Research Article

Existence of Fixed Point Results in $G$-Metric Spaces

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The purpose of this paper is to prove the existence of fixed points of contractive mapping defined on $G$-metric space where the completeness is replaced with weaker conditions. Moreover, we showed that these conditions do not guarantee the completeness of $G$-metric spaces.

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1. Introduction

The study of metric fixed point theory has been researched extensively in the past decades, since fixed point theory plays a major role in mathematics and applied sciences, such as optimization, mathematical models, and economic theories.

Different mathematicians tried to generalize the usual notion of metric space $(X, d)$ such as Gähler [1, 2] and Dhage [3–5] to extend known metric space theorems in more general setting, but different authors proved that these attempts are unvalid (for detail see [6–8]).

In 2005, Mustafa and Sims introduced a new structure of generalized metric spaces (see [9]), which are called $G$-metric spaces as generalization of metric space $(X, d)$, to develop and introduce a new fixed point theory for various mappings in this new structure. The $G$-metric space is as follows.

Definition 1.1 (see [9]). Let $X$ be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$, be a function satisfying the following:

\begin{align*}
(G1) \quad & G(x, y, z) = 0 \text{ if } x = y = z, \\
(G2) \quad & 0 < G(x, x, y); \text{ for all } x, y \in X, \text{ with } x \neq y, \\
(G3) \quad & G(x, x, y) \leq G(x, y, z), \text{ for all } x, y, z \in X \text{ with } z \neq y,
\end{align*}
(G4) \( G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots \), (symmetry in all three variables),

(G5) \( G(x, y, z) \leq G(x, a, a) + G(a, y, z) \), for all \( x, y, z, a \in X \), (rectangle inequality).

Then the function \( G \) is called a generalized metric, or, more specifically a \( G \)-metric on \( X \), and the pair \((X, G)\) is a \( G \)-metric space.

Clearly these properties are satisfied when \( G(x, y, z) \) is the perimeter of the triangle with vertices at \( x, y \), and \( z \) in \( \mathbb{R}^2 \); moreover taking \( a \) in the interior of the triangle shows that (G5) is the best possible.

If \((X, d)\) is an ordinary metric space, then \((X, d)\) can define \( G \)-metrics on \( X \) by

\[
(E_n) \quad G_n(x, y, z) = d(x, y) + d(y, z) + d(x, z),
\]

\[
(E_m) \quad G_m(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}.
\]

**Proposition 1.2** (see [9]). Let \((X, G)\) be a \( G \)-metric space. Then for any \( x, y, z, \) and \( a \in X \), it follows that

1. if \( G(x, y, z) = 0 \), then \( x = y = z \),
2. \( G(x, y, z) \leq G(x, x, y) + G(x, x, z) \),
3. \( G(x, y, y) \leq 2G(y, x, x) \),
4. \( G(x, y, z) \leq G(x, a, z) + G(a, y, z) \),
5. \( G(x, y, z) \leq (2/3)(G(x, y, a) + G(x, a, z) + G(a, y, z)) \),
6. \( G(x, y, z) \leq (G(x, a, a) + G(y, a, a) + G(z, a, a)) \).

**Proposition 1.3** (see [9]). Every \( G \)-metric space \((X, G)\) will define a metric space \((X, d_G)\) by

\[
d_G(x, y) = G(x, y, y) + G(y, y, x), \quad \forall x, y \in X.
\]

**Definition 1.4** (see [9]). Let \((X, G)\) be a \( G \)-metric space. Then for \( x_0 \in X \), \( r > 0 \), the \( G \)-ball with center \( x_0 \) and radius \( r \) is

\[
B_G(x_0, r) = \{ y \in X : G(x_0, y, y) < r \}.
\]

**Proposition 1.5** (see [9]). Let \((X, G)\) be a \( G \)-metric space. Then for any \( x_0 \in X \) and \( r > 0 \), one has

1. if \( G(x_0, x, y) < r \), then \( x, y \in B_G(x_0, r) \),
2. if \( y \in B_G(x_0, r) \), then there exists a \( \delta > 0 \) such that \( B_G(y, \delta) \subseteq B(x_0, r) \).

**Proof.** (1) follows directly from (G3), while (2) follows from (G5) with \( \delta = r - G(x_0, y, y) \). \( \square \)

It follows from (2) of the above proposition that the family of all \( G \)-balls, \( \mathcal{B} = \{ B_G(x, r) : x \in X, r > 0 \} \), is the base of a topology \( \tau(G) \) on \( X \), the \( G \)-metric topology.

**Definition 1.6** (see [9]). Let \((X, G)\) be a \( G \)-metric space, let \((x_n)\) be sequence of points of \( X \), a point \( x \in X \) is said to be the limit of the sequence \((x_n)\) if \( \lim_{n,m \to \infty} G(x, x_n, x_m) = 0 \), and we say that the sequence \((x_n)\) is \( G \)-convergent to \( x \).

Thus, if \( x_n \xrightarrow{(G)} 0 \), in a \( G \)-metric space \((X, G)\), then for any \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( G(x, x_n, x_m) < \varepsilon \), for all \( n, m \geq N \), (through this paper we mean by \( \mathbb{N} \) the set of all natural numbers).
Proposition 1.7 (see [9]). Let \((X, G)\) be a G-metric space. The sequence \((x_n) \subseteq X\) is G-convergent to \(x\) if it converges to \(x\) in the G-metric topology, \(\tau(G)\).

Proposition 1.8 (see [9]). Let \((X, G)\) be a G-metric space. Then for a sequence \((x_n) \subseteq X\) and a point \(x \in X\) the following are equivalent

1. \((x_n)\) is G-convergent to \(x\),
2. \(G(x_n, x_n, x) \to 0\) as \(n \to \infty\),
3. \(G(x_n, x, x) \to 0\) as \(n \to \infty\),
4. \(G(x_m, x_n, x) \to 0\) as \(m, n \to \infty\).

Definition 1.9 (see [9]). Let \((X, G)\) and \((X', G')\) be G-metric spaces and let \(f : (X, G) \to (X', G')\) be a function, then \(f\) is said to be G-continuous at a point \(a \in X\) if and only if, given \(\epsilon > 0\), there exists \(\delta > 0\) such that \(x, y \in X\); and \(G(a, x, y) < \delta\) implies \(G'(f(a), f(x), f(y)) < \epsilon\). A function \(f\) is G-continuous at \(X\) if and only if it is G-continuous at all \(a \in X\).

Proposition 1.10 (see [9]). Let \((X, G)\), \((X', G')\) be G-metric spaces. Then a function \(f : X \to X'\) is G-continuous at a point \(x \in X\) if and only if it is G-sequentially continuous at \(x\); that is, whenever \((x_n)\) is G-convergent to \(x\) in \(X\), \((f(x_n))\) is G-convergent to \(f(x)\).

Proposition 1.11 (see [9]). Let \((X, G)\) be a G-metric space. Then the function \(G(x, y, z)\) is jointly continuous in all three of its variables.

Definition 1.12 (see [9]). Let \((X, G)\) be a G-metric space. Then the sequence \((x_n) \subseteq X\) is said to be G-Cauchy if for every \(\epsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(G(x_n, x_m, x_l) < \epsilon\) for all \(n, m, l \geq N\).

Definition 1.13 (see [9]). A G-metric space \((X, G)\) is said to be G-complete (or complete G-metric space) if every G-Cauchy sequence in \((X, G)\) is G-convergent in \((X, G)\).

2. The Main Results

In this section we will prove several theorems in each of which we have omitted the completeness property of G-metric space and we have obtained the same conclusion as in complete G-metric space, but with assumed sufficient conditions.

Theorem 2.1. Let \((X, G)\) be a G-metric space and let \(T : X \to X\) be a mapping such that \(T\) satisfies that

\[
\begin{align*}
\text{(A1)} & \quad G(Tx, Ty, Tz) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(z, Tz, Tz) \quad \text{for all } x, y, z \in X \text{ where } 0 < a + b + c < 1, \\
\text{(A2)} & \quad T \text{ is G-continuous at a point } u \in X, \\
\text{(A3)} & \quad \text{there is } x \in X; \{T^n(x)\} \text{ has a subsequence } \{T^{n_i}(x)\} \text{ G-converges to } u. \text{ Then } u \text{ is a unique fixed point (i.e., } Tu = u). 
\end{align*}
\]

Proof. G-continuity of \(T\) at \(u\) implies that \(\{T^{n+1}(x)\}\) G-convergent to \(T(u)\). Suppose \(T(u) \neq u\), consider the two G-open balls \(B_1 = B(u, \epsilon)\) and \(B_2 = B(Tu, \epsilon)\) where \(\epsilon < (1/6) \min\{G(u, Tu, Tu), G(Tu, u, u)\}\).
Since $T^{ni}(x) \to u$ and $T^{ni+1}(x) \to Tu$, then there exist $N_1 \in \mathbb{N}$ such that if $i > N_1$ implies $T^{ni}(x) \in B_1$ and $T^{ni+1}(x) \in B_2$. Hence our assumption implies that we must have

$$G\left(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)\right) > \epsilon, \quad \forall i > N_1.$$  \hfill (2.1)

On the other hand we have from (A1),

$$G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\right) \leq aG\left(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)\right) + bG\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+2}(x)\right) + cG\left(T^{ni+2}(x), T^{ni+3}(x), T^{ni+3}(x)\right)$$ \hfill (2.2)

but, by axioms of G-metric (G3), we have

$$G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\right) \leq G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\right),$$ \hfill (2.3)

$$G\left(T^{ni+2}(x), T^{ni+2}(x), T^{ni+3}(x)\right) \leq G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\right).$$ \hfill (2.4)

So, from (2.3) and (2.4), we see (2.2) becomes

$$G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+3}(x)\right) \leq qG\left(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)\right),$$ \hfill (2.5)

where $q = a/(1 - (b + c))$ and $q < 1$, since $0 < a + b + c < 1$.

Hence (2.3) and (2.5) implies that

$$G\left(T^{ni+1}(x), T^{ni+2}(x), T^{ni+2}(x)\right) \leq qG\left(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)\right).$$ \hfill (2.6)

For $l > j > N_1$ and by repeated application of (2.6) we have

$$G\left(T^{n_l}(x), T^{n_{l+1}}(x), T^{n_{l+1}}(x)\right) \leq qG\left(T^{n_{l-1}}(x), T^{n_{l}}(x), T^{n_{l}}(x)\right) \leq q^2G\left(T^{n_{l-2}}(x), T^{n_{l-1}}(x), T^{n_{l-1}}(x)\right) \leq \cdots \leq q^{n_{l-n_j}}G\left(T^{n_{j}}(x), T^{n_{j+1}}(x), T^{n_{j+1}}(x)\right).$$ \hfill (2.7)

So, as $l \to \infty$ we have $\lim G(T^{ni}(x), T^{ni+1}(x), T^{ni+1}(x)) = 0$ which contradict (2.1), hence $Tu = u$. 


Suppose there is \( v \in X; Tv = v \), then from (A1), we have

\[
G(u, v, v) = G(Tu, Tv, Tv) \leq aG(u, Tu, Tu) + (b + c)G(v, Tv, Tv) = 0. \tag{2.8}
\]

This prove the uniqueness of \( u \).

In [10] we have proved the following theorem.

**Theorem 2.2** (see [10]). Let \((X, G)\) be a complete \(G\)-metric space and let \( T : X \rightarrow X \) be a mapping satisfies the following condition for all \( x, y, z \in X \):

\[
G(T(x), T(y), T(z)) \\
\leq aG(x, T(x), T(x)) + bG(y, T(y), T(y)) + cG(z, T(z), T(z)) + dG(x, y, z),
\]

where \( 0 \leq a + b + c + d < 1 \), then \( T \) has a unique fixed point, say \( u \), and \( T \) is \(G\)-continuous at \( u \).

We see that if we take \( d = 0 \), the following theorem becomes a direct result.

**Theorem 2.3.** Let \((X, G)\) be complete \(G\)-metric space and let \( T : X \rightarrow X \) be a mapping satisfies for all \( x, y, z \in X \)

\[
G(Tx, Ty, Tz) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(z, Tz, Tz),
\]

where \( 0 < a + b + c < 1 \), then \( T \) has a unique fixed point, say \( u \), and \( T \) is \(G\)-continuous at \( u \).

If we compare **Theorem 2.3** with **Theorem 2.1**, we see that in **Theorem 2.1** we have omitted the completeness property of the \(G\)-metric space and instead we have assumed conditions (2) and (3). However, the following examples support that conditions (2) and (3) in **Theorem 2.1** do not guarantee the completeness of the \(G\)-metric space.

**Example 2.4.** Let \( X = [0, 1], T(x) = x/4 \) and \( G(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\} \). Then \((X, G)\) is \(G\)-metric space but not complete, since the sequence \( x_n = 1 - 1/n \) is \(G\)-cauchy which is not \(G\)-convergent in \((X, G)\). However, conditions (2) and (3) in **Theorem 2.1** are satisfied.

**Theorem 2.5.** Let \((X, G)\) be a \(G\)-metric space and let \( T : X \rightarrow X \) be a \(G\)-continuous mapping satisfies the following conditions:

1. \( G(Tx, Ty, Tz) \leq k\{G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)\} \) for all \( x, y, z \in M \), where \( M \) is an everywhere dense subset of \( X \) with respect the topology of \(G\)-metric convergence and \( 0 < k < 1/6 \).
2. \( T^n(x) \rightarrow x_0 \). Then \( x_0 \) is unique fixed point.

**Proof.** The proof will follow from **Theorem 2.1**, if we can show that condition (A1) in **Theorem 2.1** holds for any \( x, y, z \in X \).
Let $x, y, z$ be any elements of $X$.

Case 1. If $x, y, z \in X \backslash M$, let $\{x_n\}, \{y_n\}$, and $\{z_n\}$ be a sequences in $M$ such that $y_n \to y$, $x_n \to x$ and $z_n \to z$. From (G5) we have

$$G(Tx, Ty, Tz) \leq G(Tx, Ty, Ty) + G(Tz, Ty, Ty),$$

(2.11)

also

$$G(Tz, Ty, Ty) \leq G(Tz, Tz_n, Tz_n) + G(Tz, Tz_n, Ty_n) + G(Ty_n, Ty, Ty)$$

(2.12)

and by (B1), we have

$$G(Tz_n, Ty_n, Ty_n) \leq k\{G(z_n, Tz_n, Tz_n) + 2G(y_n, Ty_n, Ty_n)\},$$

(2.13)

again by (G5) we have

$$G(z_n, Tz_n, Tz_n) \leq G(z_n, z, z) + G(z, Tz, Tz) + G(Tz, Tz_n, Tz_n),$$

(2.14)

$$G(y_n, Ty_n, Ty_n) \leq G(y_n, y, y) + G(y, Ty, Ty) + G(Ty, Ty_n, Ty_n).$$

So, from (2.13) and (2.14) we see that (2.12) becomes

$$G(Tz, Ty, Ty) \leq (1 + k)G(Tz, Tz_n, Tz_n) + G(Ty_n, Ty, Ty) + kG(z_n, z, z)$$

(2.15)

$$+ 2kG(y_n, y, y) + 2kG(Ty, Ty_n, Ty_n) + kG(z, Tz, Tz) + 2kG(y, Ty, Ty),$$

by the same argument we deduce that

$$G(Tx, Ty, Ty) \leq (1 + k)G(Tx, Tx_n, Tx_n) + G(Ty_n, Ty, Ty) + kG(x_n, x, x)$$

(2.16)

$$+ 2kG(y_n, y, y) + 2kG(Ty, Ty_n, Ty_n) + kG(x, Tx, Tx) + 2kG(y, Ty, Ty).$$

Hence, by (2.15) and (2.16), we have

$$G(Tx, Ty, Tz) \leq G(Tx, Ty, Ty) + G(Tz, Ty, Ty)$$

$$\leq \{ (1 + k)G(Tx, Tx_n, Tx_n) + G(Ty_n, Ty, Ty) + kG(x_n, x, x)$$

$$+ 2kG(y_n, y, y) + 2kG(Ty, Ty_n, Ty_n) + kG(x, Tx, Tx) + 2kG(y, Ty, Ty) \}$$

$$+ \{ (1 + k)G(Tz, Tz_n, Tz_n) + G(Ty_n, Ty, Ty) + kG(z_n, z, z)$$

$$+ 2kG(y_n, y, y) + 2kG(Ty, Ty_n, Ty_n) + kG(z, Tz, Tz) + 2kG(y, Ty, Ty) \}.$$  

(2.17)
Now letting $n \to \infty$ in the above inequality and using the fact that $T$ is $G$-continuous we get

\[ G(Tx, Ty, Tz) \leq k \{ G(x, Tx, Tx) + 4G(y, Ty, Ty) + G(z, Tz, Tz) \}. \]  \hspace{1cm} (2.18)

**Case 2.** If $x, y \in M$ and $z \in X \setminus M$, let $\{z_n\}$ be a sequence in $M$ such that $z_n \to z$, then by (G5), we have

\[ G(Tx, Ty, Tz) \leq G(Tx, Ty, Ty) + G(Tz, Ty, Ty) \]  \hspace{1cm} (2.19)

but by (B1), we have

\[ G(Tx, Ty, Ty) \leq k \{ G(x, Tx, Tx) + 2G(y, Ty, Ty) \}, \]  \hspace{1cm} (2.20)

and by (G5), we have

\[ G(Tz, Ty, Ty) \leq G(Tz, Tz_n, Tz_n) + G(Tz_n, Ty, Ty). \]  \hspace{1cm} (2.21)

Again by (B1), we have

\[ G(Tz_n, Ty, Ty) \leq k \{ G(z_n, Tz_n, Tz_n) + 2G(y, Ty, Ty) \}. \]  \hspace{1cm} (2.22)

Also, by (G5), we have

\[ G(z_n, Tz_n, Tz_n) \leq G(z_n, z, z) + G(z, Tz, Tz) + G(Tz, Tz_n, Tz_n). \]  \hspace{1cm} (2.23)

So, from (2.21), (2.22), and (2.23), we see that (2.19) becomes

\[ G(Tx, Ty, Tz) \]
\[ \leq k \{ G(x, Tx, Tx) + 2G(y, Ty, Ty) + G(z_n, z, z) + G(z, Tz, Tz) + G(Tz, Tz_n, Tz_n) \} \]
\[ + G(Tz, Tz_n, Tz_n) + 2kG(y, Ty, Ty). \]  \hspace{1cm} (2.24)

Now letting $n \to \infty$ in the above inequality, we get

\[ G(Tx, Ty, Tz) \leq k \{ G(x, Tx, Tx) + 4G(y, Ty, Ty) + G(z, Tz, Tz) \}. \]  \hspace{1cm} (2.25)

**Case 3.** If $y \in M$ and $x, z \in X \setminus M$, let $\{x_n\}$ and $\{z_n\}$ be sequences in $M$ such that $x_n \to x$ and $z_n \to z$, but by (G5), we have

\[ G(Tx, Ty, Tz) \leq G(Tx, Ty, Ty) + G(Tz, Ty, Ty), \]  \hspace{1cm} (2.26)

\[ G(Tx, Ty, Ty) \leq G(Tx, Tx_n, Tx_n) + G(Tx_n, Ty, Ty), \]  \hspace{1cm} (2.27)
also, from (B1), we have
\[ G(Tx_n, Ty, Ty) \leq k \{ G(x_n, Tx_n, Tx_n) + 2G(y, Ty, Ty) \}, \] (2.28)
and from (G5), we have
\[ G(x_n, Tx_n, Tx_n) \leq G(x_n, x, x) + G(x, Tx, Tx) + G(Tx, Tx_n, Tx_n). \] (2.29)
So, by (2.28) and (2.29), we have
\[ G(Tx_n, Ty, Ty) \leq 2kG(y, Ty, Ty) + kG(x_n, x, x) + kG(x, Tx, Tx) + kG(Tx, Tx_n, Tx_n), \] (2.30)
then from (2.27) and (2.30) we have
\[ G(Tx, Ty, Ty) \leq kG(x_n, x, x) + kG(x, Tx, Tx) + (1 + k)G(Tx, Tx_n, Tx_n) + 2kG(y, Ty, Ty). \] (2.31)
By the same argument we deduce that
\[ G(Tz, Ty, Ty) \leq kG(z_n, z, z) + kG(z, Tz, Tz) + (1 + k)G(Tz, Tz_n, Tz_n) + 2kG(y, Ty, Ty). \] (2.32)
Then, from (2.31) and (2.32), we see (2.26) becomes
\[
G(Tx, Ty, Tz) \leq G(Tx, Ty, Ty) + G(Tz, Ty, Ty) \\
\leq kG(x_n, x, x) + kG(x, Tx, Tx) + kG(Tx, Tx_n, Tx_n) + 2kG(y, Ty, Ty) \\
+ kG(z_n, z, z) + kG(z, Tz, Tz) + kG(Tz, Tz_n, Tz_n) + 2kG(y, Ty, Ty).
\] (2.33)
Now letting \( n \to \infty \) in the above inequality and using the fact that \( T \) is \( G \)-continuous we get
\[ G(Tx, Ty, Tz) \leq k \{ G(x, Tx, Tx) + 4G(y, Ty, Ty) + G(z, Tz, Tz) \}. \] (2.34)
So, in all cases we have for all \( x, y, z \in X \)
\[ G(Tx, Ty, Tz) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(z, Tz, Tz), \] (2.35)
where \( a = k, b = 4k, c = k, \) and \( a + b + c < 1 \) since \( 0 < k < 1/6 \), then by Theorem 2.1, \( T \) has a unique fixed point. \( \square \)
Corollary 2.6. Let \((X, G)\) be \(G\)-metric space and let \(T : X \to X\) be a mapping such that \(T\) satisfies that

\[
\begin{align*}
& (C1) \ G(Tx, Ty, Ty) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty) \text{ for all } x, y \in X \text{ where } 0 < a + b < 1, \\
& (C2) \ T \text{ is } G\text{-continuous at a point } u \in X, \\
& (C3) \text{ there is } x \in X; \{T^n(x)\} \text{ has a subsequence } \{T^{n_i}(x)\} \text{ } G\text{-converges to } u. \text{ Then } u \text{ is a unique fixed point.}
\end{align*}
\]

Proof. Let \(z = y\) in condition (A1), then we see that every mapping satisfies condition (C1) will satisfy condition (A1), so the proof follows from Theorem 2.1. \(\square\)

Corollary 2.7. Let \((X, G)\) be \(G\)-metric space and let \(T : X \to X\) be a \(G\)-continuous mapping satisfies that

\[
\begin{align*}
& (D1) \ G(Tx, Ty, Ty) \leq k\{G(x, Tx, Tx) + G(y, Ty, Ty)\} \text{ for all } x, y \in M \text{ where } M \text{ is an every} \\
& \text{where dense subset of } X \text{ (with respect the topology of } G\text{-metric convergence) and } 0 < k < 1/6, \\
& (D2) \text{ there is } x \in X \text{ such that } \{T^n(x)\} \to x_0. \text{ Then } x_0 \text{ is unique fixed point.}
\end{align*}
\]

Proof. Let \(z = y\) in condition (B1), then we see that every mapping satisfies condition (D1) will satisfy condition (B1), so the proof follows from Theorem 2.5. \(\square\)

Corollary 2.8. Let \((X, G)\) be \(G\)-metric space and let \(T : X \to X\) be a mapping such that \(T\) satisfies that

\[
\begin{align*}
& (E1) \ G(Tx, Ty, Ty) \leq kG(x, y, y) \text{ for all } x, y \in X \text{ where } 0 < k < 1/4, \\
& (E2) \ T \text{ is } G\text{-continuous at a point } u \in X, \\
& (E3) \text{ there is } x \in X; \{T^n(x)\} \text{ has a subsequence } \{T^{n_i}(x)\} \text{ } G\text{-converges to } u. \text{ Then } u \text{ is a unique fixed point.}
\end{align*}
\]

Proof. By axioms of \(G\)-metric (G5), we have

\[
G(x, y, y) \leq G(x, Tx, Tx) + G(Tx, Ty, Ty) + G(Ty, y, y),
\]

\[
G(Ty, y, y) \leq 2G(y, Ty, Ty),
\]

so, from (2.36), we see that (E1) becomes

\[
G(Tx, Ty, Ty) \leq kG(x, y, y) \leq kG(x, Tx, Tx) + kG(Tx, Ty, Ty) + 2kG(y, Ty, Ty),
\]

then \(T\) will satisfy the following condition

\[
G(Tx, Ty, Ty) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty)
\]

for all \(x, y \in X\), where \(a = k/(1-k), b = 2k/(1-k)\), and \(a + b < 1\), since \(k < 1/4\).

So, condition (C1) is satisfied and the proof follows from Corollary 2.6. \(\square\)
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References


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