Research Article

Kurosh-Amitsur Right Jacobson Radical of Type 0 for Right Near-Rings

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By a near-ring we mean a right near-ring. \( J^0_r \), the right Jacobson radical of type 0, was introduced for near-rings by the first and second authors. In this paper properties of the radical \( J^0_r \) are studied. It is shown that \( J^0_r \) is a Kurosh-Amitsur radical (KA-radical) in the variety of all near-rings \( R \), in which the constant part \( R_c \) of \( R \) is an ideal of \( R \). So unlike the left Jacobson radicals of types 0 and 1 of near-rings, \( J^0_r \) is a KA-radical in the class of all zero-symmetric near-rings. \( J^0_r \) is not \( s \)-hereditary and hence not an ideal-hereditary radical in the class of all zero-symmetric near-rings.

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1. Introduction

\( R \) denotes a right near-ring and all near-rings considered are right near-rings and not necessarily zero-symmetric.

In [1, 2], the first author studied the structure of near-rings in terms of right ideals, and showed that as in rings, matrix units determined by right ideals identify matrix near-rings. To show the importance of the right Jacobson radicals of near-rings in the extension of a form of the Wedderburn-Artin theorem of rings involving the matrix rings to near-rings, the right Jacobson radicals of type \( \nu \) were introduced and studied by the first and second authors in [3–6], \( \nu \in \{0, 1, 2, s\} \). In [6], Wedderburn-Artin theorem was extended to near-rings, and some generalizations of it were presented.

In this paper, properties of the right Jacobson radical of type 0 are studied. It is known that the left Jacobson radicals of types 0 and 1 are not KA-radicals in the class of all
zero-symmetric near-rings, and only the left Jacobson radicals of types 2 and 3 are KA-radicals in the class of all zero-symmetric near-rings. Surprisingly, $J'_0$, the right Jacobson radical of type 0, is a KA-radical in the class of all zero-symmetric near-rings. It is also shown that $J'_0$ is a KA-radical even in a bigger class of near-rings, namely, in the variety of all near-rings $R$, in which the constant part of $R$ is an ideal of $R$. Moreover, $J'_0$ is not s-hereditary, and hence not an ideal-hereditary radical in the class of all zero-symmetric near-rings.

2. Preliminaries

Near-rings considered are right near-rings and not necessarily zero-symmetric. Unless otherwise specified, $R$ stands for a right near-ring. Near-ring notions not defined here can be found in [7].

$R_0$ and $R_c$ denote the zero-symmetric part and the constant part of $R$, respectively. $\mathcal{F}$ denotes the class of near-rings $R$, in which the constant part $R_c$ of $R$ is an ideal of $R$. In [8], Fuchs has shown that the class of near-rings $\mathcal{F}$ is a variety. Obviously, $\mathcal{F}$ contains all zero-symmetric, constant, and abstract affine near-rings. Now we give here some definitions and results of [3], which will be used later.

An element $a \in R$ is called right quasiregular if and only if the right ideal of $R$ generated by the set $\{x - ax \mid x \in R\}$ is $R$. A right ideal (left ideal, ideal, subset) $K$ of $R$ is called a right quasiregular right ideal (left ideal, ideal, subset) of $R$ if each element of $K$ is right quasiregular.

A right ideal $K$ of $R$ is called right modular if there is an element $e \in R$ such that $x - ex \in K$ for all $x \in R$. In this case, we say that $K$ is right modular by $e$.

A maximal right modular right ideal of $R$ is called a right 0-modular right ideal of $R$.

$J_{1/2}^r(R)$ is the intersection of all right 0-modular right ideals of $R$, and if $R$ has no right 0-modular right ideals, then $J_{1/2}^r(R) = R$.

The largest ideal of $R$ contained in $J_{1/2}^r(R)$ is denoted by $J_0^r(R)$ and called the right Jacobson radical of $R$ of type 0.

The largest ideal contained in a right 0-modular right ideal of $R$ is called a right 0-primitive ideal of $R$. $R$ is called a right 0-primitive near-ring if $\{0\}$ is a right 0-primitive ideal of $R$.

A group $(G, +)$ is called a right $R$-group if there is a mapping $(g, r) \mapsto gr$ of $G \times R$ into $G$ such that (i) $(g + h)r = gr + hr$ and (ii) $g(rs) = (gr)s$ for all $g, h \in G$ and $r, s \in R$. A subgroup (normal subgroup) $H$ of a right $R$-group $G$ is called an $R$-subgroup (ideal) of $G$ if $hr \in H$ for all $h \in H$ and $r \in R$.

Let $G$ be a right $R$-group. An element $g \in G$ is called a generator of $G$ if $gR = G$ and $g(r + s) = gr + gs$ for all $r, s \in R$. $G$ is said to be monogenic if $G$ has a generator. $G$ is said to be simple if $G \neq \{0\}$, and $G$ and $\{0\}$ are the only ideals of $G$.

A monogenic right $R$-group $G$ is said to be a right $R$-group of type 0 if $G$ is simple.

The annihilator of a right $R$-group $G$, denoted by $(0 : G)$, is defined as $(0 : G) = \{a \in R \mid Ga = \{0\}\}$.

**Lemma 2.1.** The constant part of $R$ is right quasiregular.

**Lemma 2.2.** A nilpotent element of $R$ is right quasiregular.

**Theorem 2.3.** $J_{1/2}^r(R)$ is the largest right quasiregular right ideal of $R$.

**Theorem 2.4.** $J_0^r(R)$ is the largest right quasiregular ideal of $R$. 
Theorem 2.5. $J'_0(R)$ is the intersection of all right 0-primitive ideals of $R$.

Theorem 2.6. Let $P$ be an ideal of $R$. $P$ is a right 0-primitive ideal of $R$ if and only if $R/P$ is a right 0-primitive near-ring.

Proposition 2.7. Let $G$ be a right $R$-group of type 0 and $g_0$ a generator of $G$. Then $(0 : g_0) := \{ r \in R \mid g_0 r = 0 \}$ is a right 0-modular right ideal of $R$.

Proposition 2.8. Let $G$ be a right $R$-group. $G$ is a right $R$-group of type 0 if and only if there is a maximal right modular right ideal $K$ of $R$ such that $G$ is $R$-isomorphic to $R/K$.

Proposition 2.9. Let $P$ be an ideal of a zero-symmetric near-ring $R$. $P$ is right 0-primitive if and only if $P$ is the largest ideal of $R$ contained in $(0 : G)$ for some right $R$-group $G$ of type 0.

Let $Q$ be a mapping which assigns to each near-ring $R$ an ideal $Q(R)$ of $R$. Such mappings are called ideal-mappings. We consider the following properties which $Q$ may satisfy:

(H1) $h(Q(R)) \subseteq Q(h(R))$ for all homomorphisms $h$ of $R$;
(H2) $Q(R/Q(R)) = \{0\}$ for all $R$;
Q is $r$-hereditary if $I \cap Q(R) \subseteq Q(I)$ for all ideals $I$ of $R$;
Q is $s$-hereditary if $Q(I) \subseteq I \cap Q(R)$ for all ideals $I$ of $R$;
Q is ideal-hereditary if it is both $r$-hereditary and $s$-hereditary, that is, if $Q(I) = I \cap Q(R)$ for all ideals $I$ of $R$.
Q is idempotent if $Q(Q(R)) = Q(R)$ for all $R$;
Q is complete if $Q(I) = I$ and $I$ is an ideal of $R$ that implies $I \subseteq Q(R)$.

With $Q$ we associate two classes of near-rings $R_Q$ and $S_Q$ defined by $R_Q := \{ R \mid Q(R) = R \}$, $S_Q := \{ R \mid Q(R) = \{0\} \}$, and are called a $Q$-radical class and a $Q$-semisimple class, respectively.

An ideal-mapping $Q$ is a Hoehnke radical ($H$-radical) if it satisfies conditions (H1) and (H2).

An ideal-mapping $Q$ is a Kurosh-Amitsur radical ($KA$-radical) if it is a complete idempotent $H$-radical.

Let $\mathbb{M}$ be a class of near-rings. Classes of near-rings are always assumed to be abstract, that is, they contain the one element near-ring and are closed under isomorphic copies. With every near-ring $R$, we associate two ideals of $\mathbb{R}$, depending on $M$. These ideals are defined by the following:

$$
\mathbb{M}(R) := \Sigma\{ I \mid I \text{ is an ideal of } R, I \in \mathbb{M} \},
$$

$$
(R)\mathbb{M} := \cap\{ I \mid I \text{ is an ideal of } R, R/I \in \mathbb{M} \}.
$$

The mapping $P$ defined by $P(R) := (R)\mathbb{M}$ is always an $H$-radical and is called the $H$-radical corresponding to $\mathbb{M}$.

From Theorems 2.5 and 2.6, we have the following.

Proposition 2.10. $J'_0$ is an $H$-radical corresponding to the class of all right 0-primitive near-rings.

3. Properties of the radical $J'_0$

If $(A, +)$ is a group and $T$ is a subset of $A$, then the subgroup (normal subgroup) of $A$ generated by $T$ is denoted by $(T)_{s}((T)_{n})$. 

Remark 3.1. Let $G$ be a right $R$-group. It is clear that $H = \{ g \in G \mid gR = \{0\} \}$ is an ideal of $G$. So if $G$ is simple and $gR = \{0\}$, then $g = 0$ provided $GR \neq \{0\}$.

**Theorem 3.2.** Let $G$ be a right $R$-group of type 0. Suppose that $S$ is an invariant subnear-ring and a right ideal of $R$. If $GS \neq \{0\}$, then $G$ is also a right $S$-group of type 0.

**Proof.** Suppose that $GS \neq \{0\}$. Clearly, $G$ is a right $S$-group. Let $g \in G$ and $gS := \{ gs \mid s \in S \} \subseteq G$. Consider the normal subgroup $(gS)_n$ of $(G, +)$. Let $r \in R$, $h \in (gS)_n$. Now $h = (x_1 + \delta_1(gs_1) - x_1) + (x_2 + \delta_2(gs_2) - x_2) + \cdots + (x_k + \delta_k(gs_k) - x_k)$, $s_i \in S$, $x_i \in G$, $\delta_i \in \{1, -1\}$. Since $SR \subseteq S$, $hr = (x_1r + \delta_1(g(s_1r)) - x_1r) + (x_2r + \delta_2(g(s_2r)) - x_2r) + \cdots + (x_kr + \delta_k(g(s_kr)) - x_kr) \in (gS)_n$. So $(gS)_n$ is an ideal of the right $R$-group $G$, and hence it is also an ideal of the right $S$-group $G$. Let $0 \neq h \in G$. Suppose that $hS = \{0\}$. Since $hR \neq \{0\}$, $(hR)_n$ is a nonzero ideal of the right $R$-group $G$. Since $G$ is a simple right $R$-group, $(hR)_n = G$. So $GS = (hR)_nS \subseteq (hS)_n = \{0\}$, a contradiction to $GS \neq \{0\}$. Therefore, $hS \neq \{0\}$. Let $g_0$ be a generator of the right $R$-group $G$. So $g_0$ is a distributive element of the right $R$-group $G$ and $g_0R = G$. Clearly, $g_0$ is a distributive element of the right $S$-group $G$ and hence $g_0S$ is a subgroup of $(G, +)$. We have $(g_0S)R = g_0(SR)g_0S$. So $g_0S$ is an $R$-subgroup of $G$. Let $g \in G$ and $s \in S$. Since $g_0S = G$, $g = g_0r$ for some $r \in R$. So $g + g_0s = g + g_0s = g_0r + g_0s - g_0r = g_0(r + s - r) \in g_0S$, as $S$ is a normal subgroup of $(R, +)$. Therefore, $g_0S$ is an ideal of the right $R$-group $G$ and hence $g_0S = G$. So $g_0$ is also a generator of the right $S$-group $G$. Let $K$ be a nonzero ideal of the right $S$-group $G$. Let $0 \neq y \in K$. As seen above, $(yS)_n$ is a nonzero ideal of the right $R$-group $G$, and hence $(yS)_n = G$. Since $G = (yS)_n \subseteq K, G = K$. Therefore, $\{0\}$ and $G$ are the only ideals of the right $S$-group $G$ and hence $G$ is a right $S$-group of type 0. \hfill \Box

**Proposition 3.3.** Let $G$ be a right $R$-group of type 0 and let $T$ be a right quasiregular invariant subnear-ring of $R$. If $T$ is a right ideal of $R$, then $GT = \{0\}$.

**Proof.** Suppose that $T$ is a right ideal of $R$ and $g_0$ is a generator of $G$. So $g_0(r + s) = g_0r + g_0s$ for all $r, s \in R$ and $g_0R = G$. Now $L := \{ 0 : g_0 \} = \{ r \in R \mid g_0r = 0 \}$ is a right 0-modular right ideal of $R$. Therefore, $L$ contains the largest right quasiregular right ideal of $R$. Since $T$ is a right quasiregular right ideal of $R$, $T \subseteq L$, that is, $g_0T = \{0\}$. Let $g \in G$ and $t \in T$. Now $g = g_0r$ for some $r \in R$. $gt = g_0(rt) = 0$, as $rt \in T$. Therefore, $GT = \{0\}$. \hfill \Box

Since $R_c$ is right quasiregular in $R$, we have the following.

**Corollary 3.4.** If $R_c$ is a normal subgroup of $(R, +)$, then $GR_c = \{0\}$ for all right $R$-groups $G$ of type 0.

**Corollary 3.5.** Let $R \in \mathcal{R}$. If $G$ is a right $R$-group of type 0, then $GI_0 = \{0\}$.

**Proof.** Let $G$ be a right $R$-group of type 0. We have that $I := I_0 = \{ I \mid G \}$ is the largest right quasiregular ideal of $R$. Since $R_c$ is a right quasiregular ideal of $R$, $R_c \subseteq I$. So $I$ is an invariant ideal of $R$. Therefore, by Proposition 3.3, $GI = \{0\}$. \hfill \Box

**Proposition 3.6.** Let $R \in \mathcal{R}$. Let $I$ be an ideal of $R$ and $K := I + R_c$. If $G$ is a right $K$-group of type 0, then $G$ is a right $I$-group of type 0.

**Proof.** Suppose that $G$ is a right $K$-group of type 0 and $g_0$ is a generator of $G$. So $g_0$ is distributive over $K$ and $g_0K = G$. Let $K_c$ be the constant part of $K$. Since $K_c = R_c$ is a normal subgroup of $K$, by Corollary 3.4, $GR_c = \{0\}$. Clearly, $G$ is a right $I$-group. Now $G = g_0K = g_0(I + R_c) = g_0I$,
and hence $g_0$ is a generator of the right $I$-group $G$. Let $H$ be a nonzero ideal of the right $I$-group $G$. Let $h \in H$ and $k \in K$. $k = i + r_e$, $i \in I$, $r_e \in R_e$ and $h = g_0t$, $t \in I$. $hk = g_0((i + r_e) - ti) = g_0((t(i + r_e) - ti) + g_0(ti) = 0 + (g_0t)i = hi \in H$. Therefore, $H$ is a nonzero ideal of the right $K$-group $G$ and hence $H = G$. So $G$ is a right $I$-group of type $0$. \hfill \Box

We show now that the Hoehnke radical $J_0$ is complete in the variety $\mathcal{F}$.

**Theorem 3.7.** Let $R \in \mathcal{F}$. If $I$ is an ideal of $R$ and $J_0(I) = I$, then $I \subseteq J_0(R)$.

**Proof.** Let $I$ be an ideal of $R$ and $J_0(I) = I$. Suppose that $I \not\subseteq J_0(R)$. So $K := I + R_c$ is an ideal of $R$ and $K \not\subseteq J_0(R)$. We get a right $R$-group $G$ of type $0$ such that $GK \not\subseteq \{0\}$. Since $K$ is an invariant ideal of $R$, by Theorem 3.2, $G$ is a right $K$-group of type $0$. Therefore, by Proposition 3.6, $G$ is a right $I$-group of type $0$. This is a contradiction to the fact that $J_0(I) = I$. Therefore, $I \subseteq J_0(R)$. \hfill \Box

**Theorem 3.8.** $J_0$ is a complete Hoehnke radical in the class of all zero-symmetric near-rings.

**Theorem 3.9.** $J_0$ is a complete Hoehnke radical in the class of all zero-symmetric near-rings.

**Theorem 3.10.** Suppose that $S$ is an invariant subnear-ring of $R$. If $G$ is a right $S$-group of type $0$, then $G$ is also a right $R$-group of type $0$.

**Proof.** Suppose that $G$ is a right $S$-group of type $0$ and $g_0$ is a generator. We have that $g_0$ is distributive over $S$ and $g_0S = G$. For $g \in G$ and $r \in R$, define $gr := g_0(sr)$, if $g = g_0s$, $s \in S$. We show now that this operation is well defined. Suppose that $g = g_0s = g_0t$, $s, t \in S$. Let $r \in R$ and $h := g_0(sr) - g_0(tr)$. Now $hk = (g_0(sr) - g_0(tr))k = g_0((sr)k) - g_0((tr)k) = g_0(s(rk)) - g_0(t(rk)) = g(rk) - g(rk) = 0$ for all $k \in S$. Therefore, $hS = \{0\}$, and hence $h = 0$, that is, $g_0(sr) = g_0(tr)$. We show that $G$ is a right $R$-group of type $0$. It is clear that $G$ is a right $R$-group. $g_0 = g_0e$ for some $e \in S$. Now $G \supseteq g_0R = g_0(eR) \supseteq g_0(eS) = g_0S = G$. So $g_0R = G$. Let $p, q \in R$ and $x = g_0(p + q) - (g_0p + g_0q)$. $xs = (g_0(p + q) - (g_0p + g_0q))s = (g_0(p + q))s - ((g_0p + g_0q))s = g_0(ps + qs) - g_0(ps + g_0qs) = (g_0(ps) + g_0(qs)) - g_0(ps + g_0(qs)) = 0$ for all $s \in S$. Therefore, $x = 0$, and hence $g_0$ is a generator of the right $R$-group $G$. It can be easily verified that the action of $R$ on $G$ is an extension of the action of $S$ on $G$. So an ideal of the right $R$-group $G$ is also an ideal of the right $S$-group $G$. Since the right $S$-group $G$ has no nontrivial ideals, the right $R$-group $G$ also has no nontrivial ideals. Therefore, $G$ is also a right $R$-group of type $0$. \hfill \Box

We show now that the Hoehnke radical $J_0$ is idempotent in the variety $\mathcal{F}$.

**Theorem 3.11.** Let $R \in \mathcal{F}$. Then $J_0(J_0(R)) = J_0(R)$.

**Proof.** Let $I := J_0(R)$. $I$ is the largest right quasiregular ideal of $R$. Since $R_c$ is a right quasi-$I$-group of type $0$. So $I$ is an invariant ideal of $R$. Suppose that $J_0(I) \neq I$. So there is a right $I$-group $G$ of type $0$. By Theorem 3.10, $G$ is an $R$-group of type $0$. Now, by Corollary 3.5, $G = G_{J_0(R)} = \{0\}$. This is contrary to the fact that $G$ is an $R$-group of type $0$. Therefore, $J_0(I) = I$, that is, $J_0(J_0(R)) = J_0(R)$. \hfill \Box

From Theorems 3.7 and 3.11, we have the following.

**Theorem 3.12.** $J_0$ is a Kurosh-Amitsur radical in the variety $\mathcal{F}$.

**Theorem 3.13.** $J_0$ is a Kurosh-Amitsur radical in the class of all zero-symmetric near-rings.
Theorem 3.14. \(J_0^r\) is not \(s\)-hereditary in the class of all zero-symmetric near-rings.

Proof. Consider \(G := \mathbb{Z}_8\), the group of integers under addition modulo 8. Now \(T : G \to G\), defined by \(T(g) = 5g\), for all \(g \in G\), is an automorphism of \(G\). \(T\) fixes 0, 2, 4, and 6, and maps 1 to 5 and 3 to 7. \(A := \{I, T\}\) is an automorphism group of \(G\). \([0], [2], [4], [6], [1, 5],\) and \([3, 7]\) are the orbits. Let \(R\) be the centralizer near-ring \(\mathcal{M}_A(G)\), the near-ring of all self maps of \(G\) which fix 0 and commute with \(T\). An element of \(R\) is completely determined by its action on \([1, 2, 3, 4, 6]\).

An element \(f \in R\) maps \(2G\) into \(2G\) and \(f(1)\) and \(f(3)\) are arbitrary in \(G\). This example was considered in [9] and showed that \(P := \{(0 : 2G) = \{f \in R \mid f(h) = 0, \text{ for all } h \in 2G\}\}\) is the only nontrivial ideal of \(R\). Let \(f_0\) be the element of \(P\) which fixes all the elements in \(G - 2G\). Clearly \(f - f_0f \in (2G : G) = \{t \in R \mid t(G) \subseteq 2G\}\) for all \(f \in R\). Since \((2G : G)\) is a proper right ideal of \(R\), \(f_0\) is not right quasiregular in \(R\). So \(P\) is not a right quasiregular ideal of \(R\). Since \(R\) is a near-ring with identity, it is not right quasiregular. Therefore, \([0]\) is the largest right quasiregular ideal of \(R\), and hence \(J_0^r(R) = [0]\). So \(R\) is \(J_0^r\)-semisimple. It is shown in [9] that \(K := (4G : G)_R\) is a nonzero ideal of \(P\) and \(K^2 = [0]\). Since a nil ideal is right quasiregular, \(K\) is a right quasiregular ideal of \(P\). Therefore, \([0] \neq K \subseteq J_0^r(P)\) and hence \(P\) is not \(J_0^r\)-semisimple. So \(J_0^r\) is not \(s\)-hereditary in the class of all zero-symmetric near-rings.

Corollary 3.15. \(J_0^r\) is not \(s\)-hereditary in the class of all near-rings.

Theorem 3.16. \(J_0^r\) is not an ideal-hereditary radical in the class of all zero-symmetric near-rings.

It is not known to the authors whether \(J_0^r\) is a KA-radical in the class of all near-rings. \(J_0^r\) may fail to be idempotent and thus Kurosh-Amitsur in the class of all near-rings.

References

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