A Note on Four-Variable Reciprocity Theorem

Chandrashekar Adiga and P. S. Guruprasad

Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysore 570 006, India

Correspondence should be addressed to Chandrashekar Adiga, adiga.c@yahoo.com

Received 12 September 2009; Accepted 7 December 2009

Recommended by Teodor Bulboacă

We give new proof of a four-variable reciprocity theorem using Heine’s transformation, Watson’s transformation, and Ramanujan’s \( \psi_1 \)-summation formula. We also obtain a generalization of Jacobi’s triple product identity.

Copyright © 2009 C. Adiga and P. S. Guruprasad. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

Throughout the paper, we let \( |q| < 1 \) and we employ the standard notation:

\[
(a)_0 := (a; q)_0 = 1,
\]

\[
(a)_\infty := (a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \tag{1.1}
\]

\[
(a)_n := (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \quad -\infty < n < \infty.
\]

Ramanujan [1] stated several \( q \)-series identities in his “lost” notebook. One of the beautiful identities is the two-variable reciprocity theorem.

Theorem 1.1 (see [2]). For \( ab \neq 0 \),

\[
\rho(a, b) - \rho(b, a) = \left( \frac{1}{b} - \frac{1}{a} \right) \frac{(aq/b)_\infty(bq/a)_\infty(q)_\infty}{(-aq)_\infty(-bq)_\infty}, \tag{1.2}
\]
where

\[
\rho(a, b) := \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n}.
\]  

(1.3)

In the recent past many new proofs of (1.2) have been found. The first proof of (1.2) was given by Andrews [3] using four-free-variable identity and Jacobi’s triple product identity. Further, Andrews [4] applied (1.2) in proving Euler partition identity analogues stated in [1]. Somashekar and Fathima [5] established an equivalent version of (1.2) using Ramanujan’s \( {1 \over q_1} \) summation formula [6] and Heine’s transformation [7, 8]. Berndt et al. [9] also derived (1.2) using the same above mentioned two transformations. In fact, Berndt et al. [9] in the same paper have given two more proofs of (1.2) one employing the Rogers-Fine identity [10] and the other is purely combinatorial. Using the \( q \)-binomial theorem:

\[
\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} t^n = \frac{(at)_\infty}{(t)_\infty}, \quad |t| < 1, \quad |q| < 1,
\]  

(1.4)

Kim et al. [11] gave a much different proof of (1.2). Guruprasad and Pradeep [12] also devised a proof of (1.2) using the \( q \)-binomial theorem. Adiga and Anitha [13] established (1.2) along the lines of Ismail’s proof [14] of Ramanujan’s \( {1 \over q_1} \) summation formula. Further, they showed that the reciprocity theorem (1.2) leads to a \( q \)-integral extension of the classical gamma function. Kang [2] constructed a proof of (1.2) along the lines of Venkatachaliengar’s proof of the Ramanujan \( {1 \over q_1} \) summation formula [6, 15].

In [2] Kang proved the following three- and four-variable generalizations of (1.2). For \(|c| < |a| < 1 \) and \(|c| < |b| < 1\),

\[
\rho_3(a, b, c) - \rho_3(b, a, c) = \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(c)_n (-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n (-c/b)_{n+1}}, \quad a, \frac{c}{b} \neq q^{-n},
\]  

(1.5)

where

\[
\rho_3(a, b, c) := \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(c)_n (-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n (-c/b)_{n+1}}, \quad a, \frac{c}{b} \neq q^{-n},
\]  

(1.6)

and for \(|c|, |d| < |a|, |b| < 1\),

\[
\rho_4(a, b, c, d) - \rho_4(b, a, c, d) = \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(d)_n (cd/ab)_n (-aq/b)_{n+1} (bq/a)_n (q)_n}{(-aq)_n (-d/b)_{n+1} (-c/a)_{n+1} (-c/b)_{n+1} (-aq)_n (-bq)_n},
\]  

(1.7)
where

\[
\rho_4(a, b, c, d) := \left(1 + \frac{1}{b}\right) \sum_{n=0}^{\infty} \frac{(d)_n (c/(ab))_n (1 + cdq^{2n}/b) (-1)^n q^{n(n+1)/2} a^n b^{-n}}{(-aq)_n (-c/b)_{n+1} (-d/b)_{n+1}}, \quad a, \frac{c}{b}, \frac{d}{b} \neq q^{-n}.
\]

(1.8)

Kang [2] established (1.5) on employing Ramanujan’s \(1\psi_1\) summation formula and Jackson’s transformation of \(2\phi_1\) and \(2\phi_2\)-series. Recently (1.5) was derived by Adiga and Guruprasad [16] using \(q\)-binomial theorem and Gauss summation formula. Somashekara and Mamta [17, 18] obtained (1.5) using the two-variable reciprocity theorem (1.2), Jackson’s transformation, and again two-variable reciprocity theorem by parameter augmentation. Zhang [19] also established (1.5).

Kang [2] established (1.7) on employing Andrews’s generalization of \(1\psi_1\) summation formula, Sears’s transformation of \(3\phi_2\)-series, and a limiting case of Watson’s transformation for a terminating very well-poised \(8\phi_7\)-series [8]:

\[
\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(\gamma)_n(\delta)_n(e)_n(1 - aq^{2n})q^{n(n+3)/2}}{(aq/\beta)_n(aq/\gamma)_n(aq/\delta)_n(aq/e)_n(q)_n(1 - a)\left(-\frac{\alpha^2}{\beta\gamma\delta e}\right)^n}
\]

\[
= \frac{(aq)_{\infty}(aq/\delta e)_{\infty}}{(aq/\beta)(aq/\gamma)_{\infty}} \sum_{n=0}^{\infty} \frac{(\delta)_n(e)_n(aq/\beta)_n}{(aq/\gamma)_n(q)_n} \left(\frac{aq}{\delta e}\right)^n.
\]

(1.9)

Recently Ma [20, 21] proved a six-variable generalization and a five-variable generalization of (1.2). The main purpose of this paper is to provide a new proof of (1.7) using (1.9), Heine’s transformation:

\[
\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(\gamma)_n}{(\gamma/\delta)_{\infty}(\gamma)_{\infty}} \sum_{n=0}^{\infty} \frac{(\alpha\beta z/\gamma)_n(\beta z)_n}{(\gamma)_n(q)_n} \left(\frac{\gamma}{\beta}\right)^n, \quad |q| < 1, \ |z| < 1, \ |\gamma| < |\beta| < 1
\]

(1.10)

and Ramanujan’s \(1\psi_1\) summation formula:

\[
\psi_1(a, b; z) := \sum_{n=-\infty}^{\infty} \frac{(a)_n (b)_n}{(q)_n} z^n = \frac{(b/a)_\infty (az)_\infty (q/az)_\infty (q)_\infty}{(q/a)_\infty (b/az)_\infty (b)_\infty (z)_\infty}, \quad |q| < 1, \ \left|\frac{b}{a}\right| < |z| < 1.
\]

(1.11)

Jacobi’s triple product identity states that

\[
\sum_{n=-\infty}^{\infty} q^{n(n+1)/2} z^n = (q)_\infty (q/-q)_\infty \left(-\frac{1}{z}\right)_\infty, \quad z \neq 0, \ |q| < 1.
\]

(1.12)
Andrews [22] gave a proof of (1.12) using Euler identities. Combinatorial proofs of Jacobi’s triple product identity were given by Wright [23], Cheema [24], and Sudler [25]. We can also find a proof of (1.12) in [26]. Using (1.12), Hirschhorn [27, 28] established Jacobi’s two-square and four-square theorems.


$$
\sum_{n=0}^{\infty} (-1)^n a^n b^n q^{n(n+1)/2} (a)_{n+1}^{-} - \sum_{n=0}^{\infty} (-1)^n a^{-(n+1)} b^{n+1} q^{n(n+1)/2} (b)_{n+1}^{-} = \frac{(aq/b)_{\infty} (b/a)_{\infty} (q)_{\infty}}{(a)_{\infty} (b)_{\infty}}. \quad (1.13)
$$

Note that (1.13) which is equivalent to (1.2) may be considered as a two-variable generalization of (1.12). Corteel and Lovejoy [29, equation (1.5)] have given a bijective proof of (1.13) using representations of over partitions. All the reciprocity theorems (1.2), (1.5), and (1.7) are generalizations of Jacobi’s triple product identity (1.12).

We also obtain a generalization of Jacobi’s triple product identity (1.12) which is due to Kang [2].

2. Proof of (1.7)—The Four-Variable Reciprocity Theorem

On employing \( q \)-binomial theorem, we have

$$
\sum_{n=0}^{\infty} (-cq)_n (-dq)_n q^n = (-dq)_{\infty} \sum_{n=0}^{\infty} (-cq)_n (-bq^{n+1})_{\infty} q^n = \frac{(-dq)_{\infty}}{(-bq)_{\infty}} \sum_{n=0}^{\infty} \frac{(b/d)_m}{(q)_m} (-dq)_{m} \frac{(-cq)_n}{(-aq)_n} (q^{m+1})^n. \quad (2.1)
$$

On using Heine’s transformation (1.10) with \( \alpha = -cq, \beta = q, \gamma = -aq, z = q^{m+1} \), we have

$$
\sum_{n=0}^{\infty} (-cq)_n (q^{m+1})^n = \frac{(cq^{m+2})_{\infty}}{(q^{m+1})_{\infty}} \frac{(-a)_{\infty}}{(-aq)_{\infty}} \sum_{n=0}^{\infty} \frac{(cq^{m+2}/a)_n}{(q^{m+2})_n} (-a)^n = \frac{(q)_m (1 + a)(-a)^{-m-1}}{(cq/a)_{m+1}} \sum_{n=0}^{\infty} \frac{(cq/a)_n}{(q)_n} (-a)^n + \sum_{n=0}^{m} \frac{(cq/a)_n}{(q)_n} (-a)^n
$$

$$
= \frac{(q)_{m+1}}{(cq/a)_{m+1}} (-a)^{m-1} \sum_{n=0}^{\infty} \frac{(cq/a)_n}{(q)_n} (-a)^n - \frac{m (cq/a)_n}{(q)_n} (-a)^n
$$

$$
= \frac{(q)_m (1 + a)(-a)^{-m-1}}{(cq/a)_{m+1}} \sum_{n=0}^{\infty} (cq/a)_n (-a)^n - \frac{m (cq/a)_n}{(q)_n} \sum_{n=0}^{m} (cq/a)_n (-a)^n. \quad (2.2)
$$
Substituting this in (2.1), we obtain

\[
\sum_{n=0}^{\infty} (-c)_{n} (-d)_{n} q^{n} = \frac{(-dq)_{\infty} (-cq)_{\infty}}{(-a) (-bq)_{\infty} (-aq)_{\infty} m=0} (cq/a)_{m+1} (dq/a)^{m} \\
+ \frac{1 + a^{-1} (-dq)_{\infty}}{(-bq)_{\infty} m=0 \sum_{n=0}^{\infty} (b/d)_{m} (cq/a)_{n} (-a)_{n}}. \tag{2.3}
\]

Now,

\[
\frac{(1 + a^{-1}) (-dq)_{\infty}}{(-bq)_{\infty} m=0 \sum_{n=0}^{\infty} (b/d)_{n} (dq/a)_{m} (cq/a)_{n} (-a)_{n}} \\
= \frac{(1 + a^{-1}) (-dq)_{\infty}}{(-bq)_{\infty} m=0 \sum_{n=0}^{\infty} (b/d)_{n} (dq/a)_{m} (-a)} \sum_{n=0}^{\infty} (cq^{n+1}/a)_{m} \\
= \frac{(1 + a^{-1}) (-dq)_{\infty}}{(-bq)_{\infty} m=0 \sum_{n=0}^{\infty} (b/c)_{m} (cq/a)_{n} (-a)} \frac{1 - cq^{n+1}/a}{1 - dq/a} \sum_{n=0}^{\infty} (dq^{n+1}/a)_{m},
\]

\[
\text{on using (1.10) with } a = \frac{bq}{d}, \ \beta = q, \ \gamma = \frac{cq^{2}}{a}, \ z = \frac{dq}{a}. \tag{2.4}
\]

\[
\frac{(1 + a^{-1}) (-dq)_{\infty}}{(-bq)_{\infty} m=0 \sum_{n=0}^{\infty} (b/c)_{m} (cq/a)_{n} (-a)} \frac{1 - bq^{n+1}/a}{1 - dq^{n+1}/a} \sum_{n=0}^{\infty} (dq^{n+1}/a)_{m}
\]

\[
= (1 + a^{-1}) \sum_{m=0}^{\infty} \frac{(b/c)_{m} (-dq)_{m}}{(-bq)_{m} (dq/a)_{m+1}} (cq/a)^{m}.
\]

Substituting (2.4) in (2.3), we obtain

\[
\sum_{n=0}^{\infty} (-c)_{n} (-d)_{n} q^{n} = \frac{(-dq)_{\infty} (-cq)_{\infty}}{(-a) (-bq)_{\infty} (-aq)_{\infty} m=0} (cq/a)_{m+1} (dq/a)^{m} \\
+ \frac{1 + a^{-1}}{(-bq)_{\infty} m=0 \sum_{n=0}^{\infty} (b/c)_{m} (-dq)_{m} (cq/a)_{n} (-a)} \\
= \frac{(-dq)_{\infty} (-cq)_{\infty}}{(-a) (-bq)_{\infty} (-aq)_{\infty} m=0} \sum_{n=0}^{\infty} (b/c)_{m} (cq/a)_{n} (dq/a)^{m} \\
+ \frac{1 + a^{-1}}{(-bq)_{\infty} m=0 \sum_{n=0}^{\infty} (b/c)_{m} (-dq)_{m} (cq/a)_{n} (-a)} \tag{2.5}
\]

(Here, we used (1.10) with } a = b/d, \beta = q, \gamma = cq^{2}/a, \ z = dq/a.)
Changing \(c\) to \(-c/q\), \(d\) to \(-d/q\) in (2.5), we get

\[
\sum_{n=0}^{\infty} \frac{(c)_n (d)_n}{(-aq)_n (-bq)_n} q^n = \frac{(d)_{\infty} (c)_{\infty}}{(-a) (-bq)_{\infty} (-aq)_{\infty}} \sum_{m=0}^{\infty} \frac{(-bq/c)^m}{(-d/a)_{m+1}} \left( -\frac{c}{a} \right)^m + \left( 1 + a^{-1} \right) \sum_{m=0}^{\infty} \frac{(-bq/c)^m (d)^m}{(-aq)_{m+1}} \left( -\frac{c}{a} \right)^m.
\]  

(2.6)

Interchanging \(a\) and \(b\) in (2.6), we have

\[
\sum_{n=0}^{\infty} \frac{(c)_n (d)_n}{(-aq)_n (-bq)_n} q^n = \frac{(d)_{\infty} (c)_{\infty}}{(-b) (-bq)_{\infty} (-aq)_{\infty}} \sum_{m=0}^{\infty} \frac{(-aq/c)^m}{(-d/b)_{m+1}} \left( -\frac{c}{b} \right)^m + \left( 1 + b^{-1} \right) \sum_{m=0}^{\infty} \frac{(-aq/c)^m (d)^m}{(-aq)_{m+1}} \left( -\frac{c}{b} \right)^m.
\]  

(2.7)

Subtracting (2.6) from (2.7), we deduce that

\[
\frac{(d)_{\infty} (c)_{\infty}}{(-bq)_{\infty} (-aq)_{\infty}} \left[ \frac{1}{b} \sum_{m=0}^{\infty} \frac{(-aq/c)^m}{(-d/b)_{m+1}} \left( -\frac{c}{b} \right)^m = \frac{1}{a} \sum_{m=0}^{\infty} \frac{(-bq/c)^m}{(-d/a)_{m+1}} \left( -\frac{c}{a} \right)^m \right] = \left( 1 + b^{-1} \right) \sum_{m=0}^{\infty} \frac{(-aq/c)^m (d)^m}{(-aq)_{m+1}} \left( -\frac{c}{b} \right)^m - \left( 1 + a^{-1} \right) \sum_{m=0}^{\infty} \frac{(-bq/c)^m (d)^m}{(-bq)_{m+1}} \left( -\frac{c}{a} \right)^m.
\]  

(2.8)

Now change \(a\) to \(-b/d\), \(b\) to \(-c/a\), and \(z\) to \(-d/a\) in (1.11) to obtain

\[
\sum_{n=1}^{\infty} \frac{(-b/d)_n}{(-c/a)_n} \left( -\frac{d}{a} \right)^n + \sum_{n=0}^{\infty} \frac{(-aq/c)_n}{(-dq/b)_n} \left( -\frac{c}{b} \right)^n = \frac{(cd/ab)_{\infty} (b/a)_{\infty} (aq/b)_{\infty} (q)_{\infty}}{(-c/a)_{\infty} (-c/b)_{\infty} (-d/a)_{\infty} (-dq/b)_{\infty}}.
\]  

(2.9)

Changing \(n\) to \(n+1\) in the first summation of the above identity and then multiplying both sides by \((1+d/b)^{-1}\), we find that

\[
\frac{1}{(1+d/b)} \sum_{n=0}^{\infty} \frac{(-b/d)_{n+1}}{(-c/a)_{n+1}} \left( -\frac{d}{a} \right)^{n+1} + \sum_{n=0}^{\infty} \frac{(-aq/c)_n}{(-dq/b)_{n+1}} \left( -\frac{c}{b} \right)^n = \left( 1 - \frac{b}{a} \right) \frac{(cd/ab)_{\infty} (bq/a)_{\infty} (aq/b)_{\infty} (q)_{\infty}}{(-c/a)_{\infty} (-c/b)_{\infty} (-d/a)_{\infty} (-dq/b)_{\infty}}.
\]  

(2.10)
Using (1.10) with \( \alpha = -bq/c, \beta = q, \gamma = -dq/a, \) and \( z = -c/a \) in the first summation of the above identity and then multiplying both sides by \( 1/b, \) we get

\[
\frac{1}{b} \sum_{n=0}^{\infty} \frac{(-aq/c)_n}{(-d/b)_{n+1}} \left( -\frac{c}{b} \right)^n - \frac{1}{a} \sum_{n=0}^{\infty} \frac{(-bq/c)_n}{(-d/a)_{n+1}} \left( -\frac{c}{a} \right)^n
= \left( \frac{1}{b} - \frac{1}{a} \right) \frac{(cd/ab)_\infty (bq/a)_\infty (aq/b)_\infty (q)_\infty}{(-c/a)_\infty (-c/b)_\infty (-d/a)_\infty (-d/b)_\infty}.
\]

Substituting (2.11) in (2.8), we see that

\[
\left( \frac{1}{b} - \frac{1}{a} \right) \frac{(cd/ab)_\infty (c)_\infty (d)_\infty (bq/a)_\infty (aq/b)_\infty (q)_\infty}{(-aq)_\infty (-bq)_{c/a}_\infty (-c/a)_\infty (-d/a)_\infty (-d/b)_\infty}
= (1 + b^{-1}) \sum_{m=0}^{\infty} \frac{(-aq/c)_m (d)_m}{(-aq)_m (-d/b)_{m+1}} \left( -\frac{c}{b} \right)^m - (1 + a^{-1}) \sum_{m=0}^{\infty} \frac{(-bq/c)_m (d)_m}{(-bq)_m (-d/a)_{m+1}} \left( -\frac{c}{a} \right)^m.
\]

Now setting \( a = -cd/b, \beta = cd/ab, \gamma = c, \delta = q, \) and \( e = d \) in (1.9) and then multiplying both sides by \( 1/(1 + d/b)(1 + c/b), \) we obtain

\[
\sum_{n=0}^{\infty} \frac{(cd/ab)_n (c)_n (d)_n (1 + cdq^{2n}/b)q^{n(n+1)/2}(-1)^n a^n b^{-n}}{(-aq)_n (-c/b)_{n+1} (-d/b)_{n+1}}
= \sum_{n=0}^{\infty} \frac{(-aq/c)_n (d)_n}{(-aq)_n (-d/b)_{n+1}} \left( -\frac{c}{b} \right)^n.
\]

Interchanging \( a \) and \( b \) in (2.13), we have

\[
\sum_{n=0}^{\infty} \frac{(cd/ab)_n (c)_n (d)_n (1 + cdq^{2n}/a)q^{n(n+1)/2}(-1)^n b^n a^{-n}}{(-bq)_n (-c/a)_{n+1} (-d/a)_{n+1}}
= \sum_{n=0}^{\infty} \frac{(-bq/c)_n (d)_n}{(-bq)_n (-d/a)_{n+1}} \left( -\frac{c}{a} \right)^n.
\]

Substituting (2.13) and (2.14) in (2.12), we deduce (1.7).

**Theorem 2.1** (A four-variable generalization of Jacobi’s triple product identity). For \(|c|, |d| < |a|, |b| < 1,\)

\[
\frac{(cd/ab)_\infty (c)_\infty (d)_\infty (b/a)_\infty (aq/b)_\infty (q)_\infty}{(-a)_\infty (-b)_\infty (-c/a)_\infty (-c/b)_\infty (-d/a)_\infty (-d/b)_\infty}
= \sum_{m=0}^{\infty} \frac{(d)_m (-cq^{-m}/a)_m (-1)^m a^mb^{-m}q^{m(m+1)/2}}{(-a)_{m+1} (-d/b)_{m+1}}
- \sum_{m=0}^{\infty} \frac{(d)_m (-cq^{-m}/b)_m (-1)^m a^{-m+1}b^m q^{m(m+1)/2}}{(-b)_{m+1} (-d/a)_{m+1}}.
\]
Proof. Employing

\[
\left( -\frac{aq}{c} \right)_m = \left( \frac{a}{c} \right)^m q^{m(m-1)/2} \left( -\frac{cq^{-m}}{a} \right)_m,
\]

\[
\left( -\frac{bq}{c} \right)_m = \left( \frac{b}{c} \right)^m q^{m(m+1)/2} \left( -\frac{cq^{-m}}{b} \right)_m
\]

in the right side of (2.12) and then multiplying both sides by \( b/(1 + a)(1 + b) \), we obtain (2.15). \qed

Acknowledgment

The authors thank the anonymous referee for several helpful comments.

References

vol. 61, no. 2, pp. 156–164, 1986.
Pure and Applied Mathematics.


