Research Article

Prime Ideals and Strongly Prime Ideals of Skew Laurent Polynomial Rings

E. Hashemi

School of Mathematical Sciences, Shahrood University of Technology, P.O. Box 316-3619995161, Shahrood, Iran

Correspondence should be addressed to E. Hashemi, eb_hashemi@yahoo.com

Received 26 February 2008; Accepted 28 April 2008

Recommended by Howard Bell

We first study connections between α -compatible ideals of R and related ideals of the skew Laurent polynomials ring $R[x, x^{-1}; \alpha]$, where α is an automorphism of R . Also we investigate the relationship of $P(R)$ and $N_r(R)$ of R with the prime radical and the upper nil radical of the skew Laurent polynomial rings. Then by using Jordan's ring, we extend above results to the case where α is not surjective.

Copyright © 2008 E. Hashemi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Throughout the paper, R always denotes an associative ring with unity. We use $P(R)$, $N_r(R)$, and $N(R)$ to denote the prime radical, the upper nil radical, and the set of all nilpotent elements of R , respectively.

Recall that for a ring R with an injective ring endomorphism $\alpha : R \rightarrow R$, $R[x; \alpha]$ is the Ore extension of R . The set $\{x^j\}_{j \geq 0}$ is easily seen to be a left Ore subset of $R[x; \alpha]$, so that one can localize $R[x; \alpha]$ and form the skew Laurent polynomials ring $R[x, x^{-1}; \alpha]$. Elements of $R[x, x^{-1}; \alpha]$ are finite sum of elements of the form $x^{-j}rx^i$, where $r \in R$ and i, j are nonnegative integers. Multiplication is subject to $xr = \alpha(r)x$ and $rx^{-1} = x^{-1}\alpha(r)$ for all $r \in R$.

Now we consider Jordan's construction of the ring $A(R, \alpha)$ (see [1], for more details). Let $A(R, \alpha)$ or A be the subset $\{x^{-i}rx^i \mid r \in R, i \geq 0\}$ of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$. For each $j \geq 0$, $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{(i+j)}$. It follows that the set of all such elements forms a subring of $R[x, x^{-1}; \alpha]$ with $x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{(i+j)}$ and $(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{(i+j)}$ for $r, s \in R$ and $i, j \geq 0$. Note that α is actually an automorphism of $A(R, \alpha)$, given by $x^{-i}rx^i$ to $x^{-i}\alpha(r)x^i$, for each $r \in R$ and $i \geq 0$. We have $R[x, x^{-1}; \alpha] \cong A[x, x^{-1}; \alpha]$, by way of an isomorphism which

maps $x^{-i}rx^j$ to $\alpha^{-i}(r)x^{j-i}$. For an α -ideal I of R , put $\Delta(I) = \cup_{i \geq 0} x^{-i}Ix^i$. Hence $\Delta(I)$ is α -ideal of A . The constructions $I \rightarrow \Delta(I)$, $J \rightarrow J \cap R$ are inverses, so there is an order-preserving bijection between the sets of α -invariant ideals of R and α -invariant ideals of A .

According to Krempa [2], an endomorphism α of a ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. R is called an α -rigid ring [3] if there exists a rigid endomorphism α of R . Note that any rigid endomorphism of a ring is a monomorphism and α -rigid rings are *reduced* (i.e., R has no nonzero nilpotent element) by Hong et al. [3]. Properties of α -rigid rings have been studied in Krempa [2], Hirano [4], and Hong et al. [3, 5].

On the other hand, a ring R is called *2-primal* if $P(R) = N(R)$ (see [6]). Every reduced ring is obviously a 2-primal ring. Moreover, 2-primal rings have been extended to the class of rings which satisfy $N_r(R) = N(R)$, but the converse does not hold [7, Example 3.3]. Observe that R is a 2-primal ring if and only if $P(R) = N_r(R) = N(R)$, if and only if $P(R)$ is a *completely semiprime ideal* (i.e., $a^2 \in P(R)$ implies that $a \in P(R)$ for $a \in R$) of R . We refer to [6–12] for more detail on 2-primal rings.

Recall that a ring R is called *strongly prime* if R is prime with no nonzero nil ideals. An ideal P of R is *strongly prime* if R/P is a strongly prime ring. All (strongly) prime ideals are taken to be proper. We say an ideal P of a ring R is *minimal (strongly) prime* if P is minimal among (strongly) prime ideals of R . Note that (see [13]) $N_r(R) = \cap \{P \mid P \text{ is a minimal strongly prime ideal of } R\}$.

Recall that an ideal P of R is *completely prime* if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. Every completely prime ideal of R is strongly prime and every strongly prime ideal is prime.

According to Hong et al. [5], for an endomorphism α of a ring R , an α -ideal I is called to be *α -rigid ideal* if $a\alpha(a) \in I$ implies that $a \in I$ for $a \in R$. Hong et al. [5] studied connections between α -rigid ideals of R and related ideals of some ring extensions. Also they studied relationship of $P(R)$ and $N_r(R)$ of R with the prime radical and the upper nil radical of the Ore extension $R[x; \alpha, \delta]$ of R in the cases when either $P(R)$ or $N_r(R)$ is an α -rigid ideal of R and obtaining the following result. Let $P(R)$ (resp., $N_r(R)$) be an α -rigid δ -ideal of R . Then $P(R[x; \alpha, \delta]) \subseteq P(R)[x; \alpha, \delta]$ (resp., $N_r(R[x; \alpha, \delta]) \subseteq N_r(R)[x; \alpha, \delta]$).

In [14], the authors defined α -compatible rings, which are a generalization of α -rigid rings. A ring R is called *α -compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. In this case, clearly the endomorphism α is injective. In [14, Lemma 2.2], the authors showed that R is α -rigid if and only if R is α -compatible and reduced. Thus, the α -compatible ring is a generalization of α -rigid ring to the more general case where R is not assumed to be reduced.

Motivated by the above facts, for an endomorphism α of a ring R , we define *α -compatible ideals* in R which are a generalization of α -rigid ideals. For an ideal I , we say that I is an α -compatible ideal of R if for each $a, b \in R$, $ab \in I \Leftrightarrow a\alpha(b) \in I$. The definition is quite natural, in the light of its similarity with the notion of α -rigid ideals, where in Proposition 2.3, we will show that I is an α -rigid ideal if and only if I is an α -compatible ideal and completely semiprime.

In this paper, we first study connections between α -compatible ideals of R and related ideals of the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$, where α is an automorphism of R . Also we investigate the relationship of $P(R)$ and $N_r(R)$ of R with the prime radical and the upper nil radical of the skew Laurent polynomials. Then by using Jordan's ring, we extend above results to the case where α is not surjective.

2. Prime ideals and strongly prime ideals of skew Laurent polynomial rings

Recall that an ideal I of R is called an α -ideal if $\alpha(I) \subseteq I$; I is called α -invariant if $\alpha^{-1}(I) = I$. If I is an α -ideal, then $\bar{\alpha} : R/I \rightarrow R/I$ defined by $\bar{\alpha}(a + I) = \alpha(a) + I$ is an endomorphism. Then we have the following proposition.

Proposition 2.1. *Let I be an ideal of a ring R . Then the following statements are equivalent:*

- (1) I is an α -compatible ideal;
- (2) R/I is $\bar{\alpha}$ -compatible.

Proof. It is clear. □

Proposition 2.2. *Let I be an α -compatible ideal of a ring R . Then*

- (1) I is α -invariant;
- (2) if $ab \in I$, then $a\alpha^n(b) \in I, \alpha^n(a)b \in I$ for every positive integer n ; conversely, if $a\alpha^k(b) \in I$ or $\alpha^k(a)b \in I$ for some positive integer k , then $ab \in I$.

Proof. This follows from [15, Lemma 2.2 and Proposition 2.3]. □

Recall from [16] that a one-sided ideal I of a ring R has the *insertion of factors property* (or simply, IFP) if $ab \in I$ implies $aRb \subseteq I$ for $a, b \in R$ (Bell in 1970 introduced this notion for $I = 0$).

Proposition 2.3 (see [15, Proposition 2.4]). *Let R be a ring, I an ideal of R , and $\alpha : R \rightarrow R$ an endomorphism of R . Then the following conditions are equivalent:*

- (1) I is α -rigid ideal of R ;
- (2) I is α -compatible, semiprime and has the IFP;
- (3) I is α -compatible and completely semiprime.

For an α -ideal I of R , put $\Delta(I) = \cup_{i \geq 0} \alpha^{-i}(I)$.

Proposition 2.4. (1) *If I is an α -compatible ideal of R , then $\Delta(I)$ is an α -compatible ideal of A .*
 (2) *If J is an α -compatible ideal of A , then $J = \Delta(J_0)$ and J_0 is an α -compatible ideal of R .*
 (3) *If P is a completely (semi)prime α -compatible ideal of R , then $\Delta(P)$ is a completely (semi)prime α -compatible ideal of A .*
 (4) *If Q is a completely (semi)prime α -compatible ideal of A , then $Q = \Delta(Q_0)$ and Q_0 is a completely (semi)prime α -compatible ideal of R .*
 (5) *If P is a prime α -compatible ideal of R , then $\Delta(P)$ is a prime α -compatible ideal of A .*

Proof. (1) Since I is an α -ideal of R , $\Delta(I)$ is an ideal of A . Now, let $(x^{-i}rx^i)(x^{-j}sx^j) \in \Delta(I)$. Hence $x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j} \in \Delta(I)$ and that $\alpha^j(r)\alpha^i(s) \in I$. Thus $\alpha^j(r)\alpha^{i+1}(s) \in I$, since I is α -compatible. Consequently $(x^{-i}rx^i)\alpha(x^{-j}sx^j) \in \Delta(I)$. Therefore $\Delta(I)$ is α -compatible.

(2) Let $J_0 = J \cap J$ and $r \in J_0$. Then $\alpha^n(r) \in J_0$ for each $n \geq 0$. Hence $\alpha^n(x^{-n}rx^n) = r \in J$ for each $n \geq 0$. Thus $x^{-n}rx^n \in J$, since J is α -compatible. Therefore $\Delta(J_0) \subseteq J_0$. Now, let $x^{-m}rx^m \in J$. Then $\alpha^m(x^{-m}rx^m) \in J$ and that $r \in J$, since J is α -compatible. Thus $J \subseteq \Delta(J_0)$. Consequently, $\Delta(J_0) = J$.

(3) Let $(x^{-i}rx^i)(x^{-j}sx^j) \in \Delta(P)$. Then $x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j} \in \Delta(P)$ and that $\alpha^j(r)\alpha^i(s) \in P$. Hence $rs \in P$, by Proposition 2.2. Thus $r \in P$ or $s \in P$, since P is completely prime. Consequently, $x^{-i}rx^i \in \Delta(P)$ or $x^{-j}sx^j \in \Delta(P)$.

(4) By (2), $Q = \Delta(Q_0)$ and Q_0 is a α -compatible ideal of R . Since Q is completely (semi)prime and $Q = \Delta(Q_0)$, hence Q_0 is completely (semi)prime. Let $(x^{-i}rx^i)A(x^{-j}sx^j) \subseteq \Delta(P)$. Then $rRs \subseteq P$, by Proposition 2.2. Hence $r \in P$ or $s \in P$, since P is prime. Consequently $x^{-i}rx^i \in \Delta(P)$ or $x^{-j}sx^j \in \Delta(P)$. Therefore $\Delta(P)$ is a prime ideal of A . $\square$

Theorem 2.5. *Let P be a strongly prime α -compatible ideal of R . Then $\Delta(P)$ is a strongly prime ideal of A .*

Proof. Since P is a prime α -compatible ideal of R , hence $\Delta(P)$ is a prime ideal of A , by Proposition 2.4. We show that $\Delta(P)$ is a strongly prime ideal of A . Assume $J = I/\Delta(P)$ is a nil ideal of $A/\Delta(P)$. Let $a \in I_i$. Then $x^{-i}ax^i \in I$. Since $I/\Delta(P)$ is a nil ideal, hence $(x^{-i}ax^i)^n \in \Delta(P)$ for some $n \geq 0$. Hence $x^{-i}a^n x^i = x^{-j}px^j$ for some $p \in P$ and $j \geq 0$. Thus $\alpha^j(a^n) = \alpha^j(p) \in P$. Hence $a^n \in P$, since P is α -compatible. Then $(I_i + P)/P$ is a nil ideal of R/P for each $i \geq 0$. Hence $I_i \subseteq P$, for each $i \geq 0$. Therefore $I \subseteq \Delta(P)$. Consequently, $\Delta(P)$ is a strongly prime ideal of A . $\square$

Note that if I is an α -ideal of R , then $I[x, x^{-1}; \alpha]$ is an ideal of the skew Laurent polynomials ring $R[x, x^{-1}; \alpha]$.

Theorem 2.6. *Let α be an automorphism of R . Let I be a semiprime α -compatible ideal of R . Assume $f(x) = \sum_{i=r}^n a_i x^i$ and $g(x) = \sum_{j=s}^m b_j x^j \in R[x, x^{-1}; \alpha]$. Then the following statements are equivalent:*

- (1) $f(x)R[x, x^{-1}; \alpha]g(x) \subseteq I[x, x^{-1}; \alpha]$;
- (2) $a_i R b_j \subseteq I$ for each i, j .

Proof. (1) $\Rightarrow$ (2). Assume $f(x)R[x, x^{-1}; \alpha]g(x) \subseteq I[x, x^{-1}; \alpha]$. Then

$$(a_r x^r + \cdots + a_n x^n)c(b_s x^s + \cdots + b_m x^m) \in I[x, x^{-1}; \alpha] \quad \text{for each } c \in R. \quad (\dagger)$$

Hence $a_n \alpha^n(c b_m) \in I$. Thus $a_n c b_m \in I$, since I is α -compatible. Next, replacing c by $c b_{m-1} d a_n e$, where $c, d, e \in R$. Then $(a_r x^r + \cdots + a_n x^n) c b_{m-1} d a_n e (b_s x^s + \cdots + b_{m-1} x^{m-1}) \in I[x, x^{-1}; \alpha]$. Hence $a_n \alpha^n(c b_{m-1} d a_n e b_{m-1}) \in I$ and that $a_n c b_{m-1} d a_n e b_{m-1} \in I$, since I is α -compatible. Thus $(R a_n R b_{m-1})^2 \subseteq I$. Hence $R a_n R b_{m-1} \subseteq I$, since I is semiprime. Continuing this process, we obtain $a_n R b_k \subseteq I$, for $k = s, \dots, m$. Hence from α -compatibility of I , we get $(a_r x^r + \cdots + a_n x^n) R[x, x^{-1}; \alpha] (b_s x^s + \cdots + b_{m-1} x^{m-1}) \subseteq I[x, x^{-1}; \alpha]$. Using induction on $n + m$, we obtain $a_i R b_j \subseteq I$ for each i, j .

(2) $\Rightarrow$ (1). It follows from Proposition 2.2. $\square$

Corollary 2.7. *Let α be an automorphism on R . If I is a (semi)prime α -compatible ideal of R , then $I[x, x^{-1}; \alpha]$ is a (semi)prime ideal of $R[x, x^{-1}; \alpha]$.*

Proof. Assume that I is a prime α -compatible ideal of R . Let $f(x) = \sum_{i=r}^n a_i x^i$ and $g(x) = \sum_{j=s}^m b_j x^j \in R[x, x^{-1}; \alpha]$ such that $f(x)R[x, x^{-1}; \alpha]g(x) \subseteq I[x, x^{-1}; \alpha]$. Then $a_i R b_j \subseteq I$ for each i, j , by Theorem 2.6. Assume $g(x) \notin I[x, x^{-1}; \alpha]$. Hence $b_j \notin I$ for some j . Thus $a_i \in I$ for each $i = r, \dots, n$, since I is prime. Therefore $f(x) \in I[x, x^{-1}; \alpha]$. Consequently, $I[x, x^{-1}; \alpha]$ is a prime ideal of $R[x, x^{-1}; \alpha]$. $\square$

Theorem 2.8. *If each minimal prime ideal of R is α -compatible, then $P(R[x, x^{-1}; \alpha]) \subseteq \Delta(P(R))[x, x^{-1}; \alpha]$.*

Proof. Let P be a minimal prime ideal of R . By Proposition 2.4, $\Delta(P)$ is a α -compatible ideal of A . Assume $(a^{-i}rx^i)A(x^{-j}sx^j) \subseteq \Delta(P)$. Then $rRs \subseteq P$, since $\Delta(P)$ is α -compatible. Hence $r \in P$ or $s \in P$. Thus $a^{-i}rx^i \in \Delta(P)$ or $x^{-j}sx^j \in \Delta(P)$. Therefore $\Delta(P)$ is a prime ideal of A . Thus $\Delta(P)[x, x^{-1}; \alpha]$ is a prime ideal of $A[x, x^{-1}; \alpha]$, by Corollary 2.7. Consequently, $P(R[x, x^{-1}; \alpha]) \subseteq \Delta(P(R))[x, x^{-1}; \alpha]$. $\square$

In [14], the authors give some examples of α -compatible rings however they are not α -rigid. Note that there exists a ring R for which every nonzero proper ideal is α -compatible but R is not α -compatible. For example, let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field, and the endomorphism α of R is defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ for $a, b, c \in F$.

The following examples show that there exists α -compatible ideals which are not α -rigid.

Example 2.9 (see [15], Example 2.5). Let F be a field. Let $R = \{\begin{pmatrix} f & f_1 \\ 0 & f \end{pmatrix} \mid f, f_1 \in F[x]\}$, where $F[x]$ is the ring of polynomials over F . Then R is a subring of the 2×2 matrix ring over the ring $F[x]$. Let $\alpha : R \rightarrow R$ be an automorphism defined by $\alpha\left(\begin{pmatrix} f & f_1 \\ 0 & f \end{pmatrix}\right) = \begin{pmatrix} f & uf_1 \\ 0 & f \end{pmatrix}$, where u is a fixed nonzero element of F . Let $p(x)$ be an irreducible polynomial in $F[x]$. Let $I = \{\begin{pmatrix} 0 & f_1 \\ 0 & 0 \end{pmatrix} \mid f_1 \in \langle p(x) \rangle\}$, where $\langle p(x) \rangle$ is the principal ideal of $F[x]$ generated by $p(x)$. Then I is an α -compatible ideal of R but is not α -rigid. Indeed, since $\begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix} \alpha\left(\begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in I$, but $\begin{pmatrix} 0 & g(x) \\ 0 & 0 \end{pmatrix} \notin I$ for $g(x) \notin \langle p(x) \rangle$. Thus I is not α -rigid.

Example 2.10 (see [17], Example 2). Let $\mathbb{Z}_2$ be the field of integers modulo 2 and $A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c]$ be the free algebra of polynomials with zero constant term in noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Note that A is a ring without unity. Consider an ideal of $\mathbb{Z}_2 + A$, say I , generated by $a_0b_0, a_1b_2 + a_2b_1, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_2b_2, a_0rb_0, a_2rb_2, (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2)$ with $r \in A$ and $r_1r_2r_3r_4$ with $r_1, r_2, r_3, r_4 \in A$. Then I has the IFP. Let $\alpha : R \rightarrow R$ be an inner automorphism (i.e., there exists an invertible element $u \in R$ such that $\alpha(r) = u^{-1}ru$ for each $r \in R$). Then I is α -compatible, since I has the IFP. But I is not α -rigid, since I is not completely semiprime.

Theorem 2.11. *Let α be an automorphism of R . If each minimal prime ideal of R is α -compatible, then $P(R[x, x^{-1}; \alpha]) \subseteq P(R)[x, x^{-1}; \alpha]$.*

Proof. It follows from Corollary 2.7. $\square$

The following example shows that there exists a ring R such that all minimal prime ideals are α -compatible, but are not α -rigid.

Example 2.12 (see [15], Example 2.11). Let $R = \text{Mat}_2(\mathbb{Z}_4)$ be the 2×2 matrix ring over the ring $\mathbb{Z}_4$. Then $P(R) = \{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{ij} \in 2\mathbb{Z}\}$ is the only prime ideal of R . Let $\alpha : R \rightarrow R$ be the endomorphism defined by $\alpha\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix}$. Then α is an automorphism of R and $P(R)$ is α -compatible. However, $P(R)$ is not α -rigid, since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \alpha\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \in P(R)$, but $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \notin P(R)$.

Theorem 2.13. *Let α be an automorphism of R . If P is a completely (semi)prime α -compatible ideal of R , then $P[x, x^{-1}; \alpha]$ is a completely (semi)prime ideal of $R[x, x^{-1}; \alpha]$.*

Proof. Let P be a completely prime ideal of R . R/P is domain, hence it is a reduced ring. R/P is an $\bar{\alpha}$ -compatible ring, hence R/P is $\bar{\alpha}$ -rigid, by [14, Lemma 2.2]. Let $\overline{f(x)}, \overline{g(x)} \in R/P[x, x^{-1}; \bar{\alpha}]$ such that $\overline{f(x)}\overline{g(x)} = 0$. Then $\overline{f(x)} = 0$ or $\overline{g(x)} = 0$, by a same way as used in [3, Proposition 6]. Thus $R[x, x^{-1}; \alpha]/P[x, x^{-1}; \alpha] \cong R/P[x, x^{-1}; \bar{\alpha}]$ is domain and $P[x, x^{-1}; \alpha]$ is a completely prime ideal of $R[x, x^{-1}; \alpha]$. $\square$

Corollary 2.14. *Let α be an automorphism on R . If $P(R)$ is an α -rigid ideal of R , then $P(R[x, x^{-1}; \alpha]) \subseteq P(R)[x, x^{-1}; \alpha]$.*

Proof. $P(R)$ is α -rigid, hence $P(R)$ is a completely semiprime α -compatible ideal of R , by Proposition 2.3. Therefore $P(R[x, x^{-1}; \alpha]) \subseteq P(R)[x, x^{-1}; \alpha]$, by Theorem 2.13. $\square$

Theorem 2.15. *Let α be an automorphism of R . If P is a strongly (semi)prime α -compatible ideal of R , then $P[x, x^{-1}; \alpha]$ is a strongly (semi)prime ideal of $R[x, x^{-1}; \alpha]$.*

Proof. By Corollary 2.7, $P[x, x^{-1}; \alpha]$ is a prime ideal of $R[x, x^{-1}; \alpha]$. Hence $R[x, x^{-1}; \alpha]/P[x, x^{-1}; \alpha] \cong R/P[x, x^{-1}; \bar{\alpha}]$ is a prime ring. We claim that zero is the only nil ideal of $R/P[x, x^{-1}; \bar{\alpha}]$. Let J be a nil ideal of $R/P[x, x^{-1}; \bar{\alpha}]$. Assume I be the set of all leading coefficients of elements of J . First we show that I is an ideal of R/P . Clearly, I is a left ideal of R/P . Let $\bar{a} \in I$ and $\bar{r} \in R/P$. Then there exists $\overline{f(x)} = \bar{a}_0 + \cdots + \bar{a}_{n-1}x^{n-1} + \bar{a}x^n \in J$. Hence $(\overline{f(x)}\bar{r})^m = 0$, for some nonnegative integers m . Thus $\bar{a}\bar{a}^n(\bar{r}\bar{a}) \cdots \bar{a}^{(m-1)n}(\bar{r}\bar{a})\bar{a}^m(\bar{r}) = 0$, since it is the leading coefficient of $(\overline{f(x)}\bar{r})^m$. Therefore $(\bar{a}\bar{r})^m = 0$, since R/P is $\bar{\alpha}$ -compatible. Consequently, I is an ideal of R/P . Clearly I is a nil ideal of R/P . Hence $I = 0$ and so $J = 0$. Therefore $P[x, x^{-1}; \alpha]$ is a strongly prime ideal of $R[x, x^{-1}; \alpha]$. $\square$

Theorem 2.16. *Let α be an automorphism of R . If each minimal strongly prime ideal of R is α -compatible, then $N_r(R[x, x^{-1}; \alpha]) \subseteq N_r(R)[x, x^{-1}; \alpha]$.*

Corollary 2.17. *Let α be an automorphism of R . If $N_r(R)$ is an α -rigid ideal of R , then $N_r(R[x, x^{-1}; \alpha]) \subseteq N_r(R)[x, x^{-1}; \alpha]$.*

Proof. $N_r(R)$ is α -rigid, hence $N_r(R)$ is a completely semiprime α -compatible ideal of R , by Proposition 2.3, and that $N_r(R)$ is a strongly semiprime ideal of R . Therefore $N_r(R[x, x^{-1}; \alpha]) \subseteq N_r(R)[x, x^{-1}; \alpha]$, by Theorem 2.15. $\square$

Example 2.12 also shows that there exists a ring R such that all minimal strongly prime ideals are α -compatible, but are not α -rigid.

Theorem 2.18. *Assume each minimal prime ideal of R is α -compatible. Then the following are equivalent:*

- (1) $P(R[x, x^{-1}; \alpha])$ is completely semiprime;
- (2) $P(R[x, x^{-1}; \alpha]) = \Delta(P(R))[x, x^{-1}; \alpha]$ and $P(R)$ is completely semiprime.

Proof. (1) $\Rightarrow$ (2). Suppose that $P(R[x, x^{-1}; \alpha])$ is a completely semiprime ideal of $R[x, x^{-1}; \alpha]$. It is enough to show that $\Delta(P(R))[x, x^{-1}; \alpha] \subseteq P(R[x, x^{-1}; \alpha])$, by Theorem 2.8. Let Q be a minimal prime ideal of $R[x, x^{-1}; \alpha]$ and $P = A \cap Q$. Since $P(R[x, x^{-1}; \alpha])$ is a completely semiprime ideal of $R[x, x^{-1}; \alpha]$, P is a completely semiprime ideal of A . Clearly P is an α -invariant ideal of A . Hence $P = \Delta(P_0)$. We claim that P_0 is a minimal prime ideal of

R . Since P is completely prime, P_0 is a completely prime ideal of R . Let I be a minimal prime ideal of R such that $I \subseteq P_0$. By assumption, I is α -compatible. Hence $\Delta(I)$ is a prime α -compatible ideal of A . Thus $\Delta(I)[x, x^{-1}; \alpha]$ is a prime ideal of $R[x, x^{-1}; \alpha]$ and $\Delta(I)[x, x^{-1}; \alpha] \subseteq \Delta(P_0)[x, x^{-1}; \alpha] \subseteq Q$. Since Q is a minimal prime ideal of $R[x, x^{-1}; \alpha]$, hence $\Delta(I)[x, x^{-1}; \alpha] = \Delta(P_0)[x, x^{-1}; \alpha] = Q$. Therefore $\Delta(I) = \Delta(P_0)$ and that $I = P_0$. Consequently $\Delta(P(R))[x, x^{-1}; \alpha] \subseteq P(R[x, x^{-1}; \alpha])$ and that $P(R[x, x^{-1}; \alpha]) = \Delta(P(R))[x, x^{-1}; \alpha]$. Since $P(R[x, x^{-1}; \alpha]) = \Delta(P(R))[x, x^{-1}; \alpha]$ and $P(R[x, x^{-1}; \alpha])$ is completely semiprime, hence $P(R)$ is completely semiprime.

(2) $\Rightarrow$ (1). Since $P(R)$ is α -compatible and completely semiprime, $\Delta(P(R))$ is an α -compatible completely semiprime ideal of A . Hence $\Delta(P(R))[x, x^{-1}; \alpha]$ is a completely semiprime ideal of $A[x, x^{-1}; \alpha] = R[x, x^{-1}; \alpha]$. Thus $P(R[x, x^{-1}; \alpha]) = \Delta(P(R))[x, x^{-1}; \alpha]$ is a completely semiprime ideal of $R[x, x^{-1}; \alpha]$. $\square$

Corollary 2.19. *Let α be an automorphism of R . Let each minimal prime ideal of R be α -compatible. Then the following are equivalent:*

- (1) $P(R[x, x^{-1}; \alpha])$ is completely semiprime;
- (2) $P(R[x, x^{-1}; \alpha]) = P(R)[x, x^{-1}; \alpha]$ and $P(R)$ is completely semiprime.

Acknowledgments

The author thanks the referee for his valuable comments and suggestions. This research is supported by the Shahrood University of Technology of Iran.

References

- [1] D. A. Jordan, "Bijective extensions of injective ring endomorphisms," *Journal of the London Mathematical Society*, vol. 25, no. 3, pp. 435–448, 1982.
- [2] J. Krempa, "Some examples of reduced rings," *Algebra Colloquium*, vol. 3, no. 4, pp. 289–300, 1996.
- [3] C. Y. Hong, N. K. Kim, and T. K. Kwak, "Ore extensions of Baer and p.p.-rings," *Journal of Pure and Applied Algebra*, vol. 151, no. 3, pp. 215–226, 2000.
- [4] Y. Hirano, "On the uniqueness of rings of coefficients in skew polynomial rings," *Publicationes Mathematicae Debrecen*, vol. 54, no. 3-4, pp. 489–495, 1999.
- [5] C. Y. Hong, T. K. Kwak, and S. T. Rizvi, "Rigid ideals and radicals of Ore extensions," *Algebra Colloquium*, vol. 12, no. 3, pp. 399–412, 2005.
- [6] G. F. Birkenmeier, H. E. Heatherly, and E. K. Lee, "Completely prime ideals and associated radicals," in *Ring Theory: Proceedings of the Biennial Ohio State-Dension Conference May 1992*, S. K. Jain and S. T. Rizvi, Eds., pp. 102–129, World Scientific, River Edge, NJ, USA, 1993.
- [7] G. F. Birkenmeier, J. Y. Kim, and J. K. Park, "Polynomial extensions of Baer and quasi-Baer rings," *Journal of Pure and Applied Algebra*, vol. 159, no. 1, pp. 25–42, 2001.
- [8] Y. Hirano, "Some studies on strongly π -regular rings," *Mathematical Journal of Okayama University*, vol. 20, no. 2, pp. 141–149, 1978.
- [9] C. Y. Hong and T. K. Kwak, "On minimal strongly prime ideals," *Communications in Algebra*, vol. 28, no. 10, pp. 4867–4878, 2000.
- [10] N. K. Kim and T. K. Kwak, "Minimal prime ideals in 2-primal rings," *Mathematica Japonica*, vol. 50, no. 3, pp. 415–420, 1999.
- [11] G. Shin, "Prime ideals and sheaf representation of a pseudo symmetric ring," *Transactions of the American Mathematical Society*, vol. 184, pp. 43–60, 1973.
- [12] S.-H. Sun, "Noncommutative rings in which every prime ideal is contained in a unique maximal ideal," *Journal of Pure and Applied Algebra*, vol. 76, no. 2, pp. 179–192, 1991.
- [13] L. H. Rowen, *Ring Theory*. Vol. I, vol. 127 of *Pure and Applied Mathematics*, Academic Press, Boston, Mass, USA, 1988.

- [14] E. Hashemi and A. Moussavi, "Polynomial extensions of quasi-Baer rings," *Acta Mathematica Hungarica*, vol. 107, no. 3, pp. 207–224, 2005.
- [15] E. Hashemi, "Compatible ideals and radicals of Ore extensions," *New York Journal of Mathematics*, vol. 12, pp. 349–356, 2006.
- [16] G. Mason, "Reflexive ideals," *Communications in Algebra*, vol. 9, no. 17, pp. 1709–1724, 1981.
- [17] C. Huh, Y. Lee, and A. Smoktunowicz, "Armendariz rings and semicommutative rings," *Communications in Algebra*, vol. 30, no. 2, pp. 751–761, 2002.