Research Article

Strong Convergence Theorem of Implicit Iteration Process for Generalized Asymptotically Nonexpansive Mappings in Hilbert Space

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Let C be a nonempty closed and convex subset of a Hilbert space H , let T and $S : C \rightarrow C$ be two commutative generalized asymptotically nonexpansive mappings. We introduce an implicit iteration process of S and T defined by $x_n = \alpha_n x_0 + (1 - \alpha_n)(2/((n+1)(n+2))) \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n$, and then prove that $\{x_n\}$ converges strongly to a common fixed point of S and T . The results generalize and unify the corresponding results.

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1. Introduction

Let H be a Hilbert space, C a nonempty closed and convex subset of H . A mapping S is said to be generalized asymptotically nonexpansive if

$$\|S^n x - S^n y\| \leq k_n \|x - y\| + r_n \quad (1.1)$$

with $k_n \geq 0$, $r_n \geq 0$, $k_n \rightarrow 1$, $r_n \rightarrow 0$ ($n \rightarrow \infty$), for each $x, y \in C$, $n = 0, 1, 2, \dots$. If $k_n = 1$ and $r_n = 0$, (1.1) reduces to nonexpansive mapping; if $r_n = 0$, (1.1) reduces to asymptotically nonexpansive mapping; if $k_n = 1$, (1.1) reduces to asymptotically nonexpansive-type mapping. So, a generalized asymptotically nonexpansive mapping is much more general than many other mappings.

Browder [1] introduced the following implicit iteration process of a nonexpansive self-mapping for arbitrary $x_0 \in C$:

$$x_n = \alpha_n x_0 + (1 - \alpha_n) T x_n, \quad (1.2)$$

and then proved that $\{x_n\}$ converges strongly to the fixed point of T which is nearest to x_0 .

After that, Baillon [2] proved the first nonlinear ergodic theorem: let T be a nonexpansive self-mapping defined on a nonempty bounded closed and convex set. Then for arbitrary $x \in C$, $\{(1/(n+1))\sum_{i=0}^n T^i x\}$ converges weakly to a fixed point of T . Those results have been extended by several authors (see, e.g., [3–7]).

Recently, Shimizu and Takahashi [8] studied the following implicit iteration process of asymptotically nonexpansive mappings for arbitrary $x_0 \in C$:

$$x_n = \alpha_n x_0 + (1 - \alpha_n) \frac{1}{n+1} \sum_{i=0}^n T^i x_n, \quad (1.3)$$

where $\alpha_n = (b_n - 1)/(b_n - 1 + a)$, $b_n = (1/(n+1))\sum_{i=0}^n (1 + |1 - k_i| + e^{-i})$, $\{k_n\}$ is the asymptotical coefficient of T . And then proved that $\{x_n\}$ converges strongly to the fixed point of T which is nearest to x_0 .

They also studied the following explicit iteration process of two commutative nonexpansive self-mappings in [9]:

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n, \quad (1.4)$$

where $\{\alpha_n\} \subseteq [0, 1]$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$. And then proved that $\{x_n\}$ converges strongly to the common fixed point of S and T which is nearest to x_0 .

In this paper, we prove the strong convergence theorem of implicit iteration process for generalized asymptotically nonexpansive mappings which is defined by

$$x_n = \alpha_n x_0 + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n, \quad (1.5)$$

where $\{\alpha_n\} \subseteq [0, 1]$, $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\limsup_{n \rightarrow \infty} (1 - \alpha_n)(2/((n+1)(n+2))) \sum_{k=0}^n \sum_{i+j=k} k_i t_j < 1$.

It is well known that a Hilbert space H satisfies Opial's condition [5], that is, if a sequence $\{x_n\}$ converges weakly to an element $y \in H$ and $y \neq z$, then

$$\liminf_{n \rightarrow \infty} \|x_n - y\| < \liminf_{n \rightarrow \infty} \|x_n - z\|. \quad (1.6)$$

Throughout this paper, S and T are two commutative generalized asymptotically nonexpansive mappings. We denote by $F(S)$ and $F(T)$ the set of fixed points of S and T , respectively; and suppose that $F(S) \cap F(T) \neq \emptyset$. S and T satisfy the following conditions:

$$\begin{aligned} \|S^n x - S^n y\| &\leq k_n \|x - y\| + r_n, \\ \|T^n x - T^n y\| &\leq t_n \|x - y\| + b_n, \end{aligned} \quad (1.7)$$

where $k_n \geq 0$, $r_n \geq 0$, $t_n \geq 0$, $b_n \geq 0$, and $k_n \rightarrow 1$, $r_n \rightarrow 0$, $t_n \rightarrow 1$, $b_n \rightarrow 0$ ($n \rightarrow \infty$).

2. Auxiliary lemmas

This section collects some lemmas which will be used to prove the main results in the next section.

Lemma 2.1 (see [9]). *Letting $L_n = ((n+1)(n+2))/2$, there holds the identity in a Hilbert space H :*

$$\|y_n - v\|^2 = \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|x_{i,j} - v\|^2 - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|x_{i,j} - y_n\|^2 \quad (2.1)$$

for $\{x_{i,j}\}_{i,j=0}^\infty \subseteq H$, $y_n = (1/L_n) \sum_{k=0}^n \sum_{i+j=k} x_{i,j} \in H$, and $v \in H$.

Lemma 2.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space H , S and T two generalized asymptotically nonexpansive mappings of C into itself such that $ST = TS$. For any $x \in C$, put $F_n(x) = (2/((n+1)(n+2))) \sum_{k=0}^n \sum_{i+j=k} S^i T^j x$. Then*

$$\begin{aligned} \lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \|F_n(x) - S^l F_n(x)\| &= 0, \\ \lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \|F_n(x) - T^l F_n(x)\| &= 0. \end{aligned} \quad (2.2)$$

Proof. Put $x_{i,j} = S^i T^j x$, $v = S^l F_n(x)$ and $L_n = ((n+1)(n+2))/2$. It follows from Lemma 2.1 that

$$\begin{aligned} & \|F_n(x) - S^l F_n(x)\|^2 \\ &= \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - S^l F_n(x)\|^2 - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - F_n(x)\|^2 \\ &= \frac{1}{L_n} \sum_{k=0}^{l-1} \sum_{i+j=k} \|S^i T^j x - S^l F_n(x)\|^2 + \frac{1}{L_n} \sum_{k=l}^n \sum_{i+j=k, i \leq l-1} \|S^i T^j x - S^l F_n(x)\|^2 \\ &\quad + \frac{1}{L_n} \sum_{k=l}^n \sum_{i+j=k, i \geq l} \|S^i T^j x - S^l F_n(x)\|^2 - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - F_n(x)\|^2 \\ &\leq \frac{1}{L_n} \sum_{k=0}^{l-1} \sum_{i+j=k} \|S^i T^j x - S^l F_n(x)\|^2 + \frac{1}{L_n} \sum_{k=l}^n \sum_{i+j=k, i \leq l-1} \|S^i T^j x - S^l F_n(x)\|^2 \\ &\quad + \frac{1}{L_n} \sum_{k=l}^n \sum_{i+j=k, i \geq l} (k_l \|S^{i-l} T^j x - F_n(x)\| + r_l)^2 - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - F_n(x)\|^2 \\ &= \frac{1}{L_n} \sum_{k=0}^{l-1} \sum_{i+j=k} \|S^i T^j x - S^l F_n(x)\|^2 + \frac{1}{L_n} \sum_{k=l}^n \sum_{i+j=k, i \leq l-1} \|S^i T^j x - S^l F_n(x)\|^2 \\ &\quad + \frac{1}{L_n} \sum_{k=0}^{n-l} \sum_{i+j=k} (k_l^2 \|S^i T^j x - F_n(x)\|^2 + 2k_l r_l \|S^i T^j x - F_n(x)\| + r_l^2) \\ &\quad - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - F_n(x)\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{L_n} \sum_{k=0}^{l-1} \sum_{i+j=k} \|S^i T^j x - S^l F_n(x)\|^2 + \frac{1}{L_n} \sum_{k=l}^n \sum_{i+j=k, i \leq l-1} \|S^i T^j x - S^l F_n(x)\|^2 \\
&\quad + \frac{1}{L_n} \sum_{k=0}^{n-l} \sum_{i+j=k} (k_l^2 - 1) \|S^i T^j x - F_n(x)\|^2 + \frac{2k_l r_l}{L_n} \sum_{k=0}^{n-l} \sum_{i+j=k} \|S^i T^j x - F_n(x)\| \\
&\quad + \frac{(n+2-l)(n+1-l)}{(n+2)(n+1)} r_l^2.
\end{aligned} \tag{2.3}$$

Choose $p \in F(S) \cap F(T)$, then there exists a constant $M > 0$ such that

$$\begin{aligned}
\|S^i T^j x - p\| &\leq k_i \|T^j x - p\| + r_i \leq k_i t_j \|x - p\| + k_i b_j + r_i \leq \frac{M}{2}, \\
\|F_n(x) - p\| &\leq \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} \|S^i T^j x - p\| \leq \frac{M}{2}, \\
\|S^l F_n(x) - p\| &\leq k_l \|F_n(x) - p\| + r_l \leq \frac{M}{2}
\end{aligned} \tag{2.4}$$

for all nonnegative integer i, j, l , and n . Hence, $\|S^i T^j x - S^l F_n(x)\| \leq M$, $\|S^i T^j x - F_n(x)\| \leq M$ for all nonnegative integer i, j, l , and n . So

$$\begin{aligned}
&\sup_{x \in C} \|F_n(x) - S^l F_n(x)\|^2 \\
&\leq \frac{(l+1)l}{(n+2)(n+1)} M^2 + \frac{2(n+1-l)l}{(n+2)(n+1)} M^2 + \frac{(k_l^2 - 1)(n+2-l)(n+1-l)}{(n+2)(n+1)} M^2 \\
&\quad + \frac{2k_l r_l (n+2-l)(n+1-l)}{(n+2)(n+1)} M + \frac{(n+2-l)(n+1-l)}{(n+2)(n+1)} r_l^2 \\
&\longrightarrow 0 \quad (n \longrightarrow \infty, l \longrightarrow \infty).
\end{aligned} \tag{2.5}$$

Similarly, we can prove that

$$\lim_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{x \in C} \|F_n(x) - T^l F_n(x)\| = 0. \tag{2.6}$$

□

Remark 2.3. Lemma 2.2 extends [8, Lemma 3] and [9, Lemma 1].

Lemma 2.4. Let S and T be two continuous generalized asymptotically nonexpansive mappings defined on a nonempty bounded closed convex subset C of a Hilbert space H with $ST = TS$. Let $L_n = ((n+1)(n+2))/2$. If $\{x_n\}$ is a sequence in C such that $\{x_n\}$ converges weakly to some $x \in C$ and $\{x_n - (1/L_n) \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n\}$ converges strongly to 0, then $x \in F(S) \cap F(T)$.

Proof. We claim that $\{S^l x\}$ converges strongly to x as $l \rightarrow \infty$. If not, there exist a positive number ε_0 and a subsequence $\{l_m\}$ of $\{l\}$ such that $\|S^{l_m} x - x\| \geq \varepsilon_0$ for all m . However, we have

$$\begin{aligned}
\|x_{n_i} - S^{l_m} x\| &\leq \left\| x_{n_i} - \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} \right\| \\
&\quad + \left\| \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} - S^{l_m} \left(\frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} \right) \right\| \\
&\quad + \left\| S^{l_m} \left(\frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} \right) - S^{l_m} x \right\| \\
&\leq \left\| x_{n_i} - \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} \right\| \\
&\quad + \left\| \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} - S^{l_m} \left(\frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} \right) \right\| \quad (2.7) \\
&\quad + k_{l_m} \left\| \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} - x \right\| + r_{l_m} \\
&\leq \left\| x_{n_i} - \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} \right\| \\
&\quad + \left\| \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} - S^{l_m} \left(\frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} \right) \right\| \\
&\quad + k_{l_m} \left\| \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} - x_{n_i} \right\| + k_{l_m} \|x_{n_i} - x\| + r_{l_m}.
\end{aligned}$$

By Opial's condition, for any $y \in C$ with $y \neq x$, we have

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|. \quad (2.8)$$

Let $r = \liminf_{n \rightarrow \infty} \|x_n - x\|$ and choose a positive number ρ such that

$$\rho < \sqrt{r^2 + \frac{\varepsilon_0^2}{4}} - r. \quad (2.9)$$

Then, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\lim_{i \rightarrow \infty} \|x_{n_i} - x\| = r$ and $\|x_{n_i} - x\| < r + \rho/5$ for all i . By definition of $\{k_{l_m}\}$ and $\{r_{l_m}\}$, there exists a positive integer m_0 such that

$$k_{l_m} \|x_{n_i} - x\| < r + \frac{\rho}{5}, \quad r_{l_m} < \frac{\rho}{5} \quad (2.10)$$

for all $m > m_0$. Since

$$\lim_{n \rightarrow \infty} \left\| x_n - \frac{1}{L_n} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n \right\| = 0 \quad (2.11)$$

and $\{k_{l_m}\}$ is bounded, there exists a positive integer i_0 such that

$$\begin{aligned} \left\| x_{n_i} - \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} \right\| &< \frac{\rho}{5}, \\ k_{l_m} \left\| \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} - x_{n_i} \right\| &< \frac{\rho}{5} \end{aligned} \quad (2.12)$$

for all m and $i > i_0$. As $\{x_{n_i}\} \subset C$ is bounded and by Lemma 2.2, there exist $m_1 > m_0$ and $i_1 > 0$ such that

$$\left\| \frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} - S^{l_{m_1}} \left(\frac{1}{L_{n_i}} \sum_{k=0}^{n_i} \sum_{i+j=k} S^i T^j x_{n_i} \right) \right\| < \frac{\rho}{5} \quad (2.13)$$

for all $i > i_1$. By (2.7), (2.10), (2.12), and (2.13), we have

$$\|x_{n_i} - S^{l_{m_1}} x\| < \frac{\rho}{5} + \frac{\rho}{5} + \frac{\rho}{5} + r + \frac{\rho}{5} + \frac{\rho}{5} = r + \rho \quad (2.14)$$

for all $i > \max\{i_0, i_1\}$. However,

$$\begin{aligned} \left\| x_{n_i} - \frac{S^{l_{m_1}} x + x}{2} \right\|^2 &= \frac{1}{2} \|x_{n_i} - S^{l_{m_1}} x\|^2 + \frac{1}{2} \|x_{n_i} - x\|^2 - \frac{1}{4} \|S^{l_{m_1}} x - x\|^2 \\ &< \frac{(r + \rho)^2}{2} + \frac{(r + \rho/5)^2}{2} - \frac{\varepsilon_0^2}{4} \\ &< (r + \rho)^2 - \frac{\varepsilon_0^2}{4} < r^2 \end{aligned} \quad (2.15)$$

for all $i > \max\{i_0, i_1\}$. This contradicts with (2.8). So $\{S^l x\}$ converges strongly to x and then $x \in F(S)$. Similarly, we can get $x \in F(T)$. Hence, x is a common fixed point of S and T . $\square$

3. Main results

In this section, we prove our main theorem.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let S, T be two continuous generalized asymptotically nonexpansive mappings of C into itself such that*

- (i) $\|S^n x - S^n y\| \leq k_n \|x - y\| + r_n$, $k_n \geq 0$, $r_n \geq 0$, $k_n \rightarrow 1$, $r_n \rightarrow 0$ ($n \rightarrow \infty$),
- (ii) $\|T^n x - T^n y\| \leq t_n \|x - y\| + b_n$, $t_n \geq 0$, $b_n \geq 0$, $t_n \rightarrow 1$, $b_n \rightarrow 0$ ($n \rightarrow \infty$),
- (iii) $ST = TS$ and $F(S) \cap F(T) \neq \emptyset$.

For arbitrary $x_0 \in C$, the sequence $\{x_n\}_{n=0}^\infty$ is defined by

$$x_n = \alpha_n x_0 + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} S^i T^j x_n, \quad n \geq 0, \quad (3.1)$$

with $\alpha_n \in [0, 1]$ and $\lim_{n \rightarrow \infty} \alpha_n = 0$. If $\limsup_{n \rightarrow \infty} (1 - \alpha_n)(2/((n+1)(n+2))) \sum_{k=0}^n \sum_{i+j=k} k_i t_j < 1$, then $\{x_n\}_{n=0}^\infty$ converges strongly to the common fixed point Px_0 of S and T , where P is the metric projection of H onto $F(S) \cap F(T)$.

Proof. Firstly, we show that $\{x_n\}_{n=0}^\infty$ is bounded. By $\limsup_{n \rightarrow \infty} (1 - \alpha_n)(2/((n+1)(n+2))) \sum_{k=0}^n \sum_{i+j=k} k_i t_j < 1$, there exist $N > 0$ and $0 < a < 1$ such that $(1 - \alpha_n)(2/((n+1)(n+2))) \sum_{k=0}^n \sum_{i+j=k} k_i t_j \leq a$ for all $n > N$. Choose $p \in F(S) \cap F(T)$, then we have

$$\begin{aligned}
\|x_n - p\| &= \left\| \alpha_n(x_0 - p) + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} (S^i T^j x_n - S^i T^j p) \right\| \\
&\leq \alpha_n \|x_0 - p\| + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} (k_i t_j \|x_n - p\| + k_i b_j + r_i) \\
&= \alpha_n \|x_0 - p\| + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} k_i t_j \|x_n - p\| \\
&\quad + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} k_i b_j + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} r_i.
\end{aligned} \tag{3.2}$$

Hence

$$\begin{aligned}
(1 - a) \|x_n - p\| &\leq \alpha_n \|x_0 - p\| + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} k_i b_j \\
&\quad + (1 - \alpha_n) \frac{2}{(n+1)(n+2)} \sum_{k=0}^n \sum_{i+j=k} r_i
\end{aligned} \tag{3.3}$$

for all $n > N$. So $\{x_n\}$ is bounded.

Letting $\{x_{n_i}\}$ be any subsequence of $\{x_n\}$, there exists a subsequence $\{x_{m_t}\}$ of $\{x_{n_i}\}$ such that $\{x_{m_t}\}$ converges weakly to some $u \in C$. For $\{x_{m_t}\}$ is bounded, so is $\{2/((m_t + 1)(m_t + 2))) \sum_{k=0}^{m_t} \sum_{i+j=k} S^i T^j x_{m_t}\}$. Thus

$$\begin{aligned}
x_{m_t} - \frac{2}{(m_t + 1)(m_t + 2)} \sum_{k=0}^{m_t} \sum_{i+j=k} S^i T^j x_{m_t} &= \alpha_{m_t} \left(x_0 - \frac{2}{(m_t + 1)(m_t + 2)} \sum_{k=0}^{m_t} \sum_{i+j=k} S^i T^j x_{m_t} \right) \\
&\longrightarrow 0 (t \longrightarrow \infty).
\end{aligned} \tag{3.4}$$

It follows from Lemma 2.4 that $u \in F(S) \cap F(T)$. For

$$\alpha_{m_t} P x_0 = P x_0 - (1 - \alpha_{m_t}) \frac{2}{(m_t + 1)(m_t + 2)} \sum_{k=0}^{m_t} \sum_{i+j=k} S^i T^j P x_0, \tag{3.5}$$

we have

$$\begin{aligned}
& \alpha_{m_t} \langle x_0 - Px_0, x_{m_t} - Px_0 \rangle \\
&= \left\langle x_{m_t} - Px_0 - (1 - \alpha_{m_t}) \frac{2}{(m_t + 1)(m_t + 2)} \sum_{k=0}^{m_t} \sum_{i+j=k} (S^i T^j x_{m_t} - S^i T^j Px_0), x_{m_t} - Px_0 \right\rangle \\
&= \|x_{m_t} - Px_0\|^2 - (1 - \alpha_{m_t}) \left\langle \frac{2}{(m_t + 1)(m_t + 2)} \sum_{k=0}^{m_t} \sum_{i+j=k} (S^i T^j x_{m_t} - S^i T^j Px_0), x_{m_t} - Px_0 \right\rangle \\
&\geq \|x_{m_t} - Px_0\|^2 - (1 - \alpha_{m_t}) \frac{2}{(m_t + 1)(m_t + 2)} \\
&\quad \times \sum_{k=0}^{m_t} \sum_{i+j=k} (k_i t_j \|x_{m_t} - Px_0\| + k_i b_j + r_i) \|x_{m_t} - Px_0\| \\
&= \left(1 - (1 - \alpha_{m_t}) \frac{2}{(m_t + 1)(m_t + 2)} \sum_{k=0}^{m_t} \sum_{i+j=k} k_i t_j \right) \|x_{m_t} - Px_0\|^2 \\
&\quad - (1 - \alpha_{m_t}) \frac{2}{(m_t + 1)(m_t + 2)} \sum_{k=0}^{m_t} \sum_{i+j=k} (k_i b_j + r_i) \|x_{m_t} - Px_0\|.
\end{aligned} \tag{3.6}$$

By hypothesis, there exist $M > 0$ and $0 < a < 1$ such that $(1 - \alpha_{m_t})(2/((m_t + 1)(m_t + 2))) \sum_{k=0}^{m_t} \sum_{i+j=k} k_i t_j \leq a$ for all $t > M$. So

$$\begin{aligned}
& (1 - a) \|x_{m_t} - Px_0\|^2 \\
&\leq \alpha_{m_t} \langle x_0 - Px_0, x_{m_t} - Px_0 \rangle + (1 - \alpha_{m_t}) \frac{2}{(m_t + 1)(m_t + 2)} \sum_{k=0}^{m_t} \sum_{i+j=k} (k_i b_j + r_i) \|x_{m_t} - Px_0\|
\end{aligned} \tag{3.7}$$

for all $t > M$. We can easily prove that $\lim_{n \rightarrow \infty} (1 - \alpha_{m_t})(2/((m_t + 1)(m_t + 2))) \sum_{k=0}^{m_t} \sum_{i+j=k} (k_i b_j + r_i) \|x_{m_t} - Px_0\| = 0$. Since P is metric projection, $\langle x_0 - Px_0, x - Px_0 \rangle \leq 0$ for all $x \in F(S) \cap F(T)$. Hence

$$\begin{aligned}
\langle x_0 - Px_0, x_{m_t} - Px_0 \rangle &= \langle x_0 - Px_0, x_{m_t} - u \rangle + \langle x_0 - Px_0, u - Px_0 \rangle \\
&\leq \langle x_0 - Px_0, x_{m_t} - u \rangle.
\end{aligned} \tag{3.8}$$

Since $\{x_{m_t}\}$ converges weakly to u , $\limsup_{t \rightarrow \infty} \langle x_0 - Px_0, x_{m_t} - Px_0 \rangle \leq 0$. It follows from (3.7) that $\{x_{m_t}\}$ converges strongly to Px_0 . Hence, $\{x_n\}$ converges strongly to Px_0 . This completes the proof. $\square$

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References

- [1] F. E. Browder, "Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces," *Archive for Rational Mechanics and Analysis*, vol. 24, no. 1, pp. 82–90, 1967.
- [2] J.-B. Baillon, "Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert," *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. Série A et B*, vol. 280, no. 22, pp. 1511–1514, 1975.
- [3] N. Hirano and W. Takahashi, "Nonlinear ergodic theorems for nonexpansive mappings in Hilbert spaces," *Kodai Mathematical Journal*, vol. 2, no. 1, pp. 11–25, 1979.
- [4] S. P. Reich, "Some problems and results in fixed point theory," in *Topological Methods in Nonlinear Functional Analysis (Toronto, Ont., 1982)*, vol. 21 of *Contemporary Mathematics*, pp. 179–187, American Mathematical Society, Providence, RI, USA, 1983.
- [5] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, no. 4, pp. 591–597, 1967.
- [6] W. Takahashi, *Nonlinear Functional Analysis*, Kindaikagakusha, Tokyo, Japan, 1988.
- [7] R. Wittmann, "Approximation of fixed points of nonexpansive mappings," *Archiv der Mathematik*, vol. 58, no. 5, pp. 486–491, 1992.
- [8] T. Shimizu and W. Takahashi, "Strong convergence theorem for asymptotically nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 26, no. 2, pp. 265–272, 1996.
- [9] T. Shimizu and W. Takahashi, "Strong convergence to common fixed points of families of nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 211, no. 1, pp. 71–83, 1997.