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Research Article

New Characterization for Nonlinear Weighted Best Simultaneous Approximation

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This paper is concerned with the problem of a wide class of weighted best simultaneous approximation in normed linear spaces, and it establishes a new characterization result for the class of approximation by virtue of the notion of simultaneous regular point.

1. Introduction

The problem of best simultaneous approximation has a long history and continues to generate much interest. The problem of approximating simultaneously two continuous functions on a finite closed interval was first studied by Dunham [1]. Since then, such problems have been extended extensively, see, for example, [1–7] and references therein. In particular, characterization and uniqueness results were obtained in [7] for a wide class of best simultaneously approximating problem, which includes early results as special cases.

The setting for the problems considered here is as follows. Let $\mathbf{R}^\infty$ be a real linear space consisting of some sequences in the field of real numbers $\mathbf{R}$ and $\mathbf{e}^i = (\delta_{ij}) \in \mathbf{R}^\infty$ for each $i \in \mathbf{N}$, where $\delta_{ij} = 1$ if $j = i$, and 0 otherwise. We endow a norm $\|\cdot\|_A$ on $\mathbf{R}^\infty$ such that the norm is monotonic; that is, for $(a_v) \in \mathbf{R}^\infty$ and a real sequence (b_v), the condition that $|b_i| \le |a_i|$ for each $i = 1, 2, \ldots$, implies that $(b_1, b_2, \ldots) \in \mathbf{R}^\infty$ and $\|(b_v)\|_A \le \|(a_v)\|_A$. Let $\{\lambda_i\} \in \mathbf{R}^\infty$ be a fixed sequence of positive numbers. Let X be a complex normed linear space with the norm $\|\cdot\|$ and $G \subset X$. The problem considered here is, for a sequence (x_i) in X with $(\lambda_i \|x_i\|) \in \mathbf{R}^\infty$, finding $g_0 \in G$ such that

$$\left\| \left(\lambda_i \|x_i - g_0\| \right) \right\|_A \le \left\| \left(\lambda_i \|x_i - g\| \right) \right\|_A, \quad \forall g \in G. \tag{1.1}$$

Any element g_0 satisfying (1.1) is called a best simultaneous approximation to $\hat{x}$ from G. The set of all best simultaneous approximations to $\hat{x}$ from G is denoted by $P_G(\hat{x})$.

In order to characterize restricted Chebyshev centers of a set in normed linear spaces, the work in [8] introduced the concept of simultaneous regular point of a set. In this paper, we propose a same notion and the notion of simultaneous strongly regular point of a set for studying best simultaneous approximation to a sequence from the set in X, and establish new characterization results for this class of approximation problem. It should be remarked that our results are new even in the case when X is real (noting that results obtained in this paper is valid for real normed linear spaces) and when the approximated sequence is finite.

2. Preliminaries

Let $\mathbf{R}^\infty$ be as in Section 1 with the monotonic norm $\|\cdot\|_A$ and let $(\lambda_i) \in \mathbf{R}^\infty$ be a fixed sequence of positive numbers satisfying

$$\lim_{i\to\infty} \|(0,\ldots,0,\lambda_i,\lambda_{i+1},\ldots)\|_A = 0 \tag{2.1}$$

(noting that such a sequence (λ_i) satisfying (2.1) exists, see [7]), which plays a fundamental role in the present paper. Let X be a complex normed linear space with the norm $\|\cdot\|$. We use $(\mathbf{R}^\infty)^*$ and X^* to denote the duals of $\mathbf{R}^\infty$ and X, respectively. The inner product between $\mathbf{R}^\infty$ and $(\mathbf{R}^\infty)^*$ is denoted by $\langle\cdot,\cdot\rangle$ while $f(x)$ stands for the inner product of $x \in X$ and $f \in X^*$. Also, we denote by V and W the closed unit ball of $(\mathbf{R}^\infty)^*$ and X^*, respectively. For a set A in the dual of a Banach space, let $\overline{A}^*$ signify the weak* closure of the set A and be endowed with the weak* topology. Let $\Omega = V \times W \times W \times \cdots$, and let Ω be endowed with the product topology. Then Ω is a compact Hausdorff space.

Let

$$\mathcal{X} = \{\widehat{x} = (x_i) : (\lambda_i\|x_i\|) \in \mathbf{R}^\infty\} \tag{2.2}$$

with the norm $\|\widehat{x}\|_{\mathcal{X}} = \|(\lambda_i\|x_i\|)\|_A$ for each $\widehat{x} \in \mathcal{X}$. Then $X \subset \mathcal{X}$, where x is viewed as an element$(x,x,\ldots)$ in $\mathcal{X}$ for each $x \in X$. For simplicity, we write $\widehat{f}$ for (f_i). Thus $(a^*,\widehat{f}) \in \Omega$ means that $(a^*,f_1,f_2,\ldots) \in \Omega$. Let $\widehat{x} = (x_i) \in \mathcal{X}$. Define the function $\phi(\widehat{x})$ on Ω by

$$\phi(\widehat{x})(\omega) = \langle a^*, (\operatorname{Re}\lambda_i f_i(x_i))\rangle, \quad \forall\omega = \left(a^*,\widehat{f}\right) \in \Omega. \tag{2.3}$$

Furthermore, define

$$\phi(\widehat{x})^+(\omega) = \inf_{O\in N_\omega} \sup_{\omega'\in O} \phi(\widehat{x})(\omega'), \tag{2.4}$$

where N_ω denotes the family of all open neighborhoods of ω in Ω. Then, by (2.1) and [9, Remark 1, 2, 4], we have the following proposition (see also [7, Proposition 2.1]).

Proposition 2.1. *Let $\widehat{x} = (x_i) \in \mathcal{F}$. Then the following assertions hold:*

(i) *$\phi(\widehat{x})^+$ is upper semicontinuous on Ω,*

(ii) *$\phi(x)$ is continuous on Ω for all $x \in X$,*

(iii) *For each* $x \in X$,

$$\phi(\hat{x} - x)^+ = \phi(\hat{x})^+ - \phi(x), \tag{2.5}$$

$$\max_{\omega\in\Omega} \phi(\hat{x} - x)^+(\omega) = \max_{\omega\in\Omega}\left[\phi(\hat{x})^+(\omega) - \phi(x)(\omega)\right] = \sup_{\omega\in\Omega} \phi(\hat{x} - x)(\omega) = \|\hat{x} - x\|_{\mathcal{X}}. \tag{2.6}$$

In what follows, we always assume that G is a nonempty subset of X. Let $\hat{x} \in \mathcal{X}$ and $g \in G$. Write

$$\Omega_{\hat{x}-g} = \left\{\omega \in \Omega : \phi(\hat{x})^+(\omega) - \phi(g)(\omega) = \|\hat{x} - g\|_{\mathcal{X}}\right\}. \tag{2.7}$$

Then $\Omega^+_{\hat{x}-g}$ is a nonempty compact subset of Ω since $\phi(\hat{x})^+ - \phi(g)$ is upper-semicontinuous on the compact set Ω.

The following concept is an extension of the notion of local best approximation given in [10] to the case of best simultaneous approximation.

Definition 2.2. Let $g_0 \in G$ and $\hat{x} \in \mathcal{X}$. g_0 is called a local best simultaneous approximation to $\hat{x}$ from G if there exists an open neighborhood $U(g_0)$ (i.e., an open ball with center g_0) of g_0 such that $g_0 \in P_{G\cap U(g_0)}(\hat{x})$.

The notion of suns, introduced by Efimov and Stečhkin in [11], has proved to be very important in nonlinear approximation theory in normed linear spaces. The following definition is an extension of the notion of suns to the case of simultaneous approximations, see [4].

Definition 2.3. Let $g_0 \in G$. g_0 is called simultaneous solar point of G if for each $\hat{x} \in \mathcal{X}$, $g_0 \in P_G(\hat{x})$ implies that $g_0 \in P_G(\hat{x}_\alpha)$ for each $\alpha > 0$, here and in the sequel, $\hat{x}_\alpha = g_0 + \alpha(\hat{x} - g_0)$. G is called a simultaneous sun if each point of G is a simultaneous solar point of G.

The following notion stated in Definition 2.4 (i) is similar to the notion of simultaneous regular point in [8], which was used to characterize restricted Chebyshev centers of a set in a normed linear space.

Definition 2.4. Let $g_0 \in G$. Then g_0 is called

(i) simultaneous regular point of G if, for each $\hat{x} \in \mathcal{X}$, $g \in G$ and closed set A satisfying $\Omega_{\hat{x}-g_0} \subset A \subset \Omega$ and

$$\min_{(a^*,\hat{f})\in A} \left\langle a^*, \left(\operatorname{Re} \lambda_i f_i(g - g_0)\right)\right\rangle > 0, \tag{2.8}$$

there exists $\{g_n\} \subset G$ such that $\|g_n - g_0\| \to 0$ and

$$\phi(g_n - g_0)\left(a^*, \hat{f}\right) > \phi(\hat{x} - g_0)^+\left(a^*, \hat{f}\right) - \|\hat{x} - g_0\|_{\mathcal{X}}, \quad \forall\left(a^*, \hat{f}\right) \in A,\ n \in \mathbf{N}, \tag{2.9}$$

(ii) simultaneous strongly regular point of G if, for each $\hat{x} \in \mathcal{X}$, $g \in G$ and closed set A satisfying $\Omega_{\hat{x}-g_0} \subset A \subset \Omega$ and (2.8), there exists $\{g_n\} \subset G$ such that $\|g_n - g_0\| \to 0$ and

$$\phi(g_n - g_0)\left(a^*, \hat{f}\right) > 0, \quad \forall\left(a^*, \hat{f}\right) \in A,\ n \in \mathbf{N}, \tag{2.10}$$

(iii) G is called a simultaneous regular set (resp., simultaneous strongly regular set) of X if each point of G is a simultaneous regular point (resp., simultaneous strongly regular point) of G.

The following notions are respectively analogues to Kolmogorov Condition (cf. [10, 12]) and Papini Condition (cf. [12]) in nonlinear approximation theory in normed linear spaces.

Definition 2.5. Let $g_0 \in G$ and $\hat{x} \in \mathcal{X}$. Then g_0 is said to satisfy

(i) simultaneous Kolmogorov Condition if

$$\max\left\{\left\langle a^*, \left(\operatorname{Re} \lambda_i f_i(g_0 - g)\right)\right\rangle : \left(a^*, \hat{f}\right) \in \Omega_{\hat{x}-g_0}\right\} \geq 0, \quad \forall g \in G, \tag{2.11}$$

(ii) simultaneous Papini Condition if

$$\max\left\{\left\langle a^*, \left(\operatorname{Re} \lambda_i f_i(g - g_0)\right)\right\rangle : \left(a^*, \hat{f}\right) \in \Omega_{\hat{x}-g}\right\} \leq 0, \quad \forall g \in G. \tag{2.12}$$

Proposition 2.6. *Let $g_0 \in G$ and $\hat{x} \in \mathcal{X}$.*

(i) *If $(\hat{x}, g_0)$ satisfies simultaneous Kolmogorov Condition, then $g_0 \in P_G(\hat{x})$.*

(ii) *If $g_0 \in P_G(\hat{x})$, then $(\hat{x}, g_0)$ satisfies simultaneous Papini Condition.*

Proof. (i) Let $g \in G \setminus \{g_0\}$. Then by the assumption, there exists $(a^*, \hat{f}) \in \Omega_{\hat{x}-g_0}$ such that $\langle a^*, (\operatorname{Re} \lambda_i f_i(g_0 - g))\rangle \geq 0$. It follows from (2.6) that

$$\begin{aligned} \|\hat{x} - g\|_{\mathcal{X}} &\geq \langle a^*, (\operatorname{Re} \lambda_i f_i(x_i - g))\rangle \\ &= \langle a^*, (\operatorname{Re} \lambda_i f_i(x_i - g_0))\rangle + \langle a^*, (\operatorname{Re} \lambda_i f_i(g_0 - g))\rangle \\ &\geq \|\hat{x} - g_0\|_{\mathcal{X}}. \end{aligned} \tag{2.13}$$

This means that $g_0 \in P_G(\hat{x})$.

(ii) Let $g_0 \in P_G(\hat{x})$ and $g \in G$. Then for each $(a^*, \hat{f}) \in \Omega_{\hat{x}-g}$, one has that

$$\begin{aligned} \langle a^*, (\operatorname{Re} \lambda_i f_i(g - g_0))\rangle &= \langle a^*, (\operatorname{Re} \lambda_i f_i(g - x_i))\rangle + \langle a^*, (\operatorname{Re} \lambda_i f_i(x_i - g_0))\rangle \\ &\leq -\|\hat{x} - g\|_{\mathcal{X}} + \|\hat{x} - g_0\|_{\mathcal{X}} \leq 0. \end{aligned} \tag{2.14}$$

Hence $(\hat{x}, g_0)$ satisfies simultaneous Papini Condition. The proof is complete. □

3. Characterizations for Best Simultaneous Approximations

The relationship of best simultaneous approximations and local best simultaneous approximations is as follows.

Theorem 3.1. *Let $g_0 \in G$. Suppose that g_0 is a simultaneous solar point of G. Then for each $\hat{x} \in \mathcal{X}$, the condition that g_0 is a local best simultaneous approximation to $\hat{x}$ from G implies that $g_0 \in P_G(\hat{x})$.*

Proof. Let $\hat{x} \in \mathcal{X}$ and g_0 is a local best simultaneous approximation to $\hat{x}$ from G. Then there is an open neighborhood $U(g_0, \delta)$ of g_0 such that

$$\|\hat{x} - g_0\|_{\mathcal{X}} \leq \|\hat{x} - g\|_{\mathcal{X}}, \quad \forall g \in U(g_0, \delta) \bigcap G. \tag{3.1}$$

Clearly, we may assume that $\hat{x} \neq (g_0)$. Let

$$\alpha = \min\left\{1, \frac{\delta\|(\lambda_i)\|_A}{\left(2\|\hat{x} - g_0\|_{\mathcal{X}}\right)}\right\}. \tag{3.2}$$

We assert that $g_0 \in P_G(\hat{x}_\alpha)$. In fact, let $g \in G \setminus U(g_0, \delta)$. Then $\|g - g_0\| \geq \delta$. It follows from (3.2) that

$$\begin{aligned}
\|\hat{x}_\alpha - g_0\|_{\mathcal{X}} &\geq \|g_0 - g\|_{\mathcal{X}} - \|\hat{x}_\alpha - g_0\|_{\mathcal{X}} \\
&= \|(\lambda_i)\|_A \|g_0 - g\| - \alpha\|\hat{x} - g_0\|_{\mathcal{X}} \\
&\geq \delta\|(\lambda_i)\|_A - \frac{\delta}{2}\|(\lambda_i)\|_A \\
&\geq \alpha\|\hat{x} - g_0\|_{\mathcal{X}} = \|\hat{x}_\alpha - g_0\|_{\mathcal{X}}.
\end{aligned} \tag{3.3}$$

On the other hand, let $g \in G \cap U(g_0, \delta)$ be arbitrary. Suppose on the contrary that $\|\hat{x} - g\|_{\mathcal{X}} < \|\hat{x} - g_0\|_{\mathcal{X}}$. Then

$$\begin{aligned}
\|\hat{x} - g\|_{\mathcal{X}} &\leq \|\hat{x} - \hat{x}_\alpha\|_{\mathcal{X}} + \|\hat{x}_\alpha - g_0\|_{\mathcal{X}} \\
&< \|\hat{x} - \hat{x}_\alpha\|_{\mathcal{X}} + \|\hat{x}_\alpha - g_0\|_{\mathcal{X}} \\
&= (1 - \alpha)\|\hat{x} - g_0\|_{\mathcal{X}} + \alpha\|\hat{x} - g_0\|_{\mathcal{X}} \\
&= \|\hat{x} - g_0\|_{\mathcal{X}},
\end{aligned} \tag{3.4}$$

which contradicts (3.1). This completes the proof of the assertion. Since g_0 is a simultaneous solar point of G, one has that $g_0 \in P_G(g_0 + (1/\alpha)(\hat{x}_\alpha - g_0))$. Noting that $\hat{x} = g_0 + (1/\alpha)(\hat{x}_\alpha - g_0)$, we obtain that $g_0 \in P_G(\hat{x})$. □

The first main result of this paper is as follows.

Theorem 3.2. *Let $g_0 \in G$. Then the following statements are equivalent:*

(i) *g_0 is a simultaneous solar point of G,*

(ii) *For each $\hat{x} \in \mathcal{X}$, $g_0 \in P_G(\hat{x})$ if and only if $(\hat{x}, g_0)$ satisfies simultaneous Kolmogorov Condition,*

(iii) *g_0 is a simultaneous regular point of G.*

Proof. The equivalence of (i)$\Leftrightarrow$(ii) is exactly [7, Theorem 3.1].

(ii) $\Rightarrow$(iii) For each $\hat{x} \in \mathcal{X}$, $g \in G$ and closed set A satisfying $\Omega_{\hat{x}-g_0} \subset A \subset \Omega$ and (2.8), we obtain from (2.8) that

$$\max_{(a^*,\hat{f})\in\Omega_{\hat{x}-g_0}} \langle a^*, (\operatorname{Re}\lambda_i f_i(g_0 - g))\rangle < 0. \tag{3.5}$$

Hence $g_0 \notin P_G(\hat{x})$ by (ii). Using the equivalence of (i) and (ii) as well as Theorem 3.1, g_0 is not local best simultaneous approximation to $\hat{x}$ from G. Thus there exists a sequence $\{g_n\} \subset G$ such that $\|g_n - g_0\| \to 0$ and

$$\|\hat{x} - g_n\|_{\mathcal{X}} < \|\hat{x} - g_0\|_{\mathcal{X}}, \quad \forall n \in \mathbf{N}. \tag{3.6}$$

Let $(a^*, \hat{f}) \in A$ be arbitrary. Then by (2.5) and (2.6), one has that

$$\begin{aligned} \|\hat{x} - g_n\|_{\mathcal{X}} &\geq \phi(\hat{x} - g_n)^+\left(a^*, \hat{f}\right) \\ &= \phi(\hat{x} - g_0)^+\left(a^*, \hat{f}\right) - \phi(g_n - g_0)\left(a^*, \hat{f}\right). \end{aligned} \tag{3.7}$$

This together with (3.6) implies that (2.9) holds, which shows that g_0 is a simultaneous regular point of G.

(iii)$\Rightarrow$(ii) Let (iii) hold. By Proposition 2.6, it suffices to prove that $(\hat{x}, g_0)$ satisfies simultaneous Kolmogorov Condition for each $\hat{x} \in \mathcal{X}$ with $g_0 \in P_G(\hat{x})$. For this end, suppose on the contrary that there exists $\hat{x} \in \mathcal{X}$ with $g_0 \in P_G(\hat{x})$ such that $(\hat{x}, g_0)$ does not satisfy simultaneous Kolmogorov Condition. Then there exist $g \in G$ and $\epsilon > 0$ such that

$$\max_{(a^*,\hat{f})\in\Omega_{\hat{x}-g_0}} \langle a^*, (\operatorname{Re}\lambda_i f_i(g_0 - g))\rangle = -\epsilon < 0. \tag{3.8}$$

Let

$$U = \left\{\left(a^*, \hat{f}\right) \in \Omega : \langle a^*, (\operatorname{Re}\lambda_i f_i(g_0 - g))\rangle < -\frac{\epsilon}{2}\right\}. \tag{3.9}$$

Then U is an open subset of Ω because the function

$$\omega = \left(a^*, \hat{f}\right) \longmapsto \phi(g_0 - g)(\omega) = \langle a^*, (\operatorname{Re}\lambda_i f_i(g - g_0))\rangle \tag{3.10}$$

is continuous on Ω by Proposition 2.1(ii). Let $A = \overline{U}$. Then A is closed and $\Omega_{\widehat{x}-g_0} \subset A \subset \Omega$. Furthermore,

$$\min_{(a^*,\widehat{f})\in A} \langle a^*, (\operatorname{Re}\lambda_i f_i(g-g_0))\rangle \geq \frac{\epsilon}{2}. \tag{3.11}$$

Since g_0 is a simultaneous regular point of G, one has that $\{g_n\} \subset G$ such that $\|g_n - g_0\| \to 0$ and (2.9) holds. It follows from (2.9) and Proposition 2.1 that

$$\phi(\widehat{x}-g_n)^+\left(a^*,\widehat{f}\right) < \|\widehat{x}-g_0\|_{\mathcal{X}}, \quad \forall (a^*,\widehat{x}) \in A,\ n \in \mathbf{N}. \tag{3.12}$$

Therefore,

$$\max_{(a^*,\widehat{x})\in A} \phi(\widehat{x}-g_n)^+\left(a^*,\widehat{f}\right) < \|\widehat{x}-g_0\|_{\mathcal{X}}, \quad \forall n \in \mathbf{N}. \tag{3.13}$$

On the other hand, let $K = \Omega \setminus U$. Then K is a compact subset of Ω and $K \cap \Omega_{\widehat{x}-g_0} = \emptyset$. Thus there is $\delta > 0$ such that

$$\max_{(a^*,\widehat{x})\in K} \phi(\widehat{x}-g_0)^+\left(a^*,\widehat{f}\right) < \|\widehat{x}-g_0\|_{\mathcal{X}} - \delta. \tag{3.14}$$

For each $(a^*,\widehat{f}) \in \Omega$, since

$$\begin{aligned}\langle a^*, (\operatorname{Re}\lambda_i f_i(x_i-g_n))\rangle &= \langle a^*, (\operatorname{Re}\lambda_i f_i(x_i-g_0))\rangle + \langle a^*, (\operatorname{Re}\lambda_i f_i(g_0-g_n))\rangle \\ &\leq \langle a^*, (\operatorname{Re}\lambda_i f_i(x_i-g_0))\rangle + \|(\lambda_i)\|_A \|g_0-g_n\|,\end{aligned} \tag{3.15}$$

one has that

$$\phi(\widehat{x}-g_n)^+\left(a^*,\widehat{f}\right) \leq \phi(\widehat{x}-g_0)^+\left(a^*,\widehat{f}\right) + \|(\lambda_i)\|_A \|g_0-g_n\|. \tag{3.16}$$

Thus when n is large enough, we obtain from (3.16) and (3.14) that

$$\begin{aligned}\max_{(a^*,\widehat{x})\in K} \phi(\widehat{x}-g_n)^+\left(a^*,\widehat{f}\right) &\leq \max_{(a^*,\widehat{x})\in K} \phi(\widehat{x}-g_n)^+\left(a^*,\widehat{f}\right) + \|(\lambda_i)\|_A \|g_0-g_n\| \\ &< \|\widehat{x}-g_0\|_{\mathcal{X}}.\end{aligned} \tag{3.17}$$

This together with (2.6) and (3.13) implies that

$$\|\widehat{x}-g_n\|_{\mathcal{X}} = \max_{(a^*,\widehat{x})\in\Omega} \phi(\widehat{x}-g_n)^+\left(a^*,\widehat{f}\right) < \|\widehat{x}-g_0\|_{\mathcal{X}}, \tag{3.18}$$

which contradicts that $g_0 \in P_G(\widehat{x})$. The proof is complete. □

Corollary 3.3. *The following statements are equivalent:*

(i) *G is a simultaneous sun,*

(ii) *For each $g_0 \in G$ and $\hat{x} \in \mathcal{X}$, $g_0 \in P_G(\hat{x})$ if and only if $(\hat{x}, g_0)$ satisfies simultaneous Kolmogorov Condition,*

(iii) *G is a simultaneous regular set.*

The second main result of this paper is as follows.

Theorem 3.4. *Let $g_0 \in G$. Consider the following statements:*

(i) *g_0 is a simultaneous strongly regular point of G,*

(ii) *For each $\hat{x} \in \mathcal{X}$, the fact that $(\hat{x}, g_0)$ satisfies simultaneous Papini Condition implies that $(\hat{x}, g_0)$ satisfies simultaneous Kolmogorov Condition,*

(iii) *g_0 is a simultaneous solar point of G.*

Then (i)⇒(ii)⇒(iii).

Proof. (i)⇒(ii) Let (i) holds. Suppose on the contrary that there exists $\hat{x} \in \mathcal{X}$ such that $(\hat{x}, g_0)$ satisfies simultaneous Papini Condition but does not satisfy simultaneous Kolmogorov Condition. Then there exist $g \in G$ and $\epsilon > 0$ such that (3.8) holds. Let U and A be as in the proof of the implication (iii)⇒(ii) in Theorem 3.2, respectively. Then, A is closed, $\Omega_{\hat{x}-g_0} \subset A \subset \Omega$, and (3.11) is valid. In view of the definition of simultaneous strongly regular point, there is a sequence $\{g_n\} \subset G$ such that $\|g_n - g_0\| \to 0$ and (2.10) holds. It follows from (2.10) that

$$\max_{(a^*,\hat{f})\in A} \langle a^*, (\operatorname{Re} \lambda_i f_i(g_n - g_0))\rangle > 0, \quad \forall n \in \mathbf{N}, \tag{3.19}$$

since A is closed. Let $K = \Omega \setminus U$. Then K is a closed subset of Ω. Furthermore, let $\beta = \max_{(a^*,\hat{f})\in K} \langle a^*, (\operatorname{Re} \lambda_i f_i(x_i - g_0))\rangle$. It is easy to see that $\beta < \|\hat{x} - g_0\|_{\mathcal{X}}$. Let $\beta_0 = (1/2)(\|\hat{x} - g_0\|_{\mathcal{X}} - \beta)$ and let n be sufficiently large such that $\|g_n - g_0\|_{\mathcal{X}} < \beta_0$. Then

$$\|\hat{x} - g_n\|_{\mathcal{X}} \geq \|\hat{x} - g_0\|_{\mathcal{X}} - \|g_0 - g_n\|_{\mathcal{X}} > \|\hat{x} - g_0\|_{\mathcal{X}} - \beta_0. \tag{3.20}$$

It follows that

$$\begin{aligned}
\max_{(a^*,\hat{f})\in K} \langle a^*, (\operatorname{Re} \lambda_i f_i(x_i - g_n))\rangle &\leq \max_{(a^*,\hat{f})\in K} \langle a^*, (\operatorname{Re} \lambda_i f_i(x_i - g_0))\rangle + \|g_0 - g_n\|_{\mathcal{X}} \\
&= \beta + \|g_0 - g_n\|_{\mathcal{X}} \\
&= \left[\|\hat{x} - g_0\|_{\mathcal{X}} - \beta_0\right] - \left[\beta_0 - \|g_0 - g_n\|_{\mathcal{X}}\right] \\
&< \|\hat{x} - g_n\|_{\mathcal{X}}.
\end{aligned} \tag{3.21}$$

This shows that $K \subset \Omega \setminus \Omega_{\hat{x}-g_n}$, and hence $\Omega_{\hat{x}-g_n} \subset U$. Moreover,

$$\begin{aligned} \max_{(a^*,\hat{f})\in\Omega_{\hat{x}-g_n}} \langle a^*, (\operatorname{Re}\lambda_i f_i(g_n - g_0))\rangle &\geq \inf_{(a^*,\hat{f})\in U} \langle a^*, (\operatorname{Re}\lambda_i f_i(g_n - g_0))\rangle \\ &= \min_{(a^*,\hat{f})\in A} \langle a^*, (\operatorname{Re}\lambda_i f_i(g_n - g_0))\rangle > 0 \end{aligned} \tag{3.22}$$

thanks to (3.19). This contradicts that $(\hat{x}, g_0)$ satisfies simultaneous Papini Condition.

(ii)⇒(iii) Let $\hat{x} \in \mathcal{X}$ and $g_0 \in P_G(\hat{x})$. Then $(\hat{x}, g_0)$ satisfies simultaneous Papini Condition by Proposition 2.6. Hence $(\hat{x}, g_0)$ satisfies simultaneous Kolmogorov Condition thanks to (ii). Using Theorem 3.2 and Proposition 2.6, one has that g_0 is a simultaneous solar point of G. The proof is complete. □

By Theorem 3.4 and Proposition 2.6, we have the following result.

Corollary 3.5. *Let G be a simultaneous strongly regular set of X. Let $\hat{x} \in \mathcal{X}$ and $g_0 \in G$. Then $g_0 \in P_G(\hat{x})$ if and only if $(\hat{x}, g_0)$ satisfies simultaneous Papini Condition.*

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