Research Article

Generalized Newman Phenomena and Digit Conjectures on Primes

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We prove that the ratio of the Newman sum over numbers multiple of a fixed integer, which is not a multiple of 3, and the Newman sum over numbers multiple of a fixed integer divisible by 3 is \( o(1) \) when the upper limit of summing tends to infinity. We also discuss a connection of our results with a digit conjecture on primes.

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1. Introduction

Denote for \( x, m \in \mathbb{N} \),

\[
S_m(x) = \sum_{0 \leq n < x, n \equiv 0 \pmod{m}} (-1)^{s(n)},
\]

where \( s(n) \) is the number of 1’s in the binary expansion of \( n \). Sum (1.1) is a \textit{Newman digit sum}. From the fundamental paper of Gelfond [1], it follows that

\[
S_m(x) = O(x^{\lambda}), \quad \lambda = \frac{\ln 3}{\ln 4}.
\] (1.2)

The case \( m = 3 \) was studied in detail in [2–4].

So, from Coquet’s theorem [3, 5] it follows that

\[
-\frac{1}{3} + \frac{2}{\sqrt{3}}x^{1/3} \leq S_3(3x) \leq \frac{1}{3} + \frac{55}{3} \left(\frac{3}{65}\right)^{1/3} x^{1/3}
\] (1.3)
with a microscopic improvement [4]:

\[
\frac{2}{\sqrt{3}} x^\lambda \leq S_3(3x) \leq \frac{55}{3} \left( \frac{3}{65} \right)^{1/3} x^\lambda, \quad x \geq 2, \tag{1.4}
\]

and, moreover,

\[
\left\lfloor 2 \left( \frac{x}{6} \right)^{1/3} \right\rfloor \leq S_3(x) \leq \left\lceil \frac{55}{3} \left( \frac{x}{65} \right)^{1/3} \right\rceil. \tag{1.5}
\]

These estimates give the most exact modern limits of the so-called *Newman phenomena*. Note that Drmota and Skalba [6], using a close function \((S_m(x))\), proved that if \(m\) is a multiple of 3, then for sufficiently large \(x\),

\[
S_m(x) > 0, \quad x \geq x_0(m). \tag{1.6}
\]

In this paper, we study a general case for \(m \geq 5\) (in the cases of \(m = 2\) and \(m = 4\), we have \(|S_m(n)| \leq 1\)).

To formulate our results, put for \(m \geq 5\),

\[
\lambda_m = 1 + \log_2 b_m, \tag{1.7}
\]

\[
\mu_m = \frac{2}{2b_m - 1}, \tag{1.8}
\]

where

\[
b_m^2 = \begin{cases} 
\sin \left( \frac{\pi}{3} \left( 1 + \frac{3}{m} \right) \right) \left( \sqrt{3} - \sin \left( \frac{\pi}{3} \left( 1 + \frac{3}{m} \right) \right) \right), & \text{if } m \equiv 0 \pmod{3}, \\
\sin \left( \frac{\pi}{3} \left( 1 - \frac{1}{m} \right) \right) \left( \sqrt{3} - \sin \left( \frac{\pi}{3} \left( 1 - \frac{1}{m} \right) \right) \right), & \text{if } m \equiv 1 \pmod{3}, \\
\sin \left( \frac{\pi}{3} \left( 1 + \frac{1}{m} \right) \right) \left( \sqrt{3} - \sin \left( \frac{\pi}{3} \left( 1 + \frac{1}{m} \right) \right) \right), & \text{if } m \equiv 2 \pmod{3}.
\end{cases} \tag{1.9}
\]

Directly, one can see that

\[
\frac{\sqrt{3}}{2} > b_m \geq \begin{cases} 
0.86184088 \ldots, & \text{if } (m, 3) = 1, \\
0.8559967 \ldots, & \text{if } (m, 3) = 3.
\end{cases} \tag{1.10}
\]
and thus,

\[ \lambda_m < \lambda, \]

\[ 2.73205080 \cdots < \mu_m \leq \begin{cases} 
    2.76364572 \cdots, & \text{if } (m, 3) = 1, \\
    2.81215109 \cdots, & \text{if } (m, 3) = 3.
\end{cases} \]  

Below, we prove the following results.

**Theorem 1.1.** If \((m, 3) = 1\), then

\[ |S_m(x)| \leq 1 + \mu_m x^{\lambda_m}. \]  

**Theorem 1.2** (Generalized Newman phenomena). If \(m > 3\) is a multiple of 3, then

\[ \left| S_m(x) - \frac{3}{m} S_3(x) \right| \leq 1 + \mu_m x^{\lambda_m}. \]  

Using Theorem 1.2 and (1.5), one can estimate \(x_0(m)\) in (1.6). For example, one can prove that \(x_0(21) < e^{909}\).

**2. Explicit formula for** \(S_m(N)\)

We have

\[ S_m(N) = \sum_{n=0, m|n}^{N-1} (-1)^{s(n)} \]

\[ = \frac{1}{m} \sum_{l=0}^{m-1} \sum_{n=0}^{N-1} (-1)^{s(n)} e^{2\pi i (sl/n)} \]

\[ = \frac{1}{m} \sum_{l=0}^{m-1} \sum_{n=0}^{N-1} e^{2\pi i ((l/m)n + (1/2)s(n))}. \]  

Note that the interior sum has the form

\[ F_{\alpha}(N) = \sum_{n=0}^{N-1} e^{2\pi i (\alpha n + (1/2)s(n))}, \quad 0 \leq \alpha < 1. \]  

**Lemma 2.1.** If \(N = 2^{v_0} + 2^{v_1} + \cdots + 2^{v_r}, \quad v_0 > v_1 > \cdots > v_r \geq 0,\) then

\[ F_{\alpha}(N) = \sum_{l=0}^{r} \sum_{i=0}^{2^{v_{l-1}} - 1} e^{2\pi i (\alpha \sum_{j=0}^{l-1} 2^{v_j} + h/2) \sum_{k=0}^{v_{l-1}} (1 + e^{2\pi i (a2^k + 1/2)})}, \]  

where as usual \(\sum_{j=0}^{l-1} = 0, \prod_{k=0}^{v_{l-1}} = 1.\)
Proof. Let \( r = 0 \), then by (2.2),

\[
F_a(N) = \sum_{n=0}^{N-1} (-1)^{s(n)} e^{2\pi i a n}
\]

\[
= 1 - \sum_{j=0}^{\nu_r-1} e^{2\pi i a 2^j} + \sum_{0 \leq j_1 < j_2 \leq \nu_r-1} e^{2\pi i a (2^{j_1} + 2^{j_2})} - \ldots
\]

(2.4)

which corresponds to (2.3) for \( r = 0 \).

Assuming that (2.3) is valid for every \( N \) with \( s(N) = r + 1 \), let us consider \( N_1 = 2^{\nu_r} a + 2^{\nu_{r-1}} \) where \( a \) is odd, \( s(a) = r + 1 \), and \( \nu_{r+1} < \nu_r \). Let

\[
N = 2^{\nu_r} a = 2^{\nu_0} + \ldots + 2^{\nu_r};
\]

\[
N_1 = 2^{\nu_0} + \ldots + 2^{\nu_r} + 2^{\nu_{r-1}}.
\]

(2.5)

Notice that for \( n \in [0, 2^{\nu_{r-1}}) \), we have

\[
s(N + n) = s(N) + s(n).
\]

(2.6)

Therefore,

\[
F_a(N_1) = F_a(N) + \sum_{n=0}^{N_1} e^{2\pi i (an + (1/2)s(n))}
\]

\[
= F_a(N) + \sum_{n=0}^{2^{\nu_{r-1}}-1} e^{2\pi i (an + \alpha N + (1/2)(s(N)+s(n)))}
\]

(2.7)

\[
= F_a(N) + e^{2\pi i (a N + (1/2)s(N))} \sum_{n=0}^{2^{\nu_{r-1}}-1} e^{2\pi i (an + (1/2)s(n))}.
\]

Thus, by (2.3) and (2.4),

\[
F_a(N_1) = \sum_{h=0}^{r} e^{2\pi i (a \sum_{j=0}^{h-1} 2^j + h/2)} \prod_{k=0}^{\nu_h-1} (1 + e^{2\pi i (a 2^k + 1/2)}
\]

\[
+ e^{2\pi i (a \sum_{j=0}^{h} 2^j + (r+1)/2)} \prod_{k=0}^{\nu_{h+1}-1} (1 + e^{2\pi i (a 2^k + 1/2)})
\]

(2.8)

\[
= \sum_{h=0}^{r+1} e^{2\pi i (a \sum_{j=0}^{h-1} 2^j + h/2)} \prod_{k=0}^{\nu_h-1} (1 + e^{2\pi i (a 2^k + 1/2)})
\]

\[\square\]
Formulas (2.1)–(2.3) give an explicit expression for $S_m(N)$ as a linear combination of the products of the form

$$\prod_{k=0}^{n-1} \left(1 + e^{2\pi i (a^{2k+1}/2)}\right), \quad \alpha = \frac{t}{m}, \quad 0 \leq t \leq m - 1. \quad (2.9)$$

Remark 2.2. One can extract (2.3) from a very complicated general Gelfond formula [1], however, we prefer to give an independent proof.

3. Proof of Theorem 1.1

Note that in (2.3)

$$r \leq n_0 = \left\lfloor \frac{\ln N}{\ln 2} \right\rfloor. \quad (3.1)$$

By Lemma 2.1, we have

$$|F_{\alpha}(N)| \leq \sum_{n_0=n_0, n_1, \ldots, n_r} \left| \prod_{k=1}^{n_0} \left(1 + e^{2\pi i (a^{2k+1}/2)}\right) \right| \leq \sum_{n_0} \prod_{k=1}^{n_0} \left| 1 + e^{2\pi i (a^{2k+1}/2)} \right|. \quad (3.2)$$

Furthermore,

$$1 + e^{2\pi i (2^{k-1}a + 1/2)} = 2 \sin \left(2^{k-1}a\pi\right) \left(\sin \left(2^{k-1}a\pi\right) - i\cos \left(2^{k-1}a\pi\right)\right) \quad (3.3)$$

and, therefore,

$$\left| 1 + e^{2\pi i (2^{k-1}a + 1/2)} \right| \leq 2 \left| \sin \left(2^{k-1}a\pi\right) \right|. \quad (3.4)$$

According to (3.2), let us estimate the product

$$\prod_{k=1}^{n_0} \left(2 \left| \sin \left(2^{k-1}a\pi\right) \right| \right) = 2^n_0 \prod_{k=1}^{n_0} \left| \sin \left(2^{k-1}a\pi\right) \right|, \quad (3.5)$$

where by (2.1),

$$\alpha = \frac{t}{m}, \quad 0 \leq t \leq m - 1. \quad (3.6)$$
Repeating arguments of [1], put

\[ | \sin \left( 2^{k-1} \alpha \pi \right) | = t_k. \quad (3.7) \]

Considering the function

\[ \rho(x) = 2x \sqrt{1 - x^2}, \quad 0 \leq x \leq 1, \quad (3.8) \]

we have

\[ t_k = 2t_{k-1} \sqrt{1 - t_{k-1}^2} = \rho(t_{k-1}). \quad (3.9) \]

Note that

\[ \rho'(x) = 2 \left( \sqrt{1 - x^2} - \frac{x^2}{\sqrt{1 - x^2}} \right) \leq -1 \quad (3.10) \]

for \( x_0 \leq x \leq 1 \), where

\[ x_0 = \frac{\sqrt{3}}{2} \quad (3.11) \]

is the only positive root of the equation \( \rho(x) = x \).

Show that either

\[ t_k \leq \sin \left( \frac{\pi}{m} \left\lfloor \frac{m}{3} \right\rfloor \right) = \sin \left( \frac{\pi}{m} \left\lceil \frac{2m}{3} \right\rceil \right) = g_m < \frac{\sqrt{3}}{2} \quad (3.12) \]

or, simultaneously, \( t_k > g_m \), and

\[ t_k t_{k+1} \leq \max_{0 \leq l \leq m-1} \left( \left| \sin \left( \frac{l \pi}{m} \right) \left( \sqrt{3} - \sin \left( \frac{l \pi}{m} \right) \sin \left( \frac{l \pi}{m} \right) \right) \right| \right) \]

\[ = \begin{cases} 
\left( \sin \left( \frac{\pi}{m} \left\lfloor \frac{m}{3} \right\rfloor \right) \left( \sqrt{3} - \sin \left( \frac{\pi}{m} \left\lceil \frac{m}{3} \right\rceil \right) \right), & \text{if } m \equiv 1 \pmod{3} \\
\left( \sin \left( \frac{\pi}{m} \left\lceil \frac{m}{3} \right\rceil \right) \left( \sqrt{3} - \sin \left( \frac{\pi}{m} \left\lfloor \frac{m}{3} \right\rfloor \right) \right), & \text{if } m \equiv 2 \pmod{3} 
\end{cases} \quad (3.13) \]

Indeed, let for a fixed values of \( t \in [0, m-1] \) and \( k \in [1, n] \),

\[ t 2^{k-1} \equiv l \pmod{m}, \quad 0 \leq l \leq m - 1. \quad (3.14) \]
Then,

\[ t_k = \left| \sin \frac{lm}{m} \right| . \]  

(3.15)

Now, distinguish two cases: (1) \( t_k \leq \sqrt{3}/2 \), (2) \( t_k > \sqrt{3}/2 \).

In case (1),

\[ t_k = \frac{\sqrt{3}}{2} \Rightarrow \frac{l\pi}{m} = \frac{r\pi}{3}, \quad (r, 3) = 1, \]  

(3.16)

and since \( 0 \leq l \leq m - 1 \), then

\[ m = \frac{3l}{r}, \quad r = 1, 2. \]  

(3.17)

Because of the condition \( (m, 3) = 1 \), we have \( t_k < \sqrt{3}/2 \).

Thus, in (3.15),

\[ l \in \left[ 0, \left\lfloor \frac{m}{3} \right\rfloor \right] \cup \left[ \left\lceil \frac{2m}{3} \right\rceil, m \right], \]  

(3.18)

and (3.12) follows.

In case (2), let \( t_k > \sqrt{3}/2 = x_0 \). For \( \varepsilon > 0 \), put

\[ 1 + \varepsilon = \frac{t_k}{x_0} = \frac{2}{\sqrt{3}} \left| \sin \left( \pi 2^{k-1} \alpha \right) \right| \]  

(3.19)

such that

\[ 1 - \varepsilon = 2 - \frac{2}{\sqrt{3}} \left| \sin \left( \pi 2^{k-1} \alpha \right) \right|, \]  

(3.20)

\[ 1 - \varepsilon^2 = \frac{4}{3} \left| \sin \left( \pi 2^{k-1} \alpha \right) \right| \left( \sqrt{3} - \left| \sin \left( \pi 2^{k-1} \alpha \right) \right| \right). \]  

(3.21)

By (3.9) and (3.19), we have

\[ t_{k+1} = \rho(t_k) = \rho((1 + \varepsilon)x_0) = \rho(x_0) + \varepsilon x_0 \rho'(c), \]  

(3.22)

where \( c \in (x_0, (1 + \varepsilon)x_0) \).

Thus, according to (3.10) and taking into account that \( \rho(x_0) = x_0 \), we find

\[ t_{k+1} \leq x_0 (1 + \varepsilon), \]  

(3.23)
while by (3.19)

\[ t_k = x_0 (1 + \epsilon). \]  

(3.24)

Now, in view of (3.21) and (3.11),

\[ t_k t_{k+1} \leq | \sin \pi 2^{k-1} a | (\sqrt{3} - | \sin (\pi 2^{k-1} a) |), \]  

(3.25)

and according to (3.14), (3.15), we obtain that

\[ t_k t_{k+1} \leq h_m, \]  

(3.26)

where \( h_m \) is defined by (3.13).

Notice that from simple arguments and according to (1.9),

\[ g_m \leq \sqrt{h_m} = b_m. \]  

(3.27)

Therefore,

\[ \prod_{k=1}^{h} | \sin (\pi 2^{k-1} a) | \leq (b_m^{[h/2]})^2 \leq b_m^{h-1}. \]  

(3.28)

Now, by (3.2)–(3.4), for \( \alpha = t/m, \ t = 0, 1, \ldots, m-1 \), we have

\[ |F_{t/m}(N)| \leq \sum_{h=0}^{v_0} \prod_{k=1}^{h} \left( 1 + e^{2\pi i (\alpha 2^{k-1} + 1/2)} \right) \]

\[ \leq \sum_{h=0}^{v_0} 2^h \prod_{k=1}^{h} | \sin (2^{k-1} \alpha \pi) | \]

\[ \leq 1 + 2 \sum_{h=1}^{v_0} (2b_m)^{h-1} \]

\[ \leq 1 + 2 \frac{(2b_m)^{v_0}}{2b_m - 1}. \]

(3.29)

Note that, according to (1.7) and (3.1),

\[ (2b_m)^{v_0} = 2^{\lambda_m v_0} \leq 2^{\lambda_m \log_2 N} = N^{\lambda_m}. \]  

(3.30)

Thus, by (1.8)

\[ |F_{t/m}(N)| \leq 1 + \frac{2}{2b_m - 1} N^{\lambda_m} = 1 + \mu_m N^{\lambda_m}. \]  

(3.31)
4. Proof of Theorem 1.2

Select in (2.1) the summands which correspond to \( t = 0, \ m/3, \ 2m/3 \).
We have
\[
mS_m(N) = \sum_{n=0}^{N-1} \left( e^{\pi i (n + 3/4)} + e^{2\pi i (n+3/4)s(n)} + e^{2\pi i (2n/3+1/2)s(n)} \right) + \sum_{t=1, t \neq m/3, 2m/3}^{m-1} \sum_{n=0}^{N-1} e^{2\pi i ((t/m)n+1/2)s(n)}.
\]

(4.1)

Since the chosen summands do not depend on \( m \) and, for \( m = 3 \), the latter sum is empty, then we find
\[
mS_m(N) = 3S_3(N) + \sum_{t=1, t \neq m/3, 2m/3}^{m-1} \sum_{n=0}^{N-1} e^{2\pi i ((t/m)n+1/2)s(n)}.
\]

(4.2)

Further, the last double sum is estimated by the same way as in Section 3 such that
\[
\left| S_m(N) - \frac{3}{m} S_3(N) \right| \leq \mu_m N^{1/3}.
\]

(4.3)

Remark 4.1. Notice that from elementary arguments it follows that if \( m \geq 5 \) is a multiple of 3, then
\[
\left( \sin \frac{\pi}{m} \left( \frac{m}{3} - 1 \right) \right) \left( \sqrt{3} - \sin \frac{\pi}{m} \left( \frac{m}{3} - 1 \right) \right) \leq \left( \sin \frac{\pi}{m} \left( \frac{m+1}{3} \right) \right) \left( \sqrt{3} - \sin \frac{\pi}{m} \left( \frac{m+1}{3} \right) \right).
\]

(4.4)

The latter expression is the value of \( b_m^2 \) in this case (see (1.9)).

Example 4.2. Let us find some \( x_0 \) such that \( S_{21}(x) > 0 \) for \( x \geq x_0 \).

Supposing that \( x \) is multiple of 3 and using (1.4), we obtain that
\[
S_3(x) \geq \frac{2}{3^{\lambda+1/2}} x^4.
\]

(4.5)

Therefore, putting \( m = 21 \) in Theorem 1.2, we have
\[
S_{21}(x) \geq \frac{1}{7} S_3(x) - \mu_{21} x^{\lambda_{21}} - 1 \geq \frac{2}{7 \cdot 3^{\lambda+1/2}} x^4 - \mu_{21} x^{\lambda_{21}} - 1.
\]

(4.6)

Now, calculating \( \lambda \) and \( \lambda_m \) by (1.2) and (1.8), we find a required \( x_0 \):
\[
x_0 = \left( 3.5 \cdot 3^{\lambda+1/2} \mu_{21} \right)^{1/(-\lambda_{21})} = e^{908.379...}.
\]

(4.7)
Corollary 4.3. For \( m \) which is not a multiple of 3, denote \( U_m(x) \) the set of the positive integers not exceeding \( x \) which are multiples of \( m \) and not multiples of 3. Then,

\[
\sum_{n \in U_m(x)} (-1)^{s(n)} = -\frac{1}{m} S_3(x) + O(x^{1/3}).
\]  \tag{4.8}

In particular, for sufficiently large \( x \), we have

\[
\sum_{n \in U_m(x)} (-1)^{s(n)} < 0.
\]  \tag{4.9}

Proof. Since

\[
|U_m(x)| = S_m(x) - S_{3m}(x),
\]  \tag{4.10}

then the corollary immediately follows from Theorems 1.1, 1.2.

5. On Newman sum over primes

In [7], we put the following binary digit conjectures on primes.

Conjecture 5.1. For all \( n \in \mathbb{N}, n \neq 5, 6, \)

\[
\sum_{p \leq n} (-1)^{s(p)} \leq 0,
\]  \tag{5.1}

where the summing is over all primes not exceeding \( n \).

More precisely, by the observations, \( \sum_{p \leq n} (-1)^{s(p)} < 0 \) beginning with \( n = 31 \). Moreover, the following conjecture holds.

Conjecture 5.2.

\[
\lim_{n \to \infty} \frac{\ln \left(- \sum_{p \leq n} (-1)^{s(p)}\right)}{\ln n} = \frac{\ln 3}{\ln 4}.
\]  \tag{5.2}

A heuristic proof of Conjecture 5.2 was given in [8]. For a prime \( p \), denote \( V_p(x) \) the set of positive integers not exceeding \( x \) for which \( p \) is the least prime divisor. Show that the correctness of Conjectures 5.1 (for \( n \geq n_0 \)) follows from the following very plausible statement, especially in view of the above estimates.

Conjecture 5.3. For sufficiently large \( n \), we have

\[
\left| \sum_{5 \leq p \leq \sqrt{n}} \sum_{j \in V_p(n), p \nmid j} (-1)^{s(j)} \right| < \sum_{j \in V_3(n)} (-1)^{s(j)} = S_3(n) - S_6(n).
\]  \tag{5.3}
Indeed, in the “worst case” (really is not satisfied), in which for all \( n \geq p^2 \)

\[
\sum_{j \in V_p(n), j > p} (-1)^{s(j)} < 0, \quad p \geq 5, \tag{5.4}
\]

we have a decreasing but positive sequence of sums:

\[
\sum_{j \in V_3(n), j > 3} (-1)^{s(j)} + \sum_{j \in V_5(n), j > 5} (-1)^{s(j)}, \ldots,
\]

\[
\sum_{j \in V_3(n), j > 3} (-1)^{s(j)} + \sum_{s:p<\sqrt{n}} \sum_{j \in V_p(n), j > p} (-1)^{s(j)} > 0. \tag{5.5}
\]

Hence, the “balance condition” for odd numbers [8]

\[
\left| \sum_{j \leq n, j \text{ is odd}} (-1)^{s(j)} \right| \leq 1 \tag{5.6}
\]

must be ensured permanently by the excess of the odious primes. This explains Conjecture 5.1.

It is very interesting that for some primes \( p \) the inequality (5.4), indeed, is satisfied for all \( n \geq p^2 \). Such primes we call “resonance primes.” Our numerous observations show that all resonance primes not exceeding 1000 are

\[
11, 19, 41, 67, 107, 173, 179, 181, 307, 313, 421, 431, 433, 587, 601, 631, 641, 647, 727, 787. \tag{5.7}
\]

In conclusion, note that for \( p \geq 3 \), we have

\[
\lim_{n \to \infty} \frac{|V_p(n)|}{n} = \frac{1}{p} \prod_{2 \leq q < p} \left( 1 - \frac{1}{q} \right) \tag{5.8}
\]

such that

\[
\lim_{n \to \infty} \left( \sum_{p \geq 3} \frac{|V_p(n)|}{n} \right) = \frac{1}{2}. \tag{5.9}
\]

Thus, using Theorems 1.1, 1.2 in the form

\[
S_m(n) = \begin{cases} 
  o(S_3(n)), & (m,3) = 1, \\
  \frac{3}{m} S_3(n) (1 + o(1)), & m \mid m,
\end{cases} \tag{5.10}
\]
and inclusion-exclusion for \( p \geq 5 \), we find
\[
\sum_{j \in V_p(n)} (-1)^{s(j)} = -\frac{3}{2p} \prod_{2 \leq q < p, q \neq 3} \left(1 - \frac{1}{q}\right) S_3(n)(1 + o(1))
\]
\[
= -\frac{3}{2p} \prod_{2 \leq q < p} \left(1 - \frac{1}{q}\right) S_3(n)(1 + o(1)).
\]

Now, in view of (1.5), we obtain the following absolute result as an approximation of Conjectures 5.1, 5.2.

**Theorem 5.4.** For every prime number \( p \geq 5 \) and sufficiently large \( n \geq n_p \), we have
\[
\sum_{j \in V_p(n)} (-1)^{s(j)} < 0
\]
and, moreover,
\[
\lim_{n \to \infty} \frac{\ln \left(-\sum_{j \in V_p(n)} (-1)^{s(j)}\right)}{\ln n} = \frac{\ln 3}{\ln 4}.
\]

**References**


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