Research Article

An Extension of Stolarsky Means to the Multivariable Case

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We give an extension of well-known Stolarsky means to the multivariable case in a simple and applicable way. Some basic inequalities concerning this matter are also established with applications in Analysis and Probability Theory.

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1. Introduction

There is a huge amount of papers investigating properties of the so-called Stolarsky (or extended) two-parametric mean value, defined for positive values of x, y , as

$$E_{r,s}(x, y) := \left(\frac{r(x^s - y^s)}{s(x^r - y^r)} \right)^{1/(s-r)}, \quad (1.1)$$

$$rs(r-s)(x-y) \neq 0.$$

E means can be continuously extended on the domain

$$\{(r, s; x, y) \mid r, s \in \mathbb{R}; x, y \in \mathbb{R}_+\} \quad (1.2)$$

by the following:

$$E_{r,s}(x,y) = \begin{cases} \left(\frac{r(x^s - y^s)}{s(x^r - y^r)} \right)^{1/(s-r)} & rs(r-s) \neq 0; \\ \exp\left(-\frac{1}{s} + \frac{x^s \log x - y^s \log y}{x^s - y^s}\right), & r = s \neq 0; \\ \left(\frac{x^s - y^s}{s(\log x - \log y)} \right)^{1/s}, & s \neq 0, r = 0; \\ \sqrt{xy}, & r = s = 0; \\ x, & y = x > 0, \end{cases} \quad (1.3)$$

and in this form are introduced by Keneth Stolarsky in [1].

Most of the classical two-variable means are special cases of the class E . For example, $E_{1,2} = (x + y)/2$ is the arithmetic mean, $E_{0,0} = \sqrt{xy}$ is the geometric mean, $E_{0,1} = (x - y)/(\log x - \log y)$ is the logarithmic mean, $E_{1,1} = (x^x/y^y)^{1/(x-y)}/e$ is the identric mean, and so forth. More generally, the r th power mean $((x^r + y^r)/2)^{1/r}$ is equal to $E_{r,2r}$.

Recently, several papers are produced trying to define an extension of the class E to n , $n > 2$ variables. Unfortunately, this is done in a highly artificial mode (cf. [2–4]), without a practical background. Here is an illustration of this point; recently Merikowski [4] has proposed the following generalization of the Stolarsky mean $E_{r,s}$ to several variables:

$$E_{r,s}(X) := \left[\frac{L(X^s)}{L(X^r)} \right]^{1/(s-r)}, \quad r \neq s, \quad (1.4)$$

where $X = (x_1, \dots, x_n)$ is an n -tuple of positive numbers and

$$L(X^s) := (n-1)! \int_{E_{n-1}} \prod_{i=1}^n x_i^{su_i} du_1 \cdots du_{n-1}. \quad (1.5)$$

The symbol E_{n-1} stands for the Euclidean simplex which is defined by

$$E_{n-1} := \{(u_1, \dots, u_{n-1}) : u_i \geq 0, 1 \leq i \leq n-1; u_1 + \cdots + u_{n-1} \leq 1\}. \quad (1.6)$$

In this paper, we give another attempt to generalize Stolarsky means to the multivariable case in a simple and applicable way. The proposed task can be accomplished by founding a “weighted” variant of the class E , wherefrom the mentioned generalization follows naturally.

In the sequel, we will need notions of the weighted geometric mean $G = G(p, q; x, y)$ and weighted r th power mean $S_r = S_r(p, q; x, y)$, defined by

$$G := x^p y^q; \quad S_r := (px^r + qy^r)^{1/r}, \quad (1.7)$$

where

$$p, q, x, y \in \mathbb{R}_+; \quad p + q = 1; \quad r \in \mathbb{R} / \{0\}. \quad (1.8)$$

Note that $(S_r)^r > (G)^r$ for $x \neq y$, $r \neq 0$, and $\lim_{r \rightarrow 0} S_r = G$.

1.1. Weighted Stolarsky Means

We introduce here a class W of weighted two-parameters means which includes the Stolarsky class E as a particular case. Namely, for $p, q, x, y \in \mathbb{R}_+$, $p + q = 1$, $rs(r-s)(x-y) \neq 0$, we define

$$W = W_{r,s}(p, q; x, y) := \left(\frac{r^2 (S_s)^s - (G)^s}{s^2 (S_r)^r - (G)^r} \right)^{1/(s-r)} = \left(\frac{r^2 px^s + qy^s - x^{ps}y^{qs}}{s^2 px^r + qy^r - x^{pr}y^{qr}} \right)^{1/(s-r)}. \quad (1.9)$$

Various properties concerning the means W can be established; some of them are the following:

$$\begin{aligned} W_{r,s}(p, q; x, y) &= W_{s,r}(p, q; x, y); \\ W_{r,s}(p, q; x, y) &= W_{r,s}(q, p; y, x); \quad W_{r,s}(p, q; y, x) = xy W_{r,s}(p, q; x^{-1}, y^{-1}); \\ W_{ar,as}(p, q; x, y) &= (W_{r,s}(p, q; x^a, y^a))^{1/a}, \quad a \neq 0. \end{aligned} \quad (1.10)$$

Note that

$$\begin{aligned} W_{2r,2s}\left(\frac{1}{2}, \frac{1}{2}; x, y\right) &= \left(\frac{r^2 x^{2s} + y^{2s} - 2(\sqrt{xy})^{2s}}{s^2 x^{2r} + y^{2r} - 2(\sqrt{xy})^{2r}} \right)^{1/2(s-r)} \\ &= \left(\frac{r^2 (x^s - y^s)^2}{s^2 (x^r - y^r)^2} \right)^{1/2(s-r)} = E(r, s; x, y). \end{aligned} \quad (1.11)$$

In the same manner, we get

$$\begin{aligned} W_{r,s}\left(\frac{2}{3}, \frac{1}{3}; x^3, y^3\right) &= \left(\frac{2x^s + y^s}{2x^r + y^r} \right)^{1/(s-r)} (E(r, s; x, y))^2; \\ W_{r,s}\left(\frac{3}{4}, \frac{1}{4}; x^4, y^4\right) &= \left(\frac{3x^{2s} - (xy)^s + y^{2s}}{3x^{2r} - (xy)^r + y^{2r}} \right)^{1/(s-r)} (E(r, s; x, y))^2. \end{aligned} \quad (1.12)$$

The weighted means from the class W can be extended continuously to the domain

$$D = \{(r, s; x, y) \mid r, s \in \mathbb{R}; x, y \in \mathbb{R}_+\}. \quad (1.13)$$

This extension is given by

$$W_{r,s}(p, q; x, y) = \begin{cases} \left(\frac{r^2 px^s + qy^s - x^{ps}y^{qs}}{s^2 px^r + qy^r - x^{pr}y^{qr}} \right)^{1/(s-r)}, & rs(r-s)(x-y) \neq 0; \\ \left(2 \frac{px^s + qy^s - x^{ps}y^{qs}}{pq s^2 \log^2(x/y)} \right)^{1/s}, & s(x-y) \neq 0, r=0; \\ \exp \left(\frac{-2}{s} + \frac{px^s \log x + qy^s \log y - (p \log x + q \log y) x^{ps} y^{qs}}{px^s + qy^s - x^{ps} y^{qs}} \right), & s(x-y) \neq 0, r=s; \\ x^{(p+1)/3} y^{(q+1)/3}, & x \neq y, r=s=0; \\ x, & x=y. \end{cases} \quad (1.14)$$

Note that those means are homogeneous of order 1, that is, $W_{r,s}(p, q; tx, ty) = tW_{r,s}(p, q; x, y)$, $t > 0$, symmetric in r, s , $W_{r,s}(p, q; x, y) = W_{s,r}(p, q; x, y)$ but are not symmetric in x, y unless $p = q = 1/2$.

1.2. Multivariable Case

A natural generalization of weighted Stolarsky means to the multivariable case gives

$$W_{r,s}(\mathbf{p}; \mathbf{x}) = \begin{cases} \left(\frac{r^2 \left(\sum p_i x_i^s - \left(\prod x_i^{p_i} \right)^s \right)}{s^2 \left(\sum p_i x_i^r - \left(\prod x_i^{p_i} \right)^r \right)} \right)^{1/(s-r)}, & rs(s-r) \neq 0; \\ \left(\frac{2}{s^2} \frac{\sum p_i x_i^s - \left(\prod x_i^{p_i} \right)^s}{\sum p_i \log^2 x_i - \left(\sum p_i \log x_i \right)^2} \right)^{1/s}, & r=0, s \neq 0; \\ \exp \left(\frac{-2}{s} + \frac{\sum p_i x_i^s \log x_i - \left(\sum p_i \log x_i \right) \left(\prod x_i^{p_i} \right)^s}{\sum p_i x_i^s - \left(\prod x_i^{p_i} \right)^s} \right), & r=s \neq 0; \\ \exp \left(\frac{\sum p_i \log^3 x_i - \left(\sum p_i \log x_i \right)^3}{3 \left(\sum p_i \log^2 x_i - \left(\sum p_i \log x_i \right)^2 \right)} \right), & r=s=0, \end{cases} \quad (1.15)$$

where $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$, $n \geq 2$, $\mathbf{p}$ is an arbitrary positive weight sequence associated with $\mathbf{x}$ and $W_{r,s}(\mathbf{p}; \mathbf{x}_0) = a$ for $\mathbf{x}_0 = (a, a, \dots, a)$.

We also write $\Sigma(\cdot)$, $\Pi(\cdot)$ instead of $\sum_1^n(\cdot)$, $\prod_1^n(\cdot)$.

The above formulae are obtained by an appropriate limit process, implying continuity. For example, applying

$$t^s = 1 + s \log t + \frac{s^2}{2} \log^2 t + \frac{s^3}{6} \log^3 t + o(s^3) \quad (s \rightarrow 0), \quad (1.16)$$

we get

$$\begin{aligned} W_{0,0}(\mathbf{p}; \mathbf{x}) &= \lim_{s \rightarrow 0} W_{s,0}(\mathbf{p}; \mathbf{x}) = \lim_{s \rightarrow 0} \left(\frac{2}{s^2} \frac{\sum p_i x_i^s - \left(\prod x_i^{p_i} \right)^s}{\sum p_i \log^2 x_i - (\sum p_i \log x_i)^2} \right)^{1/s} \\ &= \lim_{s \rightarrow 0} \left(\frac{2}{s^2 (\sum p_i \log^2 x_i - (\sum p_i \log x_i)^2)} \right. \\ &\quad \times \left(\left(\sum p_i + s \sum p_i \log x_i + \left(\frac{s^2}{2} \right) \sum p_i \log^2 x_i + \left(\frac{s^3}{6} \right) \sum p_i \log^3 x_i \right) \right. \\ &\quad \left. - \left(\sum p_i + s \log \left(\prod x_i^{p_i} \right) + \left(\frac{s^2}{2} \right) \log^2 \left(\prod x_i^{p_i} \right) \right. \right. \\ &\quad \left. \left. + \left(\frac{s^3}{6} \right) \log^3 \left(\prod x_i^{p_i} \right) + o(s^3) \right) \right)^{1/s} \\ &= \lim_{s \rightarrow 0} \left(1 + \frac{\sum p_i \log^3 x_i - (\sum p_i \log x_i)^3}{3 (\sum p_i \log^2 x_i - (\sum p_i \log x_i)^2)} s (1 + o(1)) \right)^{1/s} \\ &= \exp \left(\frac{\sum p_i \log^3 x_i - (\sum p_i \log x_i)^3}{3 (\sum p_i \log^2 x_i - (\sum p_i \log x_i)^2)} \right). \end{aligned} \quad (1.17)$$

Remark 1.1. Analogously to the former considerations, one can define a class of Stolarsky means in n variables $E_{r,s}(\mathbf{x}; n)$ as

$$E_{r,s}(\mathbf{x}; n) := W_{nr,ns}(\mathbf{p}_0, \mathbf{x}), \quad (1.18)$$

where $\mathbf{p}_0 = \{1/n\}_1^n$.

Therefore,

$$E_{r,s}(\mathbf{x}; n) = \left(\frac{r^2 \sum_1^n x_i^{ns} - n \prod_1^n x_i^s}{s^2 \sum_1^n x_i^{nr} - n \prod_1^n x_i^r} \right)^{1/n(s-r)}, \quad rs(r-s) \neq 0. \quad (1.19)$$

Details are left to the readers.

2. Results

The following basic assertion is of importance.

Proposition 2.1. *The expressions $W_{r,s}(\mathbf{p}; \mathbf{x})$ are actual means, that is, for arbitrary weight sequence $\mathbf{p}$ one has*

$$\min\{x_1, x_2, \dots, x_n\} \leq W_{r,s}(\mathbf{p}; \mathbf{x}) \leq \max\{x_1, x_2, \dots, x_n\}. \quad (2.1)$$

Our main result is contained in the following.

Proposition 2.2. *The means $W_{r,s}(\mathbf{p}, \mathbf{x})$ are monotone increasing in both variables r and s .*

Passing to the continuous variable case, we get the following definition of the class $\overline{W}_{r,s}(p, x)$.

Assuming that all integrals exist,

$$\begin{aligned} & \overline{W}_{r,s}(\mathbf{p}, \mathbf{x}) \\ &= \begin{cases} \left(\frac{r^2 \left(\int p(t) x^s(t) dt - \exp\left(s \int p(t) \log x(t) dt\right) \right)}{s^2 \left(\int p(t) x^r(t) dt - \exp\left(r \int p(t) \log x(t) dt\right) \right)} \right)^{1/(s-r)}, & rs(s-r) \neq 0; \\ \left(\frac{2 \int p(t) x^s(t) dt - \exp\left(s \int p(t) \log x(t) dt\right)}{s^2 \int p(t) \log^2 x(t) dt - \left(\int p(t) \log x(t) dt \right)^2} \right)^{1/s}, & r = 0, s \neq 0; \\ \exp \left(\frac{-2}{s} + \frac{\int p(t) x^s(t) \log x(t) dt - \left(\int p(t) \log x(t) dt \right) \exp\left(s \int p(t) \log x(t) dt\right)}{\int p(t) x^s(t) dt - \exp\left(s \int p(t) \log x(t) dt\right)} \right), & r = s \neq 0; \\ \exp \left(\frac{\int p(t) \log^3 x(t) dt - \left(\int p(t) \log x(t) dt \right)^3}{3 \left(\int p(t) \log^2 x(t) dt - \left(\int p(t) \log x(t) dt \right)^2 \right)} \right), & r = s = 0, \end{cases} \end{aligned} \quad (2.2)$$

where $x(t)$ is a positive integrable function and $p(t)$ is a nonnegative function with $\int p(t) dt = 1$.

From our former considerations, a very applicable assertion follows.

Proposition 2.3. $\overline{W}_{r,s}(\mathbf{p}, \mathbf{x})$ is monotone increasing in either r or s .

3. Applications

3.1. Applications in Analysis

As an illustration of the above, we give the following proposition.

Proposition 3.1. The function $w(s)$, defined by

$$w(s) := \begin{cases} \left(\frac{12}{(\pi s)^2} (\Gamma(1+s) - e^{-\gamma s}) \right)^{1/s}, & s \neq 0; \\ \exp\left(-\gamma - \frac{4\xi(3)}{\pi^2}\right), & s = 0, \end{cases} \quad (3.1)$$

is monotone increasing for $s \in (-1, \infty)$.

In particular, for $s \in (-1, 1)$, one has

$$\Gamma(1-s)e^{-\gamma s} + \Gamma(1+s)e^{\gamma s} - \frac{\pi s}{\sin(\pi s)} \leq 1 - \frac{(\pi s)^4}{144}, \quad (3.2)$$

where $\Gamma(\cdot)$, $\xi(\cdot)$, γ stands for the Gamma function, Zeta function, and Euler's constant, respectively.

3.2. Applications in Probability Theory

For a random variable X and an arbitrary probability distribution with support on $(-\infty, +\infty)$, it is well known that

$$Ee^X \geq e^{EX}. \quad (3.3)$$

Denoting the central moment of order k by $\mu_k = \mu_k(X) := E(X - EX)^k$, we improve this inequality to the following propositions.

Proposition 3.2. For an arbitrary probability law with support on $\mathbb{R}$, one has

$$Ee^X \geq \left(1 + \left(\frac{\mu_2}{2} \right) \exp\left(\frac{\mu_3}{3\mu_2} \right) \right) e^{EX}. \quad (3.4)$$

Proposition 3.3. *One also has that*

$$\left(\frac{Ee^{sX} - e^{sEX}}{s^2 \sigma_X^2 / 2} \right)^{1/s} \quad (3.5)$$

is monotone increasing in s .

3.3. Shifted Stolarsky Means

Especially interesting is studying the *shifted Stolarsky means* E^* , defined by

$$E_{r,s}^*(x, y) := \lim_{p \rightarrow 0^+} W_{r,s}(p, q; x, y). \quad (3.6)$$

Their analytic continuation to the whole (r, s) plane is given by

$$E_{r,s}^*(x, y) = \begin{cases} \left(\frac{r^2(x^s - y^s(1 + s \log(x/y)))}{s^2(x^r - y^r(1 + r \log(x/y)))} \right)^{1/(s-r)}, & rs(r-s)(x-y) \neq 0; \\ \left(\frac{2}{s^2} \frac{x^s - y^s(1 + s \log(x/y))}{\log^2(x/y)} \right)^{1/s}, & s(x-y) \neq 0, r=0; \\ \exp\left(\frac{-2}{s} + \frac{(x^s - y^s) \log x - sy^s \log y \log(x/y)}{x^s - y^s(1 + s \log(x/y))} \right), & s(x-y) \neq 0, r=s; \\ x^{1/3} y^{2/3}, & r=s=0; \\ x, & x=y. \end{cases} \quad (3.7)$$

Main results concerning the means E^* are contained in the following propositions.

Proposition 3.4. *Means $E_{r,s}^*(x, y)$ are monotone increasing in either r or s for each fixed $x, y \in \mathbb{R}^+$.*

Proposition 3.5. *Means $E_{r,s}^*(x, y)$ are monotone increasing in either x or y for each $r, s \in \mathbb{R}$.*

A well known result of Qi ([5]) states that the means $E_{r,s}(x, y)$ are logarithmically concave for each fixed $x, y > 0$ and $r, s \in [0, +\infty)$; also, they are logarithmically convex for $r, s \in (-\infty, 0]$.

According to this, we propose the following proposition.

Open Question

Is there any compact interval $I, I \subset \mathbb{R}$ such that the means $E_{r,s}^*(x, y)$ are logarithmically convex (concave) for $r, s \in I$ and each $x, y \in \mathbb{R}^+$?

A partial answer to this problem is given in what follows.

Proposition 3.6. *On any interval I which includes zero and $r, s \in I$,*

- (i) $E_{r,s}^*(x, y)$ *are not logarithmically convex (concave);*
- (ii) $W_{r,s}(p, q; x, y)$ *are logarithmically convex (concave) if and only if $p = q = 1/2$.*

4. Proofs

For the proof of Proposition 2.1, we apply the following assertion on Jensen functionals $J_f(\mathbf{p}, \mathbf{x})$ from [6].

Theorem 4.1. *Let $f, g : I \rightarrow \mathbb{R}$ be twice continuously differentiable functions. Assume that g is strictly convex and ϕ is a continuous and strictly monotonic function on I . Then the expression*

$$\phi^{-1}\left(\frac{J_n(\mathbf{p}, \mathbf{x}; f)}{J_n(\mathbf{p}, \mathbf{x}; g)}\right) \quad (n \geq 2) \quad (4.1)$$

represents a mean value of the numbers $x_1, \dots, x_n$, that is,

$$\min\{x_1, \dots, x_n\} \leq \phi^{-1}\left(\frac{J_n(\mathbf{p}, \mathbf{x}; f)}{J_n(\mathbf{p}, \mathbf{x}; g)}\right) \leq \max\{x_1, \dots, x_n\} \quad (4.2)$$

if and only if the relation

$$f''(t) = \phi(t)g''(t) \quad (4.3)$$

holds for each $t \in I$.

Recall that the Jensen functional $J_n(\mathbf{p}, \mathbf{x}; f)$ is defined on an interval $I, I \subseteq \mathbb{R}$ by

$$J_n(\mathbf{p}, \mathbf{x}; f) := \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right), \quad (4.4)$$

where $f : I \rightarrow \mathbb{R}, \mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$, and $\mathbf{p} = \{p_i\}_1^n$ is a positive weight sequence.

The famous Jensen's inequality asserts that

$$J_n(\mathbf{p}, \mathbf{x}; f) \geq 0, \quad (4.5)$$

whenever f is a (strictly) convex function on I , with the equality case if and only if $x_1 = x_2 = \dots = x_n$.

Proof of Proposition 2.1. Define the auxiliary function $h_s(x)$ by

$$h_s(x) := \begin{cases} \frac{e^{sx} - sx - 1}{s^2}, & s \neq 0; \\ \frac{x^2}{2}, & s = 0. \end{cases} \quad (4.6)$$

Since

$$h'_s(x) = \begin{cases} \frac{e^{sx} - 1}{s}, & s \neq 0; \\ x, & s = 0, \end{cases} \quad (4.7)$$

$$h''_s(x) = e^{sx}, \quad s \in \mathbb{R},$$

we conclude that $h_s(x)$ is a continuously twice differentiable convex function on $\mathbb{R}$.

Denoting $f(t) := h_s(t)$, $g(t) := h_r(t)$, we realize that the condition (4.3) of Theorem 4.1 is fulfilled with $\phi(t) = e^{(s-r)t}$. Hence, applying Theorem 4.1, we obtain that $\log W_{r,s}(\mathbf{p}, e^x)$ represents a mean value, which is equivalent to the assertion of Proposition 2.1. $\square$

Proof of Proposition 2.2. We prove first a global theorem concerning log-convexity of the Jensen's functional with a parameter, which can be very usable (cf. [7]).

Theorem 4.2. *Let $f_s(x)$ be a twice continuously differentiable function in x with a parameter s . If $f''_s(x)$ is log-convex in s for $s \in I := (a, b)$; $x \in K := (c, d)$, then the Jensen functional*

$$J_f(w, x; s) = J(s) := \sum w_i f_s(x_i) - f_s\left(\sum w_i x_i\right), \quad (4.8)$$

is log-convex in s for $s \in I$, $x_i \in K$, $i = 1, 2, \dots$, where $w = \{w_i\}$ is any positive weight sequence.

At the beginning, we need some preliminary lemmas.

Lemma 4.3. *A positive function f is log-convex on I if and only if the relation*

$$f(s)u^2 + 2f\left(\frac{s+t}{2}\right)uw + f(t)w^2 \geq 0 \quad (4.9)$$

holds for each real u, w and $s, t \in I$.

This assertion is nothing more than the discriminant test for the nonnegativity of second-order polynomials. Other well known assertions are the following (cf [8, pages 74, 97-98]) lemmas.

Lemma 4.4 (Jensen's inequality). *If $g(x)$ is twice continuously differentiable and $g''(x) \geq 0$ on K , then $g(x)$ is convex on K and the inequality*

$$\sum w_i g(x_i) - g\left(\sum w_i x_i\right) \geq 0 \quad (4.10)$$

holds for each $x_i \in K$, $i = 1, 2, \dots$, and any positive weight sequence $\{w_i\}$, $\sum w_i = 1$.

Lemma 4.5. *For a convex f , the expression*

$$\frac{f(s) - f(r)}{s - r} \quad (4.11)$$

is increasing in both variables.

Proof of Theorem 4.2. Consider the function $F(x)$ defined as

$$F(x) = F(u, v, s, t; x) := u^2 f_s(x) + 2uv f_{(s+t)/2}(x) + v^2 f_t(x), \quad (4.12)$$

where $u, v \in \mathbb{R}$; $s, t \in I$ are real parameters independent of the variable $x \in K$.

Since

$$F''(x) = u^2 f''_s(x) + 2uv f''_{(s+t)/2}(x) + v^2 f''_t(x), \quad (4.13)$$

and by assuming $f''_s(x)$ is log-convex in s , it follows from Lemma 4.3 that $F''(x) \geq 0$, $x \in K$.

Therefore, by Lemma 4.4, we get

$$\sum w_i F(x_i) - F\left(\sum w_i x_i\right) \geq 0, \quad x_i \in K, \quad (4.14)$$

which is equivalent to

$$u^2 J(s) + 2uv J\left(\frac{s+t}{2}\right) + v^2 J(t) \geq 0. \quad (4.15)$$

According to Lemma 4.3 again, this is possible only if $J(s)$ is log-convex and the proof is done. $\square$

Now, the proof of Proposition 2.2 easily follows.

From the above, we see that $h_s(x)$ is twice continuously differentiable and that $h''_s(x)$ is a log-convex function for each real s, x .

Applying Theorem 4.2, we conclude that the form

$$\Phi_h(w, x; s) = \Phi(s) := \begin{cases} \frac{\sum w_i e^{sx_i} - e^{s \sum w_i x_i}}{s^2}, & s \neq 0, \\ \frac{\sum w_i x_i^2 - (\sum w_i x_i)^2}{2}, & s = 0, \end{cases} \quad (4.16)$$

is log-convex in s .

By Lemma 4.5, with $f(s) = \log \Phi(s)$, we find out that

$$\frac{\log \Phi(s) - \log \Phi(r)}{s - r} = \log \left(\frac{\Phi(s)}{\Phi(r)} \right)^{1/(s-r)} \quad (4.17)$$

is monotone increasing either in s or r . Therefore, by changing variable $x_i \rightarrow \log x_i$, we finally obtain the proof of Proposition 2.2. $\square$

Proof of Proposition 2.3. The assertion of Proposition 2.3 follows from Proposition 2.2 by the standard argument (cf. [8, pages 131–134]). Details are left to the reader. $\square$

Proof of Proposition 3.1. The proof follows putting $x(t) = t$, $p(t) = e^{-t}$, $t \in (0, +\infty)$ and applying Proposition 2.2. with $r = 0$. Corresponding integrals are

$$\int_0^\infty e^{-t} \log t = -\gamma; \quad \int_0^\infty e^{-t} \log^2 t = \gamma^2 + \frac{\pi^2}{6}; \quad \int_0^\infty e^{-t} \log^3 t = -\gamma^3 - \frac{\gamma \pi^2}{2} - 2\zeta(3), \quad (4.18)$$

with

$$\Gamma(1-s)\Gamma(1+s) = \frac{\pi s}{\sin(\pi s)}. \quad (4.19)$$

$\square$

Proof of Proposition 3.2. By Proposition 2.3, we get

$$W_{0,1}(\mathbf{p}, e^x) \geq W_{0,0}(\mathbf{p}, e^x), \quad (4.20)$$

that is,

$$\frac{Ee^X - e^{EX}}{\mu_2/2} \geq \exp\left(\frac{EX^3 - (EX)^3}{3\mu_2}\right). \quad (4.21)$$

Using the identity $EX^3 - (EX)^3 = \mu_3 + 3\mu_2 EX$, we obtain the proof of Proposition 3.2. $\square$

Proof of Proposition 3.3. This assertion is straightforward consequence of the fact that $W_{0,s}(\mathbf{p}, e^x)$ is monotone increasing in s . $\square$

Proof of Proposition 3.4. Direct consequence of Proposition 2.2. $\square$

Proof of Proposition 3.5. This is left as an easy exercise to the readers. $\square$

Proof of Proposition 3.6. We prove only part (ii). The proof of (i) goes along the same lines.

Suppose that $0 \in (a, b) := I$ and that $E_{r,s}(p, q; x, y)$ are log-convex (concave) for $r, s \in I$ and any fixed $x, y \in \mathbb{R}^+$. Then there should be an $s, s > 0$ such that

$$F_s(p, q; x, y) := W_{0,s}(p, q; x, y)W_{0,-s}(p, q; x, y) - (W_{0,0}(p, q; x, y))^2 \quad (4.22)$$

is of constant sign for each $x, y > 0$.

Substituting $(x/y)^s := e^w$, $w \in \mathbb{R}$, after some calculations, we get that the above is equivalent to the assertion that $F(p, q; w)$ is of constant sign, where

$$F(p, q; w) := pe^w + q - e^{pw} - e^{(2/3)(1+p)w}(pe^{-w} + q - e^{-pw}). \quad (4.23)$$

Developing in power series in w , we get

$$F(p, q; w) = \frac{1}{1620}pq(1+p)(2-p)(1-2p)w^5 + O(w^6). \quad (4.24)$$

Therefore, $F(p, q; w)$ can be of constant sign for each $w \in \mathbb{R}$ only if $p = 1/2 (= q)$.

Suppose now that I is of the form $I := [0, a)$ or $I := (-a, 0]$, $a > 0$. Then there should be an $s, s \neq 0, s \in I$ such that

$$W_{0,0}(p, q; x, y)W_{0,2s}(p, q; x, y) - (W_{0,s}(p, q; x, y))^2 \quad (4.25)$$

is of constant sign for each $x, y \in \mathbb{R}^+$.

Proceeding as before, this is equivalent to the assertion that $G(p, q; w)$ is of constant sign with

$$G(p, q; w) := p^3q^3w^6e^{(2/3)(p+1)w}(pe^{2w} + q - e^{2pw}) - (pe^w + q - e^{pw})^4. \quad (4.26)$$

However,

$$G(p, q; w) = \frac{2}{405}p^4q^4(1+p)(1+q)(q-p)w^{11} + O(w^{12}). \quad (4.27)$$

Hence, we conclude that $G(p, q; w)$ can be of constant sign for sufficiently small w , $w \in \mathbb{R}$ only if $p = q = 1/2$. Combining this with Feng Qi theorem, the assertion from Proposition 3.6 follows. $\square$

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