Research Article

Convergence to Common Fixed Point for Generalized Asymptotically Nonexpansive Semigroup in Banach Spaces

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Let $K$ be a nonempty closed convex subset of a reflexive and strictly convex Banach space $E$ with a uniformly Gâteaux differentiable norm, $\mathcal{G} = \{T(h) : h \geq 0\}$ a generalized asymptotically nonexpansive self-mapping semigroup of $K$, and $f : K \to K$ a fixed contractive mapping with contractive coefficient $\beta \in (0, 1)$. We prove that the following implicit and modified implicit viscosity iterative schemes $\{x_n\}$ defined by $x_n = \alpha_n f(x_n) + (1 - \alpha_n) T(t_n) x_n$ and $y_n = \beta_n f(x_{n-1}) + (1 - \beta_n) x_{n-1}$ strongly converge to $p \in F$ as $n \to \infty$ and $p$ is the unique solution to the following variational inequality:

$$\langle f(p) - p, j(y - p) \rangle \leq 0 \text{ for all } y \in F.$$ 

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1. Introduction

Let $C$ be a closed convex subset of a Hilbert space $H$ and $T$ a nonexpansive mapping from $C$ into itself. We denote by $F(T)$ the set of fixed points of $T$. Let $F(T)$ be nonempty and $u$ an element of $C$. For each $t$ with $0 < t < 1$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tu + (1 - t)Tx$. Browder [1] showed that $\{x_t\}$ defined by $x_t = tu + (1 - t)Tx_t$ converges strongly to the element of $F(T)$ which is nearest to $u$ in $F(T)$ as $t \to 0$.

In 2004, for a contraction $f : C \to C$ and a nonexpansive mapping $T : C \to C$, Xu [2] proposed the following viscosity approximation method in Banach space:

$$x_t = tf(x_t) + (1 - t)Tx_t, \quad t \in (0, 1), \quad t \to 0, \quad (1.1)$$

and Song and Xu [3] studied the convergence of the following implicit viscosity iterative scheme:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n) T(t_n) x_n, \quad (1.2)$$

where $\{\alpha_n\} \subset (0, 1)$. 

On the other hand, for a fixed Lipschitz strongly pseudocontractive mapping \( f \) and a continuous pseudocontractive mapping \( T \), Song and Chen [4] proposed the following motivated implicit viscosity iterative scheme:

\[
\begin{align*}
    x_n &= \alpha_n y_n + (1 - \alpha_n)Tx_n, \\
y_n &= \beta_n f(x_{n-1}) + (1 - \beta_n)x_{n-1}.
\end{align*}
\] (1.3)

In this paper, we will still study the implicit viscosity iterative scheme (1.2) and propose the following iterative scheme:

\[
\begin{align*}
    x_n &= \alpha_n y_n + (1 - \alpha_n)T(t_n)x_n, \\
y_n &= \beta_n f(x_{n-1}) + (1 - \beta_n)x_{n-1},
\end{align*}
\] (1.4)

where \( \{T(h) : h \geq 0\} \) is a generalized asymptotically nonexpansive self-mappings semigroup and \( f \) a fixed contractive mapping with contractive coefficient \( \beta \in (0, 1) \).

**2. Preliminaries**

Throughout this paper, we assume that \( E \) is a Banach space and \( K \) a nonempty closed convex subset of \( E \). Let \( E^* \) be a dual space of \( E \), \( J : E \rightarrow 2^{E^*} \) the normalized duality mapping defined by

\[
J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\| \cdot \|f\|, \quad \|x\| = \|f\| \},
\] (2.1)

where \( \langle \cdot, \cdot \rangle \) denotes the generalized duality pairing.

**Definition 2.1** (see [5]). A mapping \( T : E \rightarrow E \) is said to be total asymptotically nonexpansive if there exist nonnegative real sequences \( \{k_n^{(1)}\} \) and \( \{k_n^{(2)}\} \), \( n > 0 \), with \( k_n^{(1)} \) and \( k_n^{(2)} \rightarrow 0 \) as \( n \rightarrow \infty \), and strictly increasing and continuous functions \( \phi : R^+ \rightarrow R^+ \) with \( \phi(0) = 0 \) such that

\[
\|T^n x - T^n y\| \leq \|x - y\| + k_n^{(1)} \phi(\|x - y\|) + k_n^{(2)} \quad \forall x, y \in K.
\] (2.2)

**Remark 2.2.** If \( \phi(\lambda) = \lambda \), the total asymptotically nonexpansive mapping coincides with generalized asymptotically nonexpansive mapping. In addition, for all \( n \in N \), if \( k_n^{(2)} = 0 \), then generalized asymptotically nonexpansive mapping coincide with asymptotically nonexpansive mapping; if \( k_n^{(1)} = 0 \), \( k_n^{(2)} = \max\{0, p_n\} \), where \( p_n := \sup_{x,y \in K} (\|Tx - Ty\| - \|x - y\|) \), then generalized asymptotically nonexpansive mapping coincide with asymptotically nonexpansive mapping in the intermediate sense; if \( k_n^{(1)} = 0 \) and \( k_n^{(2)} = 0 \), then we obtain from (2.2) the class of nonexpansive mapping.

**Remark 2.3.** In [5], for the total asymptotically nonexpansive mapping, the authors assume that there exist \( M, M^* > 0 \) such that \( \phi(\lambda) \leq M^* \lambda \) for all \( \lambda \geq M \), so for \( M_0 = \max\{\phi(M), M^*\} \), \( \phi(\lambda) \leq M_0(1 + \lambda) \) for all \( \lambda \geq 0 \), then the total asymptotically nonexpansive mapping studied by [5] coincides with generalized asymptotically nonexpansive mapping.
A (one-parameter) generalized asymptotically nonexpansive semigroup is a family $\mathcal{F} = \{T(h) : h \geq 0\}$ of self-mapping of $K$ such that

\begin{itemize}
  \item[(i)] $T(0)x = x$ for $x \in K$;
  \item[(ii)] $T(s + t)x = T(s)T(t)x$ for $t, s \geq 0$ and $x \in K$;
  \item[(iii)] $\lim_{t \to 0} T(t)x = x$ for $x \in K$;
  \item[(iv)] for each $h \geq 0$, $T(h)$ is generalized asymptotically nonexpansive, that is,
  \begin{equation}
  \|T(h)x - T(h)y\| \leq (1 + k_h^{(1)})\|x - y\| + k_h^{(2)} \quad \forall \ x, y \in K.
  \end{equation}
\end{itemize}

We will denote by $F$ the common fixed point set of $\mathcal{F}$, that is,

\begin{equation}
F := \text{Fix}(\mathcal{F}) = \{x \in K : T(h)x = x, \ h \geq 0\} = \bigcap_{h \geq 0} \text{Fix}(T(h)).
\end{equation}

**Definition 2.4.** A Banach space $E$ is said to be strictly convex if $\|x + y\|/2 < 1$ for $\|x\| = \|y\| = 1$ and $x \neq y$.

**Definition 2.5.** Let $U = \{x \in E : \|x\| = 1\}$, the norm of $E$ is said to be uniformly Gâteaux differentiable, if for each $y \in U$, $\lim_{t \to 0}(\|x + ty\| - \|x\|)/t$ exists uniformly for $x \in U$.

**Definition 2.6.** Let $\mu$ be a continuous linear functional on $l^\infty$ and let $(a_0, a_1, \ldots) \in l^\infty$. One writes $\mu_n(a_n)$ instead of $\mu((a_0, a_1, \ldots))$. One calls $\mu$ a Banach limit when $\mu$ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for each $(a_0, a_1, \ldots) \in l^\infty$.

For a Banach limit $\mu$, one knows that $\lim_{n \to \infty} a_n \leq \mu_n(a_n) \leq \lim_{n \to \infty} a_n$ for every $a = (a_0, a_1, \ldots) \in l^\infty$. So if $a = (a_0, a_1, \ldots) \in l^\infty$, $b = (b_0, b_1, \ldots) \in l^\infty$ and $a_n - b_n \to 0$ as $n \to \infty$, one has $\mu_n(a_n) = \mu_n(b_n)$.

**Definition 2.7.** Let $K$ be a nonempty closed convex subset of a Banach space $E$, $\mathcal{F} = \{T(h) : h \geq 0\}$ a continuous operator semigroup on $K$. Then $\mathcal{F}$ is said to be uniformly asymptotically regular (in short, u.a.r.) on $K$ if for all $h \geq 0$ and any bounded subset $C$ of $K$, $\lim_{t \to \infty} \sup_{x \in C}\|T(h)(T(t)x) - T(t)x\| = 0$.

**Lemma 2.8** (see [6]). Let $E$ be a Banach space with a uniformly Gâteaux differentiable norm, then the normalized duality mapping $j : E \to 2^{E^*}$ defined by (2.1) is single-valued and uniformly continuous from the norm topology of $E$ to the weak* topology of $E^*$ on each bounded subset of $E$.

The single-valued normalized duality mapping is denoted by $j$.

**Lemma 2.9.** Let $E$ be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, $K$ a nonempty closed convex subset of $E$. Suppose that $\{x_n\}$ is a bounded sequence in $K$, $\{T(h) : h \geq 0\}$ a continuous generalized asymptotically nonexpansive semigroup from $K$ into itself such that $\lim_{n \to \infty} \|x_n - T(h)x_n\| = 0$ for all $h \geq 0$. Define the set

\begin{equation}
K^* = \left\{x \in K : \mu_n\|x_n - x\|^2 = \min_{y \in K} \mu_n\|x_n - y\|^2\right\}.
\end{equation}

If $F \neq \emptyset$, then $K^* \cap F \neq \emptyset$. 

Proof. Set \( g(y) = \mu_n \|x_n - y\|^2 \), then \( g(y) \) is a convex and continuous function, and \( g(y) \to \infty \) as \( \|y\| \to \infty \). Using [7, Theorem 1.3.11], there exists \( x \in K \) such that \( g(x) = \inf_{y \in K} g(y) \) by the reflexivity of \( E \), that is, \( K^* \) is nonempty. Clearly, \( K^* \) is closed convex by the convexity and continuity of \( g(y) \).

Since \( \lim_{h \to \infty} \|x_n - T(h)x_n\| = 0 \), \( \lim_{h \to \infty} k_n^{(i)} = 0 \) \((i = 1, 2)\), and \( g(y) \) is continuous for all \( z \in K^* \), we have

\[
g \left( \lim_{h \to \infty} T(h)z \right) = \lim_{h \to \infty} g(T(h)z)
\]

\[
= \lim_{h \to \infty} \mu_n \|x_n - T(h)z\|^2
\]

\[
\leq \lim_{h \to \infty} \mu_n \|T(h)x_n - T(h)z\|^2
\]

\[
\leq \lim_{h \to \infty} \mu_n \left( (1 + k_n^{(1)}) \|x_n - z\| + k_n^{(2)} \right)^2
\]

\[
= \mu_n \|x_n - z\|^2.
\]

Hence \( \lim_{h \to \infty} T(h)z \in K^* \).

Let \( p \in F \). Since \( K^* \) is closed convex set, there exists a unique \( v \in K^* \) such that

\[
\|p - v\| = \min_{x \in K} \|p - x\|.
\]

Since \( p = \lim_{h \to \infty} T(h)p \) and \( \lim_{h \to \infty} T(h)v \in K^* \),

\[
\|p - \lim_{h \to \infty} T(h)v\| = \|\lim_{h \to \infty} T(h)p - \lim_{h \to \infty} T(h)v\|
\]

\[
= \lim_{h \to \infty} \|T(h)p - T(h)v\|
\]

\[
\leq \lim_{h \to \infty} (1 + k_n^{(1)}) \|p - v\| + k_n^{(2)}
\]

\[
= \|p - v\|.
\]

Therefore, \( \lim_{h \to \infty} T(h)v = v \). Since \( T(s + t)x = T(s)T(t)x \) for all \( x \in K \), then we have

\[
v = \lim_{t \to \infty} T(t)v = \lim_{t \to \infty} T(s + t)v = \lim_{t \to \infty} T(s)T(t)v = T(s)\lim_{t \to \infty} T(t)v = T(s)v
\]

for all \( s \geq 0 \). Therefore \( v \in F \) and the proof is complete.

**Lemma 2.10** (see [8]). Let \( K \) be a nonempty convex subset of a Banach space \( E \) with a uniformly Gâteaux differentiable norm, and \( \{x_n\} \) a bounded sequence of \( E \). If \( z_0 \in K \), then

\[
\mu_n \|x_n - z_0\|^2 = \min_{y \in K} \mu_n \|x_n - y\|^2
\]

if and only if

\[
\mu_n \langle y - z_0, J(x_n - z_0) \rangle \leq 0 \quad \forall y \in K.
\]

**Lemma 2.11** (see [9]). Let \( \{a_n\} \) be a sequence of nonnegative real numbers satisfying the following conditions:

\[
a_{n+1} \leq (1 - \lambda_n)a_n + b_n + c_n \quad \forall n \geq n_0,
\]

where \( n_0 \) is some nonnegative integer, \( \lambda_n \in [0, 1] \) with \( \sum_{n=1}^{\infty} \lambda_n = \infty \), \( \limsup_{n \to \infty} (b_n/\lambda_n) \leq 0 \), and \( \sum_{n=1}^{\infty} c_n < \infty \). Then \( a_n \to 0 \) as \( n \to \infty \).
3. Implicit iteration scheme

**Theorem 3.1.** Let $E$ be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, $K$ a nonempty closed convex subset of $E$, $\mathcal{F} = \{ T(h) : h \geq 0 \}$ a u.a.r generalized asymptotically nonexpansive semigroup from $K$ into itself with sequences $\{ k_n^{(1)} \}, \{ k_n^{(2)} \}, h \geq 0$, such that $F \neq \emptyset$, and $f : K \to K$ a fixed contractive mapping with contractive coefficient $\beta \in (0, 1)$. If $\{ x_n \}$ is given by (1.2), where $\lim_{n \to \infty} t_n = \infty$, $\alpha_n \in (0, 1)$, $\lim_{n \to \infty} \alpha_n = 0$ and $\lim_{n \to \infty} (k_n^{(i)} / \alpha_n) = 0$ ($i = 1, 2$), then $\{ x_n \}$ converges strongly to some common fixed point $p$ of $F$ such that $p$ is the unique solution in $F$ to variational inequality:

$$ (f(p) - p, j(y - p)) \leq 0 \quad \forall y \in F. $$  

**Proof.** For any fixed $y \in F$,

$$ ||x_n - y||^2 = (\alpha_n (f(x_n) - y) + (1 - \alpha_n)(T(t_n)x_n - y), j(x_n - y)) $$

$$ = \alpha_n (f(x_n) - f(y), j(x_n - y)) + \alpha_n (f(y) - y, j(x_n - y)) $$

$$ + (1 - \alpha_n)(T(t_n)x_n - T(t_n)y, j(x_n - y)) $$

$$ \leq \alpha_n \beta ||x_n - y||^2 + \alpha_n (f(y) - y, j(x_n - y)) $$

$$ + (1 - \alpha_n)||x_n - y||[(1 + k_n^{(1)})||x_n - y|| + k_n^{(2)}] $$

$$ = (1 - \alpha_n(1 - \beta) + (1 - \alpha_n)k_n^{(1)})||x_n - y||^2 + \alpha_n (f(y) - y, j(x_n - y)) $$

$$ + (1 - \alpha_n)(1 - \beta)k_n^{(2)}||x_n - y||. $$

Let $d_n^{(i)} = (k_n^{(i)} / \alpha_n)$ ($i = 1, 2$). Since $\lim_{n \to \infty} (k_n^{(i)} / \alpha_n) = 0$ for all $\varepsilon \in (0, 1 - \beta)$, there exists $N \in \mathbb{N}$ such that $k_n^{(i)} / \alpha_n < \varepsilon < 1 - \beta < (1 - \beta) / (1 - \alpha_n)$ for all $n \geq N$.

Furthermore,

$$ ||x_n - y||^2 \leq \frac{(f(y) - y, j(x_n - y))}{1 - \beta - (1 - \alpha_n)k_n^{(1)}d_n^{(2)}} + \frac{(1 - \alpha_n)d_n^{(2)} ||x_n - y||}{1 - \beta - (1 - \alpha_n)d_n^{(1)}} $$

for all $n \geq N$. That is, $||x_n - y|| \leq (||f(y) - y|| + (1 - \alpha_n)d_n^{(2)}) / (1 - \beta - (1 - \alpha_n)d_n^{(1)})$ for all $n \geq N$. Thus $\{ x_n \}$ is bounded, so are $\{ T(t_n)x_n \}$ and $\{ f(x_n) \}$. This imply that

$$ \lim_{n \to \infty} ||x_n - T(t_n)x_n|| = \lim_{n \to \infty} ||T(t_n)x_n - f(x_n)|| = 0. $$

Since $\{ T(h) \}$ is u.a.r and $\lim_{n \to \infty} t_n = \infty$, then for all $h \geq 0$,

$$ \lim_{n \to \infty} ||T(h)T(t_n)x_n - T(t_n)x_n|| \leq \lim_{n \to \infty} \sup_{x \in C} ||T(h)T(t_n)x - T(t_n)x|| = 0, $$

where $C$ is any bounded subset of $K$ containing $\{ x_n \}$. Since $\{ T(h) \}$ is continuous, hence

$$ ||x_n - T(h)x_n|| \leq ||x_n - T(t_n)x_n|| + ||T(t_n)x_n - T(h)(T(t_n)x_n)|| $$

$$ + ||T(h)(T(t_n)x_n) - T(h)x_n|| \to 0. $$
That is, for all \( h \geq 0 \), \( \lim_{n \to \infty} \| x_n - T(h)x_n \| = 0 \). We claim that the set \( \{ x_n \} \) is sequentially compact. Indeed, define the set
\[
K^* = \left\{ x \in K : \mu_n \| x_n - x \|^2 = \min_{y \in K} \mu_n \| x_n - y \|^2 \right\}.
\]
(3.7)

By Lemma 2.9, we can found \( p \in K^* \cap F \). Using Lemma 2.10, we get that
\[
\mu_n \langle y - p, j(x_n - p) \rangle \leq 0 \quad \forall y \in K.
\]
(3.8)

It follows from (3.3) that
\[
\mu_n \| x_n - p \|^2 \leq \mu_n \frac{\langle f(p) - p, j(x_n - p) \rangle}{1 - \beta - (1 - \alpha_n)d_n^{(1)}} + \mu_n \frac{(1 - \alpha_n)d_n^{(2)} \| x_n - p \|}{1 - \beta - (1 - \alpha_n)d_n^{(1)}} \to 0.
\]
(3.9)

Then we have \( \mu_n \| x_n - p \| = 0 \).

Hence, there exists a subsequence \( \{ x_{n_k} \} \) of \( \{ x_n \} \) which strongly converges to \( p \in F \) as \( k \to \infty \).

Next we show that \( p \) is a solution in \( F \) to the variational inequality (3.1). In fact, for any fixed \( y \in F \), there exists a constant \( Q > 0 \) such that \( \| x_n - y \| \leq Q \), then
\[
\| x_n - y \|^2 = \langle \alpha_n(f(x_n) - y) + (1 - \alpha_n)(T(t_n)x_n - y), j(x_n - y) \rangle \\
= \alpha_n \langle f(x_n) - f(p) + p - x_n, j(x_n - y) \rangle + \alpha_n \langle f(p) - p, j(x_n - y) \rangle \\
+ \alpha_n \langle x_n - y, j(x_n - y) \rangle + (1 - \alpha_n)(T(t_n)x_n - T(t_n)y, j(x_n - y) \rangle \\
\leq \alpha_n(\beta + 1)\| x_n - p \|Q + \alpha_n (1 - \alpha_n)d_n^{(1)}Q^2 + (1 - \alpha_n)d_n^{(2)}Q.
\]
(3.10)

Therefore,
\[
\langle f(p) - p, j(y - x_n) \rangle \leq (\beta + 1)\| x_n - p \|Q + (1 - \alpha_n)d_n^{(1)}Q^2 + (1 - \alpha_n)d_n^{(2)}Q.
\]
(3.11)

Taking limit as \( n_k \to \infty \) in two sides of (3.11), by Lemma 2.8 and \( \{ x_{n_k} \} \to p \) as \( k \to \infty \), we obtain
\[
\langle f(p) - p, j(y - p) \rangle \leq 0 \quad \forall y \in F.
\]
(3.12)

That is, \( p \in F \) is a solution of variational inequality (3.1).

Suppose that \( p, q \in F \) satisfy (3.1), we have
\[
\langle f(p) - p, j(q - p) \rangle \leq 0,
\]
(3.13)
\[
\langle f(q) - q, j(p - q) \rangle \leq 0.
\]
(3.14)

Combining (3.13) and (3.14), it follows that
\[
(1 - \beta)\| p - q \|^2 \leq \langle (p - q) - f(p) + f(q), j(p - q) \rangle \leq 0.
\]
(3.15)

Hence \( p = q \), that is, \( p \in F \) is the unique solution of variational inequality (3.1), so each cluster point of sequence \( \{ x_n \} \) is equal to \( p \). Therefore, \( \{ x_n \} \) converges to \( p \) and the proof is complete. \( \square \)
Remark 3.2. Let $E$, $K$, $F$, $f$, $\{\alpha_n\}$, and $\{t_n\}$ be as in Theorem 3.1, $\mathcal{F} = \{T(h) : h \geq 0\}$ a u.a.r nonexpansive semigroup from $K$ into itself, then our result coincides with Theorem 3.2 in [3].

4. Modified implicit iteration scheme

Theorem 4.1. Let $E$ be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, $K$ a nonempty closed convex subset of $E$, $\mathcal{F} = \{T(h) : h \geq 0\}$ a u.a.r. generalized asymptotically nonexpansive semigroup from $K$ into itself with sequences $\{k_n^{(1)}\}$, $\{k_n^{(2)}\}$, $h \geq 0$, such that $F \neq \emptyset$, and $f : K \rightarrow K$ a fixed contraction mapping with contractive coefficient $\beta \in (0, 1)$. If $\{x_n\}$ is given by (1.4), where $\lim_{n \rightarrow \infty} t_n = \infty$, $\alpha_n, \beta_n \in (0, 1]$, $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, and $\sum_{n=1}^{\infty} (k_n^{(1)} / \alpha_n) < \infty$, $\sum_{n=1}^{\infty} (k_n^{(2)} / \alpha_n) < \infty$, then $\{x_n\}$ converges strongly to some common fixed point $p$ of $F$ such that $p$ is the unique solution in $F$ to variational inequality (3.1).

Proof. For any fixed $y \in F$,

$$\|x_n - y\| = \|\alpha_n(y_n - y) + (1 - \alpha_n)(T(t_n)x_n - y)\|$$

$$\leq (1 - \alpha_n)\|T(t_n)x_n - y\| + \alpha_n\|y_n - y\|$$

$$\leq (1 - \alpha_n)[(1 + k_n^{(1)})\|x_n - y\| + k_n^{(2)}] + \alpha_n\|y_n - y\|. \tag{4.1}$$

Let $d_n^{(i)} = (k_n^{(i)} / \alpha_n)$ ($i = 1, 2$). Hence,

$$\|x_n - y\| \leq \frac{(1 - \alpha_n)d_n^{(2)}}{1 - (1 - \alpha_n)d_n^{(1)}} \frac{\|y_n - y\|}{1 - \alpha_n} + \frac{\beta_n\|f(x_{n-1}) - y\| + (1 - \beta_n)d_n^{(1)}\|x_{n-1} - y\|}{1 - d_n^{(1)}}$$

$$\leq \frac{d_n^{(2)}}{1 - d_n^{(1)}} \beta_n\|f(x_{n-1}) - f(y)\| + \frac{\beta_n\|f(y) - y\|}{1 - \beta_n} \frac{(1 - \beta_n)d_n^{(1)}\|x_{n-1} - y\|}{1 - d_n^{(1)}} \tag{4.2}$$

$$\leq \frac{(1 - \beta_n)d_n^{(1)}\|x_{n-1} - y\|}{1 - d_n^{(1)}} + \frac{\beta_n\|f(y) - y\| + d_n^{(2)}}{1 - d_n^{(1)}}$$

$$\leq \frac{1}{1 - d_n^{(1)}} \max \left\{\|x_{n-1} - y\| + d_n^{(2)} \frac{\|f(y) - y\| + d_n^{(2)}}{1 - \beta}\right\} \frac{\|x_n - y\|}{1 - d_n^{(1)}}.$$ 

By induction, we get that

$$\|x_n - y\| \leq \frac{1}{1 - d_n^{(1)}} \max \left\{\frac{\|x_{n-1} - y\| + d_n^{(2)} + d_n^{(2)} + d_{n-1}^{(2)} + \|f(y) - y\| + d_{n-1}^{(2)} + \|f(y) - y\| + d_1^{(2)}}{(1 - d_{n-1}^{(1)})(1 - \beta)} \frac{1}{1 - \beta}\right\}$$

$$\cdots$$

$$\leq \frac{1}{(1 - d_1^{(1)}) \cdots (1 - d_1^{(1)})} \times \max \left\{\|x_1 - y\| + \sum_{i=1}^{\infty} d_i^{(2)} \frac{\|f(y) - y\| + d_i^{(2)}}{1 - \beta}, \cdots, \frac{\|f(y) - y\| + d_1^{(2)}}{1 - \beta}\right\}. \tag{4.3}$$
Since \( \sum_{n=1}^{\infty} a_n(i) < \infty \) (\( i = 1, 2 \)), we know from Abel–Dini theorem that there exists \( r > 0 \) such that \( \lim_{n \to \infty} (1 - a_n(1)) \cdot (1 - a_n(2)) = r \). Thus \( \{x_n\} \) is bounded, so are \( \{T(t_n)x_n\} \), \( \{f(x_n)\} \) and \( \{y_n\} \). This imply that

\[
\lim_{n \to \infty} \|x_n - T(t_n)x_n\| = \lim_{n \to \infty} \|y_n - T(t_n)x_n\| = 0. \tag{4.4}
\]

Since \( \{T(h) : h \geq 0\} \) is u.a.r. and \( \lim_{n \to \infty} t_n = \infty \) for all \( h \geq 0 \),

\[
\lim_{n \to \infty} \|T(h)T(t_n)x_n - T(t_n)x_n\| \leq \limsup_{n \to \infty} \|T(h)T(t_n)x - T(t_n)x\| = 0, \tag{4.5}
\]

where \( C \) is any bounded subset of \( K \) containing \( \{x_n\} \). Since \( T(h) \) is continuous, hence

\[
\|x_n - T(h)x_n\| \leq \|x_n - T(t_n)x_n\| + \|T(t_n)x_n - T(h)(T(t_n)x_n)\| + \|T(h)(T(t_n)x_n) - T(h)x_n\| \rightarrow 0.
\]

That is, for all \( h \geq 0 \),

\[
\lim_{n \to \infty} \|x_n - T(h)x_n\| = 0. \tag{4.7}
\]

From Theorem 3.1, there exists the unique solution \( p \in F \) to the variational inequality (3.1). Since \( p = T(h)p \) for all \( h \geq 0 \), we have

\[
\|x_{n+1} - p\|^2 = \alpha_{n+1}(y_{n+1} - p, j(x_{n+1} - p)) + (1 - \alpha_{n+1})(T(t_{n+1})x_{n+1} - p, j(x_{n+1} - p))
\]

\[
= \alpha_{n+1}(\beta_{n+1}f(x_n) + (1 - \beta_{n+1})x_n - p, j(x_{n+1} - p))
\]

\[
+ (1 - \alpha_{n+1})(T(t_{n+1})x_{n+1} - p, j(x_{n+1} - p))
\]

\[
\leq \alpha_{n+1}\beta_{n+1}(f(x_n) - f(p), j(x_{n+1} - p)) + \alpha_{n+1}\beta_{n+1}(f(p) - p, j(x_{n+1} - p))
\]

\[
+ \alpha_{n+1}(1 - \beta_{n+1})(x_n - p, j(x_{n+1} - p))
\]

\[
+ (1 - \alpha_{n+1})\|x_{n+1} - p\|[\|1 + k^{(1)}_{t_{n+1}}\|\|x_{n+1} - p\| + k^{(2)}_{t_{n+1}}]
\]

\[
\leq \frac{\alpha_{n+1}\beta_{n+1}}{2} \|x_n - p\|^2 + \|x_{n+1} - p\|^2 + \frac{\alpha_{n+1}(1 - \beta_{n+1})}{2} \|x_n - p\|^2 + \|x_{n+1} - p\|^2
\]

\[
+ (1 - \alpha_{n+1})\|x_{n+1} - p\|^2 + (1 - \alpha_{n+1})k^{(1)}_{t_{n+1}} \|x_{n+1} - p\|^2 + (1 - \alpha_{n+1})k^{(2)}_{t_{n+1}} \|x_{n+1} - p\|
\]

\[
+ \alpha_{n+1}\beta_{n+1}(f(p) - p, j(x_{n+1} - p))
\]

\[
= \frac{\alpha_{n+1}}{2} \|x_{n+1} - p\|^2 + (1 - \alpha_{n+1})\|x_{n+1} - p\|^2 + \frac{\alpha_{n+1}}{2}(1 + \beta_{n+1}^2 - \beta_{n+1}) \|x_n - p\|^2
\]

\[
+ (1 - \alpha_{n+1})k^{(1)}_{t_{n+1}} \|x_{n+1} - p\|^2 + (1 - \alpha_{n+1})k^{(2)}_{t_{n+1}} \|x_{n+1} - p\|
\]

\[
+ \alpha_{n+1}\beta_{n+1}(f(p) - p, j(x_{n+1} - p)). \tag{4.8}
\]

Therefore

\[
\|x_{n+1} - p\|^2 \leq (1 - (1 - \beta_{n+1}^2)\beta_{n+1})\|x_n - p\|^2 + 2\beta_{n+1}(f(p) - p, j(x_{n+1} - p))
\]

\[
+ 2(1 - \alpha_{n+1})(a_n^{(1)}\|x_{n+1} - p\| + a_n^{(2)}) \|x_{n+1} - p\|. \tag{4.9}
\]
That is,
\[
\|x_{n+1} - p\|^2 \leq (1 - \lambda_n)\|x_n - p\|^2 + b_n + c_n, \tag{4.10}
\]
where \(\lambda_n = (1 - \beta^2)\beta_{n+1}\), \(b_n = 2\beta_{n+1} (f(p) - p, j(x_{n+1} - p))\) and \(c_n = 2(1 - \alpha_{n+1})(d_{n+1}^{(1)}\|x_{n+1} - p\| + d_{n+1}^{(2)}\|x_{n+1} - p\|).\) Since \(\sum_{n=1}^{\infty} \lambda_n = \infty, \sum_{n=1}^{\infty} d_{n+1}^{(i)} < \infty (i = 1, 2),\) \(\|x_{n+1} - p\|\) is bounded, we have \(\sum_{n=1}^{\infty} c_n < \infty.\) So we only need to show that \(\lim_{n \to \infty} (b_n / \lambda_n) \leq 0,\) that is,
\[
\lim_{n \to \infty} (f(p) - p, j(x_{n+1} - p)) \leq 0. \tag{4.11}
\]
Let \(z_m = a_m f(z_m) + (1 - a_m) T(t_m) z_m,\) where \(t_m\) and \(a_m\) satisfy the condition of Theorem 3.1. Then it follows from Theorem 3.1 that \(p = \lim_{m \to \infty} z_m.\)

Since
\[
\|x_{n+1} - z_m\|^2 = (1 - \alpha_m)\|(T(t_m)z_m - x_{n+1}, f(z_m - x_{n+1})) + \alpha_m(f(z_m) - x_{n+1}, f(z_m - x_{n+1}))
\]
\[
= (1 - \alpha_m)((T(t_m)z_m - T(t_m)x_{n+1}, f(z_m - x_{n+1}))) + (T(t_m)x_{n+1} - x_{n+1}, f(z_m - x_{n+1}))
\]
\[
+ \alpha_m(f(z_m) - z_m - (f(p) - p), j(z_m - x_{n+1})) + \alpha_m(f(p) - p, j(z_m - x_{n+1}))
\]
\[
+ \alpha_m(z_m - x_{n+1}, j(z_m - x_{n+1}))
\]
\[
\leq \|x_{n+1} - z_m\|^2 + (1 - \alpha_m)\|(k_m^{(1)} Q + k_m^{(2)} Q) + (1 - \alpha_m)\|T(t_m)x_{n+1} - x_{n+1}\|Q
\]
\[
+ \alpha_m(f(p) - p, j(z_m - x_{n+1})) + \alpha_m(1 + \beta)\|z_m - p\|Q.
\]
\[
\tag{4.12}
\]
Furthermore,
\[
\langle f(p) - p, j(x_{n+1} - z_m) \rangle \leq \frac{1 - \alpha_m}{\alpha_m} (k_m^{(1)} Q + k_m^{(2)} Q) + \frac{\|T(t_m)x_{n+1} - x_{n+1}\|Q}{\alpha_m} + (1 + \beta)\|z_m - p\|Q,
\]
\[
\tag{4.13}
\]
where \(Q\) is a constant such that \(Q \geq \|z_m - x_{n+1}\|\). Hence, taking upper limit as \(n \to \infty\) firstly, and then as \(m \to \infty\) in (4.13), we have
\[
\lim_{m \to \infty} \lim_{n \to \infty} (f(p) - p, j(x_{n+1} - z_m)) \leq 0. \tag{4.14}
\]
On the other hand, since \(p = \lim_{m \to \infty} z_m\) and by Lemma 2.8, we have
\[
\langle f(p) - p, j(x_{n+1} - z_m) \rangle \to \langle f(p) - p, j(x_{n+1} - p) \rangle \text{ uniformly.} \tag{4.15}
\]
Thus given \(\varepsilon > 0\), there exists \(N \geq 1\) such that if \(m > N\) for all \(n\) we have
\[
\langle f(p) - p, j(x_{n+1} - p) \rangle < \langle f(p) - p, j(x_{n+1} - z_m) \rangle + \varepsilon. \tag{4.16}
\]
Hence, taking upper limit as \(n \to \infty\) firstly, and then as \(m \to \infty\) in two sides of (4.16), we get that
\[
\lim_{n \to \infty} (f(p) - p, j(x_{n+1} - p)) \leq \lim_{m \to \infty} \lim_{n \to \infty} (f(p) - p, j(x_{n+1} - z_m)) + \varepsilon \leq \varepsilon. \tag{4.17}
\]
For the arbitrariness of \(\varepsilon,\) (4.11) holds. By Lemma 2.11, \(x_n \to p\) and the proof is complete. 
\(\square\)
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References

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