infty} \frac{(\lambda + 1)(n + 1)}{[(\lambda + 1)(n + 1) + |b|(1 - \alpha)]} a_k z^k \right\} > 0 \quad (z \in \mathbb{U}) \quad (2.6)$$

Now, since

$$[1 + \lambda(k - 1)] C(n, k) \quad (\lambda \geq 0, \ n \in \mathbb{N}_0) \quad (2.7)$$
Theorem 2.1. The inequality

\[ \text{Re} \left\{ 1 + \sum_{k=1}^{\infty} \frac{(\lambda + 1)(n + 1)}{[(\lambda + 1)(n + 1) + |b|(1 - \alpha)]} a_k z^k \right\} \]

Taking

\[ > 1 - \frac{1}{[(\lambda + 1)(n + 1) + |b|(1 - \alpha)]^r} \]

This proves the inequality (2.6), and hence also the subordination result (2.1) asserted by Theorem 2.1. The inequality (2.2) follows from (2.1) by taking

\[ g(z) = \frac{z}{1 - z} \in K. \] (2.9)

Next, we consider the function

\[ f_1(z) = z - \frac{|b|(1 - \alpha)}{(\lambda + 1)(n + 1)} z^2 \quad (0 \leq \alpha < 1; \lambda \geq 0; b \in \mathbb{C} \setminus \{0\}; n \in \mathbb{N}_0) \]

which is a member of the class \( H_n^*(\alpha, \lambda, b) \). Then by using (2.1), we have

\[ \frac{(\lambda + 1)(n + 1)}{2[(\lambda + 1)(n + 1) + |b|(1 - \alpha)]} f_1(z) < \frac{z}{1 - z} \quad (z \in U). \]

It can be easily verified for the function \( f_1(z) \) defined by (2.10) that

\[ \inf_{z \in U} \left\{ \text{Re} \left( \frac{(\lambda + 1)(n + 1)}{2[(\lambda + 1)(n + 1) + |b|(1 - \alpha)]} f_1(z) \right) \right\} = -\frac{1}{2} \quad (z \in U) \] (2.12)

which completes the proof of Theorem 2.1.

3. Some applications

Taking \( n = 0 \) in Theorem 2.1, we obtain the following.

**Corollary 3.1.** If the function \( f \) defined by (1.1) satisfies

\[ \sum_{k=2}^{\infty} \left| \frac{1 + \lambda(k - 1)}{2[(\lambda + 1)(n + 1) + |b|(1 - \alpha)]} a_k \right| \leq m \quad (\lambda \geq 0, m > 0) \] (3.1)
then for every function $g$ in $K$, one has
\[
\frac{\lambda + 1}{2(\lambda + m + 1)} (f * g)(z) < g(z), \quad (z \in \mathbb{U}),
\]
\[
\Re f(z) > -\left(1 + \frac{m}{\lambda + 1}\right).
\]  
(3.2)

The constant $(\lambda + 1)/2(\lambda + m + 1)$ cannot be replaced by larger one.

Putting $\lambda = 0$ in Theorem 2.1, we have the following corollary.

**Corollary 3.2.** If the function $f$ defined by (1.1) satisfies
\[
\sum_{k=2}^{\infty} C(n,k) |a_k| \leq m, \quad m > 0, \quad n \in \mathbb{N}_0,
\]  
(3.3)

where $C(n,k)$ is defined by (1.10), then for every function $g$ in $K$, one has
\[
\frac{(n + 1)}{2(n + m + 1)} (f * g)(z) < g(z), \quad (z \in \mathbb{U}),
\]
\[
\Re f(z) > -\left(1 + \frac{m}{n + 1}\right).
\]  
(3.4)

The constant $(n + 1)/2(n + m + 1)$ cannot be replaced by larger one.

Next, letting $\lambda = 1$ and $n = 0$, in Theorem 2.1, we obtain the following corollary.

**Corollary 3.3.** If the function $f$ satisfies
\[
\sum_{k=2}^{\infty} k |a_k| \leq m, \quad (m > 0),
\]  
(3.5)

then for every function $g$ in $K$, one has
\[
\frac{1}{(m + 2)} (f * g)(z) < g(z), \quad (z \in \mathbb{U}),
\]
\[
\Re f(z) > -\left(1 + \frac{m}{2}\right).
\]  
(3.6)

The constant $1/(m + 2)$ cannot be replaced by a larger one.

**Remark 3.4.** Putting $\lambda = 1$, $m = 1$, and $n = 0$, in Theorem 2.1, we obtain the result due to Singh [17].

Also, by taking $\lambda = 0$ and $n = 0$, in Theorem 2.1, we have the following.

**Corollary 3.5.** If the function $f$ satisfies
\[
\sum_{k=2}^{\infty} |a_k| \leq m, \quad (m > 0),
\]  
(3.7)
then for every function \( g \) in \( K \), one has
\[
\frac{1}{2(m+1)}(f*g)(z) < g(z) \quad (z \in \mathbb{U}),
\]
\[
\text{Re } f(z) > -(1 + m).
\]

The constant \( 1/2(m + 1) \) cannot be replaced by a larger one.

It is clearly from the proof of Theorem 2.1 that the function \( f(z) = z - mz^2 \) \((m > 0, \ z \in \mathbb{U})\) is the extremal function of Corollary 3.5. Also, the following example gives a nonpolynomial extremal function for the same corollary.

Example 3.6. Let the function \( h \) be defined by
\[
h(z) = \frac{(m+1)z}{(m+1)+mz} \quad (m > 0, \ z \in \mathbb{U}),
\]
the above function is analytic in \( \mathbb{U} \) and it is equivalent to
\[
h(z) = z + \sum_{k=2}^{\infty} \left( \frac{-m}{m+1} \right)^{k-1} z^k.
\]

Then we have
\[
\sum_{k=2}^{\infty} \left| \left( \frac{-m}{m+1} \right)^{k-1} \right| = m.
\]

Therefore, the Taylor-Maclaurin coefficients of the function \( h \) satisfy the condition in Corollary 3.5. Moreover, it can be easily verified that
\[
\inf_{z \in \mathbb{U}} \text{Re } h(z) = h(-1) = -(m + 1).
\]

Then, the constant \(-(m + 1)\) cannot be replaced by a larger one. Therefore, the function \( h \) is the extremal function of Corollary 3.5.

References


