Research Article

Harmonic Maps and Stability on f -Kenmotsu Manifolds

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The purpose of this paper is to study some submanifolds and Riemannian submersions on an f -Kenmotsu manifold. The stability of a φ -holomorphic map from a compact f -Kenmotsu manifold to a Kählerian manifold is proven.

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1. Introduction

In Section 2, we give preliminaries on f -Kenmotsu manifolds. The concept of f -Kenmotsu manifold, where f is a real constant, appears for the first time in the paper of Jannsens and Vanhecke [1]. More recently, Olszak and Roşca [2] defined and studied the f -Kenmotsu manifold by the formula (2.3), where f is a function on M such that $df \wedge \eta = 0$. Here, η is the dual 1-form corresponding to the characteristic vector field ξ of an almost contact metric structure on M . The condition $df \wedge \eta = 0$ follows in fact from (2.3) if $\dim M \geq 5$. This does not hold in general if $\dim M = 3$.

A 1-Kenmotsu manifold is a Kenmotsu manifold (see Kenmotsu [3, 4]. Theorem 2.1 provides a geometric interpretation of an f -Kenmotsu structure.

In Section 3, we initiate a study of harmonic maps when the domain is a compact f -Kenmotsu manifold and the target is a Kähler manifold.

Ianus and Pastore [5, 6] defined a (φ, J) -holomorphic map between an almost contact metric manifold $M(\varphi, \eta, \xi, g)$ and an almost Hermitian manifold $N(J, h)$ as a smooth map $F : M \rightarrow N$ such that the condition $F_* \circ \varphi = J \circ F_*$ is satisfied. Then, the formula $J(\tau(F)) = F_*(\operatorname{div} \varphi) - \operatorname{Tr}_g \beta$ holds, where $\tau(F)$ is the tension field of F and $\beta(X, Y) = (\tilde{\nabla}_X J)(F_* Y)$, $\tilde{\nabla}$ being the connection induced in the pull-back bundle $F^*(TN)$ (see [7]). It is easy to see that in our assumptions $\operatorname{div} \varphi = 0$ and $\operatorname{Tr}_g \beta = 0$ so that a (φ, J) -holomorphic map between an

f -Kenmotsu manifold M and a Kähler manifold N is a harmonic map. If M is a compact manifold, a second-order elliptic operator J_F , called the Jacobi operator, is associated to the harmonic map F . It is well known that the spectrum of J_F consists only of a discrete set of an infinite number of eigenvalues with finite multiplicities, bounded by the first one. We define the *Morse index* of the harmonic map F as the sum of multiplicities of negative eigenvalues of the Jacobi operator J_F [8, 9]. A harmonic map is called *stable* if the Morse index is zero. We have proven that any (φ, J) -holomorphic map from a compact f -Kenmotsu manifold to a Kähler manifold is a stable harmonic map (see [10]).

2. f -Kenmotsu manifolds

A differentiable $(2n + 1)$ -dimensional manifold M is said to have a (φ, ξ, η) -structure or an almost contact structure if there exist a tensor field φ of type $(1, 1)$, a vector field ξ , and a 1-form η on M satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad (2.1)$$

where I denotes the identity transformation.

It seems natural to include also $\varphi\xi = 0$ and $\eta \circ \varphi = 0$; both can be derived from (2.1).

Let g be an associated Riemannian metric on M such that

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y). \quad (2.2)$$

Putting $Y = \xi$ in (2.2) and using (2.1), we get $\eta(X) = g(X, \xi)$, for any vector field X on M .

In this paper, we denote by $C^\infty(M)$ and $\Gamma(E)$ the algebra of smooth functions on M and the $C^\infty(M)$ -module of smooth sections of a vector bundle E , respectively. All manifolds are assumed to be connected and of class C^∞ . Tensors fields, distribution, and so on are assumed to be of class C^∞ if not stated otherwise.

We say that M is an f -Kenmotsu manifold if there exists an almost contact metric structure (φ, ξ, η, g) on M satisfying

$$(\tilde{\nabla}_X \varphi)Y = f(g(\varphi X, Y)\xi - \eta(Y)\varphi X) \quad (2.3)$$

for $X, Y \in \Gamma(TM)$, where f is a smooth function on M such that $df \wedge \eta = 0$.

A 1-Kenmotsu manifold is a Kenmotsu manifold [2, 3].

The following theorem provides a geometric interpretation of any f -Kenmotsu structure.

Theorem 2.1 (Olszak-Roşca). *Let M be an almost contact metric manifold. Then, M is f -Kenmotsu if and only if it satisfies the following conditions:*

- (a) *the distribution $D = \text{Ker } \eta$ is integrable and any leaf of the foliation $\mathcal{F}$ corresponding to D is a totally umbilical hypersurface with constant mean curvature;*
- (b) *the almost Hermitian structure (J, g) induced on an arbitrary leaf is Kähler;*
- (c) $\nabla_\xi \xi = 0$ and $L_\xi \varphi = 0$.

Moreover, we have

$$\tilde{\nabla}_X \xi = f(X - \eta(X)\xi) \quad (2.4)$$

which gives $\operatorname{div} \xi = 2nf$.

The characteristic vector field of an f -Kenmotsu manifold also satisfies

$$R(X, Y)\xi = f^2(\eta(X)Y - \eta(Y)X). \quad (2.5)$$

Levy proven that a second-order symmetric parallel nonsingular tensor on a space of constant curvature is a constant multiple of the metric tensor [11]. On the other hand, Sharma proven that there is no nonzero skew-symmetric second-order parallel tensor on a Sasakian manifold [12]. For an f -Kenmotsu manifold we have the following theorem.

Theorem 2.2. *There is no nonzero parallel 2-form on an f -Kenmotsu manifold.*

Proof. We omit it. □

A plane section p in $T_x \widetilde{M}$, $x \in \widetilde{M}$, of a Kenmotsu manifold ($f = 1$) is called a φ -section if it spanned by a vector X orthogonal to ξ and φX . A connected Kenmotsu manifold $\widetilde{M}$ is called a *Kenmotsu space form* and it is denoted by $\widetilde{M}(c)$ if it has the constant φ -sectional curvature c . The curvature tensor of a Kenmotsu space form $\widetilde{M}(c)$ is given by

$$\begin{aligned} 4R(X, Y)Z = & (c - 3)\{g(Y, Z)X - g(X, Z)Y\} \\ & + (c + 1)\{\eta(X)\eta(Z)Y + -\eta(Y)\eta(Z)X + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\eta \\ & + g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} \end{aligned} \quad (2.6)$$

for any $X, Y, Z \in \Gamma(T\widetilde{M})$.

Now, let $M(J, g')$ be a $2m$ -dimensional almost Hermitian manifold. A surjective map $\pi : \widetilde{M} \rightarrow M$ is called a *contact-complex Riemannian submersion* if it is a Riemannian submersion and satisfies [10]

$$\pi_* \circ \varphi = J \circ \pi_*. \quad (2.7)$$

In [13], we have proven the following theorem.

Theorem 2.3. *Let $\pi : \widetilde{M} \rightarrow M$ be a contact-complex Riemannian submersion from a $(2m + 1)$ -dimensional Kenmotsu manifold $\widetilde{M}$ to a $2m$ -dimensional almost Hermitian manifold M . Then, M is a Kählerian manifold. Moreover, $\widetilde{M}$ is a Kenmotsu space form if and only if M is a complex space form.*

3. Harmonic maps and stability

Let (M, g) and (N, h) be two Riemannian manifolds and $F : M \rightarrow N$ a differentiable map. Then, the second fundamental form α_F of F is defined by

$$\alpha_F(X, Y) = \tilde{\nabla}_X F_* Y - F_*(\nabla_X Y), \quad (3.1)$$

where ∇ is the Levi-Civita connection on M and $\tilde{\nabla}$ is the connection induced by F on the bundle $F^{-1}(TN)$, which is the pull-back of the Levi-Civita connection ∇' on N , and satisfies the following formula (see [8]):

$$\tilde{\nabla}_X F_* Y - \tilde{\nabla}_Y F_* X = F_*([X, Y]), \quad X, Y \in \Gamma(TM). \quad (3.2)$$

The tension field $\tau(F)$ of F is defined as the trace of the second fundamental form α_F , that is $\tau(F)_x = \sum \alpha_F(e_i, e_i)(x)$, where $(e_1, \dots, e_m)$ is an orthonormal basis for $T_x M$ at $x \in M$.

In what follows, we will use Einstein summation convention, so we will omit the sigma symbol.

We say that a map $F : M \rightarrow N$ is a *harmonic map* $\tau(F) = 0$ at $x \in M$.

Examples. (1) If M is the circle S^1 , a map $F : S^1 \rightarrow (N, g)$ is harmonic if and only if it is a geodesic parametrized proportionally to arc length. (2) If $N = \mathbb{R}$, a harmonic map $F : (M, g) \rightarrow \mathbb{R}$ is a harmonic function. (3) A holomorphic map between two Kähler manifolds is harmonic [8]. For examples in the contact metric geometry, see [5, 6, 14].

Now let us consider a variation $F_{s,t} \in C^\infty(M, N)$, with $s, t \in (-\varepsilon, \varepsilon)$ and $F_{0,0} = F$. If the corresponding variation vector fields are denoted by V and W , the Hessian of F is given by

$$H_F(V, W) = \int_M h(J_F(V), W) \mathcal{U}_g, \quad (3.3)$$

where $\mathcal{U}_g$ is the canonical measure associated to the Riemannian metric g and $J_F(V)$ is a second-order self-adjoint operator acting on $\Gamma(F^{-1}(TN))$ by

$$J_F(V) = \sum_i (\tilde{\nabla}_{\nabla_{e_i}} e_i - \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i}) V - \sum_i R'(V, F_* e_i) F_* e_i, \quad (3.4)$$

where R' is the curvature operator on (N, h) .

We say that a map $f : (M, \varphi, \xi, \eta, g) \rightarrow (N, J, h)$ from an almost contact metric manifold to an almost Hermitian manifold is a (φ, J) -holomorphic map if and only if $F_* \circ \varphi = J \circ F_*$.

If $M(\varphi, \xi, \eta, g)$ is a Sasaki manifold and $N(J, h)$ is a Kähler manifold, then any (φ, J) -holomorphic map from M to N is a harmonic map [14].

Then, we can prove the same result for any (φ, J) -holomorphic map from an f -Kenmotsu manifold to a Kähler manifold (see also [15]).

Our main result is the following.

Theorem 3.1. *Let $M(\varphi, \xi, \eta, g)$ be a compact f -Kenmotsu manifold and let $N(J, h)$ be a Kähler manifold. Then, any (φ, J) -holomorphic map $F : M \rightarrow N$ is stable.*

If M is compact, the spectrum of J_F consists only of a discrete set of an infinite number of eigenvalues with finite multiplicities, bounded below by the first one. We define the *Morse index* of the harmonic map $F : M \rightarrow N$ as the *sum of multiplicities of negative eigenvalues of the Jacobi operator J_F* . Equivalently, the Morse index of F equals the dimension of the largest subspace of $\Gamma(F^{-1}(TN))$ on which the Hessian H_F is negative definite (see [8, 9]).

We recall the following formula (see [5, 9]):

$$H_F(V, W) = \int_M (h(\tilde{\nabla}_{e_a} V, \tilde{\nabla}_{e_a} W) + h(R'(F_* e_a, V) F_* e_a, W)) \mathcal{U}_g, \quad (3.5)$$

where we omitted the summation symbol for repeated indices $a = 1, \dots, n$, $n = \dim M$ [5].

Now, let $(e_1, \dots, e_m; f_1, \dots, f_m, \xi)$ be a local orthonormal φ -basis on $M(\varphi, \xi, \eta, g)$ such that $f_i = \varphi e_i$, $i = 1, \dots, m$.

From the (φ, J) -holomorphicity of F and by $\varphi \xi = 0$, we have $F_* \xi = 0$. Thus, from (3.5), we obtain the following.

Lemma 3.2. *Let $F : M \rightarrow N$ be a (φ, J) -holomorphic map from an f -Kenmotsu manifold M to a Kähler manifold N . Then, one has*

$$\begin{aligned} H_F(V, V) &= \int_M (h(\tilde{\nabla}_{e_i} V, \tilde{\nabla}_{e_i} V) + h(\tilde{\nabla}_{f_i} V, \tilde{\nabla}_{f_i} V)) \mathcal{U}_g + \int_M (h(R'(F_* e_i, V) F_* e_i, V) + h(R'(F_* f_i, V) F_* f_i, V)) \mathcal{U}_g. \end{aligned} \quad (3.6)$$

Lemma 3.3. *Let T be a vector field on M such that*

$$g(T, X) = h(\tilde{\nabla}_{\varphi X} V, JV) \quad (3.7)$$

for any $X \in \Gamma(D)$, where $D = \text{Ker } \eta$ and $g(T, \xi) = 0$. Then,

$$\text{div}(T) = h(R'(F_* e_i, F_* f_i) V, JV) + 2h(\tilde{\nabla}_{e_i} JV, \tilde{\nabla}_{f_i} V). \quad (3.8)$$

Proof. Let

$$\begin{aligned} h(R'(F_* e_i, F_* f_i) V, JV) &= h(\tilde{\nabla}_{e_i} \tilde{\nabla}_{f_i} V - \tilde{\nabla}_{f_i} \tilde{\nabla}_{e_i} V - \tilde{\nabla}_{[e_i, f_i]} V, JV) \\ &= e_i h(\tilde{\nabla}_{f_i} V, JV) - h(\tilde{\nabla}_{f_i} V, \tilde{\nabla}_{e_i} JV) - f_i h(\tilde{\nabla}_{e_i} V, JV) \\ &\quad + h(\tilde{\nabla}_{e_i} V, \tilde{\nabla}_{f_i} JV) - h(\tilde{\nabla}_{\nabla_{e_i} f_i} V, JV) + h(\tilde{\nabla}_{\nabla_{f_i} e_i} V, JV). \end{aligned} \quad (3.9)$$

By using (3.7) and (2.3), we obtain

$$\begin{aligned} \text{div}(T) &= g(\nabla_{e_i} T, e_i) + g(\nabla_{f_i} T, f_i) + g(\nabla_{\xi} T, \xi) \\ &= e_i g(T, e_i) - g(T, \nabla_{e_i} e_i) + f_i g(T, f_i) - g(T, \nabla_{f_i} f_i) \\ &= e_i h(\tilde{\nabla}_{f_i} V, JV) - f_i h(\tilde{\nabla}_{e_i} V, JV) + h(\tilde{\nabla}_{\nabla_{f_i} e_i} V, JV) + h(\tilde{\nabla}_{\nabla_{e_i} f_i} V, JV) \end{aligned} \quad (3.10)$$

and (3.8) follows. $\square$

Proposition 3.4. *Let $M(\varphi, \xi, \eta, g)$ be a compact f -Kenmotsu manifold. Then, the function f satisfies*

$$\int_M f \mathcal{U}_g = 0. \quad (3.11)$$

Proof. We have

$$\operatorname{div}(\xi) = g(e_i, \nabla_{e_i} \xi) + g(f_i, \nabla_{f_i} \xi) + g(\xi, \nabla_{\xi} \xi). \quad (3.12)$$

Using (2.1)–(2.4), we obtain $\operatorname{div}(\xi) = -2nf$. Since M is a compact manifold (without boundary), using Stokes's theorem, we have

$$\int_M \operatorname{div}(\xi) \mathcal{U}_g = 0, \quad (3.13)$$

so that (3.11) follows from (3.13). $\square$

Now we are ready to prove Theorem 3.1. Since F is a (φ, J) -holomorphic map, by using the curvature Kähler identity $R'(U, V)JW = JR'(U, V)W$ on $N(J, h)$ and Bianchi's identity, we have

$$R'(F_*e_i, V)F_*e_i + R'(F_*f_i, V)F_*f_i = -JR'(F_*e_i, F_*f_i)V. \quad (3.14)$$

For any $V \in \Gamma(F^{-1}(TN))$, we define the operator

$$DV : \Gamma(TM) \longrightarrow \Gamma(F^{-1}(TN)) \quad (3.15)$$

by the formula

$$DV(X) = \tilde{\nabla}_{\varphi X} V - J\tilde{\nabla}_X V, \quad (3.16)$$

for any $X \in \Gamma(TM)$ (see [5]).

Using Lemmas 3.2, 3.3, and (3.14), by a straightforward calculation, we obtain

$$H_F(V, V) = \frac{1}{2} \int_M (h(DV(e_i), DV(e_i)) + h(DV(f_i), DV(f_i))) \mathcal{U}_g \quad (3.17)$$

because $\int_M \operatorname{div}(T) \mathcal{U}_g = 0$.

Thus, we have $H_F(V, V) \geq 0$ for any $V \in \Gamma(F^{-1}(TN))$, so that F is a stable harmonic map.

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