Research Article

Some Properties of Fractional Calculus and Linear Operators Associated with Certain Subclass of Multivalent Functions

Sh. Khosravianarab,¹ S. R. Kulkarni,² and O. P. Ahuja³

¹ Department of Mathematics, Pune University, 411007 Pune, India

² Department of Mathematics, Fergusson College, 411004 Pune, India

³ Department of Mathematical Sciences, Kent State University, 14111 Claridon Troy Road, Burton, OH 44021-9500, USA

Correspondence should be addressed to Sh. Khosravianarab, shkhosravian@yahoo.com

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We investigate several distortion inequalities involving fractional calculus, Ruscheweyh derivatives, and some well-known integral operators. In special cases, the results presented in this paper provide new approaches to several previously known results.

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1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions f of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \quad (1.1)$$

which are analytic in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also, let $\mathcal{K}(p)$ denote the subclass of $\mathcal{A}(p)$ consisting of all functions f of the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}, a_{p+k} \geq 0), \quad (1.2)$$

For functions $f, g \in \mathcal{A}(p)$, given by

$$f(z) := z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad g(z) := z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}, \quad (1.3)$$

the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}. \quad (1.4)$$

The Ruscheweyh derivative of f of order $\delta + p - 1$ is defined by

$$D^{\delta+p-1} f(z) := \frac{z^p}{(1-z)^{\delta+p}} * f(z) = z^p + \sum_{k=1}^{\infty} \frac{(\delta+p)_k}{(1)_k} a_{p+k} z^{p+k}, \quad (1.5)$$

where f is given by (1.1) and $\delta > -p$. The Ruscheweyh derivative $D^{\delta+p-1}$ has been studied by several authors; for example, see [1, 2].

For $\beta < 1$, $\gamma \geq 0$, $p \in \mathbb{N}$, and $\delta > -p$, let $\mathcal{T}_{\delta,p,\gamma}(\beta)$ consist of functions $f \in \mathcal{K}(p)$ so that

$$\Re \left\{ e^{i\eta} \left((1-\gamma) \frac{D^{\delta+p-1} f(z)}{z^p} + \gamma \frac{(D^{\delta+p-1} f(z))'}{p z^{p-1}} - \beta \right) \right\} > 0, \quad z \in \mathcal{U}, \quad (1.6)$$

for some $\eta \in \mathbb{R}$. In [3], the authors obtained four containment results for the class $\mathcal{T}_{\delta,p,\gamma}(\beta)$. We denote $\mathcal{T}_{0,1,\gamma}(\beta) = \mathcal{T}_{\gamma}(\beta)$. The class $\mathcal{T}_{\gamma}(\beta)$ was studied by Swaminathan [4–6], Barnard et al. [7], Kim and Rønning [8], and others.

In the present paper, we investigate several distortion inequalities involving fractional calculus, Ruscheweyh derivative, and some well-known integral operators defined on the class $\mathcal{T}_{\delta,p,\gamma}(\beta)$. In special cases, the results presented here provide new approaches to some previously known results.

Remark 1.1. Throughout this section, we assume that $\delta + p \geq 1$.

2. Definitions and Lemmas

For the function f given by (1.2), we define

$$q_{\delta,p,\gamma}(z) = (1-\gamma) \frac{D^{\delta+p-1} f(z)}{z^p} + \gamma \frac{(D^{\delta+p-1} f(z))'}{p z^{p-1}}. \quad (2.1)$$

It is easy to verify that

$$q_{\delta,p,\gamma}(z) = 1 - \sum_{k=1}^{\infty} \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k} a_{p+k} z^k, \quad a_{p+k} \geq 0. \quad (2.2)$$

Lemma 2.1. Let the function f be given by (1.2). Then $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$ if and only if

$$\sum_{k=1}^{\infty} \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k} a_{p+k} \leq \frac{1}{2} \left(\left| 1 + e^{i\eta} (1-\beta) \right| - \left| 1 - e^{i\eta} (1-\beta) \right| \right) \quad (2.3)$$

for some $\eta \in \mathbb{R}$.

Proof. Using the fact that $\operatorname{Re}(\omega) \geq 0$ if and only if $|1 + \omega| \geq |1 - \omega|$, it suffices to show that

$$\left| 1 + e^{i\eta} (q_{\delta,p,\gamma}(z) - \beta) \right| - \left| 1 - e^{i\eta} (q_{\delta,p,\gamma}(z) - \beta) \right| \geq 0, \quad (2.4)$$

where $q_{\delta,p,\gamma}(z)$ is defined by (2.1). Letting

$$\mathcal{A}_k = \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k}, \quad (2.5)$$

and assuming (2.3), we obtain

$$\begin{aligned} & \left| 1 + e^{i\eta} (q_{\delta,p,\gamma}(z) - \beta) \right| - \left| 1 - e^{i\eta} (q_{\delta,p,\gamma}(z) - \beta) \right| \\ &= \left| 1 + e^{i\eta} (1-\beta) - e^{i\eta} \sum_{k=1}^{\infty} \mathcal{A}_k a_{p+k} z^k \right| - \left| 1 - e^{i\eta} (1-\beta) + e^{i\eta} \sum_{k=1}^{\infty} \mathcal{A}_k a_{p+k} z^k \right| \\ &\geq \left| 1 + e^{i\eta} (1-\beta) \right| - \left| e^{i\eta} \sum_{k=1}^{\infty} \mathcal{A}_k a_{p+k} z^k \right| - \left| 1 - e^{i\eta} (1-\beta) \right| - \left| e^{i\eta} \sum_{k=1}^{\infty} \mathcal{A}_k a_{p+k} z^k \right| \\ &\geq \left| 1 + e^{i\eta} (1-\beta) \right| - \left| 1 - e^{i\eta} (1-\beta) \right| - 2 \sum_{k=1}^{\infty} \mathcal{A}_k a_{p+k} \geq 0, \end{aligned} \quad (2.6)$$

where $z \in \partial \mathcal{U} = \{z; \ z \in \mathbb{C} \text{ and } |z| = 1\}$. By (2.3), the desired inequality (2.4) follows at once. Conversely, if $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then

$$\operatorname{Re} \left\{ e^{i\eta} (q_{\delta,p,\gamma}(z) - \beta) \right\} > 0, \quad (2.7)$$

or, equivalently (2.4). This yields

$$\left| 1 + e^{i\eta} \left(1 - \sum_{k=1}^{\infty} \mathcal{A}_k a_{p+k} z^k - \beta \right) \right| \geq \left| 1 - e^{i\eta} \left(1 - \sum_{k=1}^{\infty} \mathcal{A}_k a_{p+k} z^k - \beta \right) \right|, \quad (2.8)$$

which implies that

$$\left| 1 + e^{i\eta} (1-\beta) - e^{i\eta} \sum_{k=1}^{\infty} \mathcal{A}_k a_{p+k} z^k \right| \geq \left| 1 - e^{i\eta} (1-\beta) + e^{i\eta} \sum_{k=1}^{\infty} \mathcal{A}_k a_{p+k} z^k \right|. \quad (2.9)$$

Squaring the above inequality, choosing the value of z on the half line $z = re^{-i\theta}$ ($0 \leq r < 1$), and letting $r \rightarrow 1^-$ through this line, we obtain

$$\begin{aligned} & \left| 1 + e^{i\eta}(1 - \beta) \right|^2 + \left| e^{i\eta} \right|^2 \left| \sum_{k=1}^{\infty} A_k a_{p+k} z^k \right|^2 - 2 \left| 1 + e^{i\eta}(1 - \beta) \right| \left| \sum_{k=1}^{\infty} A_k a_{p+k} z^k \right| \\ & \geq \left| 1 - e^{i\eta}(1 - \beta) \right|^2 + \left| e^{i\eta} \right|^2 \left| \sum_{k=1}^{\infty} A_k a_{p+k} z^k \right|^2 + 2 \left| 1 - e^{i\eta}(1 - \beta) \right| \left| \sum_{k=1}^{\infty} A_k a_{p+k} z^k \right|. \end{aligned} \quad (2.10)$$

Hence we get

$$\begin{aligned} & 2 \left| \sum_{k=1}^{\infty} A_k a_{p+k} z^k \right| \left(\left| 1 - e^{i\eta}(1 - \beta) \right| + \left| 1 + e^{i\eta}(1 - \beta) \right| \right) \leq \left| 1 + e^{i\eta}(1 - \beta) \right|^2 - \left| 1 - e^{i\eta}(1 - \beta) \right|^2 \\ & = \left(\left| 1 + e^{i\eta}(1 - \beta) \right| - \left| 1 - e^{i\eta}(1 - \beta) \right| \right) \left(\left| 1 + e^{i\eta}(1 - \beta) \right| + \left| 1 - e^{i\eta}(1 - \beta) \right| \right), \end{aligned} \quad (2.11)$$

which reduces to

$$2 \sum_{k=1}^{\infty} A_k a_{p+k} \leq \left| 1 + e^{i\eta}(1 - \beta) \right| - \left| 1 - e^{i\eta}(1 - \beta) \right|. \quad (2.12)$$

So the desired inequality (2.3) follows upon using (2.5). $\square$

Setting $\eta = \delta = 0$ and $p = 1$ in Lemma 2.1, we get the following result.

Corollary 2.2 ([5, Theorem 2.4]). *Let $f(z)$ be of the form (1.2). Then necessary and sufficient condition for f to be in $\mathcal{T}_\gamma(\beta)$ is*

$$\sum_{k=2}^{\infty} [1 + \gamma(k-1)] a_k \leq 1 - \beta, \quad a_k \geq 0. \quad (2.13)$$

Throughout this paper, we define

$$E_{\eta, \beta} := \left| 1 + e^{i\eta}(1 - \beta) \right| - \left| 1 - e^{i\eta}(1 - \beta) \right|. \quad (2.14)$$

As an immediate consequence of Lemma 2.1, we have the following corollary.

Corollary 2.3. *Let the function f be defined by (1.2). If $f \in \mathcal{T}_{\delta, p, \gamma}(\beta)$, then*

$$a_{p+k} \leq \frac{p(1)_k E_{\eta, \beta}}{2(\delta + p)_k (p + k\gamma)} \quad (2.15)$$

for some $\eta \in \mathbb{R}$.

Let $\alpha_1, \alpha_2, \dots, \alpha_p$ and $\beta_1, \beta_2, \dots, \beta_q$ ($p, q \in \mathbb{N} \cup \{0\}$, $p \leq q + 1$) be complex numbers such that $\beta_k \neq 0, -1, -2, \dots$ for $k \in \{1, 2, \dots, q\}$. The generalized hypergeometric function ${}_pF_q$ is given by

$${}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!}, \quad z \in \mathcal{U}, \quad (2.16)$$

where $(x)_n$ denotes the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)(x+2) \cdots (x+n-1), \quad \text{for } n \in \mathbb{N}, \quad (x)_0 = 1. \quad (2.17)$$

The operator ${}_pF_q$ has recently been studied by several authors; for example, [3, 5]. For $p = q + 1 = 2$, the above series give rise to the Gaussian hypergeometric series $F(a, b; c; z)$.

In [9], Hohlov introduced the convolution operator $H_{a,b;c}$ by

$$H_{a,b;c}(f)(z) := zF(a, b; c; z) * f(z), \quad f \in \mathbb{A}(1). \quad (2.18)$$

Motivated by the operator $H_{a,b;c}$, the authors in [3] defined the convolution operators G_f^p and $H_{a,b;c}^{p,d}$ as follows:

$$G_f^p(a, b; z) := G(z) := \left(\sum_{k=1}^{\infty} \frac{(1+a)(1+b)}{(k+a)(k+b)} z^{k+p-1} \right) * f(z), \quad (a > -1, b > -1), \quad (2.19)$$

$$H_{a,b;c}^{p,d}(f)(z) = H(z) = z^p {}_3F_2\left(a, b, 1 + \frac{p}{d}; \frac{p}{d}, c; z\right) * f(z), \quad (2.20)$$

where $f \in \mathcal{A}(p)$, $p \in \mathbb{N}$, and $d \geq 0$. For $p = 1$, the operator $G_f^1(a, b; z)$ was introduced in [7].

In Section 3, we will make use of the following well-known fractional calculus operators $D_z^{-\mu}$, D_z^{μ} , and $D_z^{n+\mu}$. For an analytic function f defined in a simply connected region of the complex z -plane containing the origin, these operators are defined as follows (See [1, 10]):

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(t)}{(z-t)^{1-\mu}} dt \quad (\mu > 0), \quad (2.21)$$

where multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t > 0$;

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \int_0^z \frac{f(t)}{(z-t)^{\mu}} dt \quad (0 \leq \mu < 1), \quad (2.22)$$

where the multiplicity of $(z-t)^{-\mu}$ is removed, as in the definition of $D_z^{-\mu} f(z)$;

$$D_z^{n+\mu} f(z) := \frac{d^n}{dz^n} D_z^{\mu} f(z) \quad (0 \leq \mu < 1; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \quad (2.23)$$

By virtue of (2.21), (2.22), (2.23) and in terms of Gamma function, it is wellknown (see for details [11]) that

$$\begin{aligned} D_z^{-\mu} z^k &= \frac{\Gamma(k+1)}{\Gamma(k+\mu+1)} z^{k+\mu} \quad (k \in \mathbb{N}, \mu > 0), \\ D_z^{\mu} z^k &= \frac{\Gamma(k+1)}{\Gamma(k-\mu+1)} z^{k-\mu} \quad (k \in \mathbb{N}, 0 \leq \mu < 1), \\ D_z^{q+\mu} z^k &= \frac{d^q}{dz^q} D_z^{\mu} z^k = \frac{\Gamma(k+1)}{\Gamma(k-q-\mu+1)} z^{k-(q+\mu)}, \end{aligned} \quad (2.24)$$

where $q \in \mathbb{N}_0$, $k \in \mathbb{N}$, $0 \leq \mu < 1$, and $q \leq k$ for $\mu = 0$.

In Section 4, we will investigate the integral operator $J_{\delta,p}$ defined by

$$(J_{\delta,p}f)(z) = \frac{\delta+p}{z^p} \int_0^z t^{\delta-1} f(t) dt, \quad (2.25)$$

where $f \in \mathcal{A}(p)$, $\delta > -p$, and $p \in \mathbb{N}$. For $p = 1$ and $\delta = 0$, the operator was first defined by Bernardi [12]. Later on several authors studied the operator $J_{\delta,p}$; for example, see [1, 5].

3. Distortion Inequalities of Convolution Operators

Theorem 3.1. *Let the function f defined by (1.2) be in the class $\mathcal{T}_{\delta,p,\gamma}(\beta)$. Then*

$$\begin{aligned} |q_{\delta,p,\gamma}(z)| &\geq 1 - \frac{1}{2} E_{\eta,\beta} |z|, \\ |q_{\delta,p,\gamma}(z)| &\leq 1 + \frac{1}{2} E_{\eta,\beta} |z| \end{aligned} \quad (3.1)$$

for some $\eta \in \mathbb{R}$. Here, $q_{\delta,p,\gamma}(z)$ and $E_{\eta,\beta}$ are defined, respectively, by (2.1) and (2.14).

Proof. From (2.2), we have

$$\begin{aligned} |q_{\delta,p,\gamma}(z)| &\geq 1 - \left| \sum_{k=1}^{\infty} \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k} a_{p+k} z^k \right| \\ &\geq 1 - \sum_{k=1}^{\infty} \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k} a_{p+k} |z|^k \\ &\geq 1 - |z| \sum_{k=1}^{\infty} \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k} a_{p+k}. \end{aligned} \quad (3.2)$$

Making use of Lemma 2.1, we get

$$|q_{\delta,p,\gamma}(z)| \geq 1 - \frac{1}{2} E_{\eta,\beta} |z| \quad (3.3)$$

for some $\eta \in \mathbb{R}$. Similarly,

$$|q_{\delta,p,\gamma}(z)| \leq 1 + |z| \sum_{k=1}^{\infty} \frac{(\delta+p)_k (1+(k\gamma/p))}{(1)_k} a_{p+k} \leq 1 + \frac{1}{2} E_{\eta,\beta} |z| \quad (3.4)$$

for some $\eta \in \mathbb{R}$. This completes the proof. $\square$

We next obtain distortion inequalities for the fractional operators D_z^μ and $D_z^{-\mu}$.

Theorem 3.2. Suppose $\mu \leq (1+k+p)/(k+b+2)$ and $p < b+1$. If $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then for some $\eta \in \mathbb{R}$, one has

$$\begin{aligned} |D_z^\mu G(z)| &\geq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} |z|^{p-\mu} \left(1 - \frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1-\mu+p)(\delta+p)(\gamma+p)} |z| E_{\eta,\beta} \right), \\ |D_z^\mu G(z)| &\leq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} |z|^{p-\mu} \left(1 + \frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1-\mu+p)(\delta+p)(\gamma+p)} |z| E_{\eta,\beta} \right), \end{aligned} \quad (3.5)$$

where $0 \leq \mu < 1$, $z \in \mathcal{U}$, $p \in \mathbb{N}$, and the operator $G(z) := G_f^p(a, b; z)$ was defined by (2.19).

Proof. By using (2.19), we deduce that

$$D_z^\mu G(z) = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} z^{p-\mu} - \sum_{k=1}^{\infty} \frac{(1+a)(1+b)\Gamma(k+p+1)}{(k+a+1)(k+b+1)\Gamma(k+p-\mu+1)} a_{p+k} z^{p+k-\mu}. \quad (3.6)$$

Then

$$\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)} z^{\mu-p} D_z^\mu G(z) = 1 - \sum_{k=1}^{\infty} \theta(k) a_{p+k} z^k, \quad (3.7)$$

where

$$\theta(k) = \frac{(1+a)(1+b)\Gamma(k+p+1)\Gamma(p-\mu+1)}{(k+a+1)(k+b+1)\Gamma(k+p-\mu+1)\Gamma(p+1)} \quad (k, p \in \mathbb{N}, 0 \leq \mu < 1). \quad (3.8)$$

Since $\theta(k)$ is a decreasing function of k , when $\mu \leq (1+k+p)/(k+b+2)$, then

$$0 < \theta(k) \leq \theta(1) = \frac{(1+a)(1+b)(1+p)}{(2+a)(2+b)(1-\mu+p)}. \quad (3.9)$$

Also, according to Lemma 2.1 and $\delta + p \geq 1$, we have

$$\begin{aligned} \frac{1}{p}(\delta + p)(p + \gamma) \sum_{k=1}^{\infty} a_{p+k} &= \frac{\Gamma(\delta + p + 1)(1 + (\gamma/p))}{\Gamma(\delta + p)} \sum_{k=1}^{\infty} a_{p+k} \\ &\leq \sum_{k=1}^{\infty} \frac{(\delta + p)_k (1 + (k\gamma/p))}{(1)_k} a_{p+k} \\ &\leq \frac{1}{2} E_{\eta, \beta} \end{aligned} \quad (3.10)$$

for some $\eta \in \mathbb{R}$. Then

$$\sum_{k=1}^{\infty} a_{p+k} \leq \frac{p}{2(\delta + p)(p + \gamma)} E_{\eta, \beta} \quad (3.11)$$

for some $\eta \in \mathbb{R}$. From (3.7) and (3.9), we obtain

$$\frac{\Gamma(p - \mu + 1)}{\Gamma(p + 1)} z^{\mu-p} D_z^{\mu} G(z) \geq 1 - \theta(1) \sum_{k=1}^{\infty} a_{p+k} z^k. \quad (3.12)$$

In view of (3.11), we conclude that

$$\left| \frac{\Gamma(p - \mu + 1)}{\Gamma(p + 1)} z^{\mu-p} D_z^{\mu} G(z) \right| \geq 1 - \frac{p(1 + a)(1 + b)(1 + p)}{2(2 + a)(2 + b)(1 - \mu + p)(\delta + p)(\gamma + p)} |z| E_{\eta, \beta} \quad (3.13)$$

for some $\eta \in \mathbb{R}$, and

$$\left| \frac{\Gamma(p - \mu + 1)}{\Gamma(p + 1)} z^{\mu-p} D_z^{\mu} G(z) \right| \leq 1 + \frac{p(1 + a)(1 + b)(1 + p)}{2(2 + a)(2 + b)(1 - \mu + p)(\delta + p)(\gamma + p)} |z| E_{\eta, \beta} \quad (3.14)$$

for some $\eta \in \mathbb{R}$, which yield (3.5). $\square$

By letting $p = 1$ and $b = a - 1 > 0$ in Theorem 3.2, we deduce the following consequence.

Corollary 3.3. *If $f \in \mathcal{T}_{\delta, 1, b}(\beta)$, then for $0 \leq \mu < 1$, $z \in \mathcal{U}$, $\delta \geq 0$ and some $\eta \in \mathbb{R}$*

$$\begin{aligned} \left| D_z^{\mu} G(z) \right| &\geq \frac{1}{\Gamma(2 - \mu)} |z|^{1-\mu} \left(1 - \frac{1}{(3 + b)(2 - \mu)(1 + \delta)} |z| E_{\eta, \beta} \right), \\ \left| D_z^{\mu} G(z) \right| &\leq \frac{1}{\Gamma(2 - \mu)} |z|^{1-\mu} \left(1 + \frac{1}{(3 + b)(2 - \mu)(1 + \delta)} |z| E_{\eta, \beta} \right). \end{aligned} \quad (3.15)$$

Theorem 3.4. Let $\mu > 0$, $z \in \mathcal{U}$ and $p \in \mathbb{N}$. If $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then

$$\left| D_z^{-\mu} G(z) \right| \geq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} |z|^{p+\mu} \left(1 - \frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1-\mu+p)(\delta+p)(\gamma+p)} |z|^{E_{\eta,\beta}} \right), \quad (3.16)$$

$$\left| D_z^{-\mu} G(z) \right| \leq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} |z|^{p+\mu} \left(1 + \frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1-\mu+p)(\delta+p)(\gamma+p)} |z|^{E_{\eta,\beta}} \right) \quad (3.17)$$

for some $\eta \in \mathbb{R}$. The operator $G(z) := G_f^p(a, b; z)$ was defined by (2.19).

Proof. In view of (2.19) and (2.22), we have

$$\frac{\Gamma(p+\mu+1)}{\Gamma(p+1)} z^{-(\mu+p)} D_z^{-\mu} G(z) = 1 - \sum_{k=1}^{\infty} \tau(k) a_{p+k} z^k, \quad (3.18)$$

where

$$\tau(k) = \frac{(1+a)(1+b)\Gamma(k+p+1)\Gamma(p+\mu+1)}{(k+a+1)(k+b+1)\Gamma(k+p+\mu+1)\Gamma(p+1)}. \quad (3.19)$$

Since τ is a decreasing function of k , it follows that

$$0 < \tau(k) \leq \tau(1) = \frac{(1+a)(1+b)(1+p)}{(2+a)(2+b)(1+\mu+p)}. \quad (3.20)$$

By using (3.11), (3.18), and (3.20), we get

$$\begin{aligned} \left| \frac{\Gamma(p+\mu+1)}{\Gamma(p+1)} z^{-(\mu+p)} D_z^{-\mu} G(z) \right| &\geq 1 - \frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1+\mu+p)(\delta+p)(\gamma+p)} |z|^{E_{\eta,\beta}}, \\ \left| \frac{\Gamma(p+\mu+1)}{\Gamma(p+1)} z^{-(\mu+p)} D_z^{-\mu} G(z) \right| &\leq 1 + \frac{p(1+a)(1+b)(1+p)}{2(2+a)(2+b)(1+\mu+p)(\delta+p)(\gamma+p)} |z|^{E_{\eta,\beta}} \end{aligned} \quad (3.21)$$

for some $\eta \in \mathbb{R}$. The last two inequalities yield (3.16) and (3.17), respectively. $\square$

Letting $\delta = 0$, $p = 1$, and $a = b + 1 = \mu + 2$ in Theorem 3.4, we get the following result.

Corollary 3.5. Let $G_f^1(\mu + 2, \mu + 1; z)$ be defined by (2.19). If $f \in \mathcal{T}_\gamma(\beta)$, then

$$\begin{aligned} \left| D_z^{-\mu} G_f^1(\mu + 2, \mu + 1; z) \right| &\geq \frac{1}{\Gamma(\mu + 2)} |z|^{\mu+1} \left(1 - \frac{1}{(\mu + 4)(\gamma + 1)} |z| E_{\eta, \beta} \right), \\ \left| D_z^{-\mu} G_f^1(\mu + 2, \mu + 1; z) \right| &\leq \frac{1}{\Gamma(\mu + 2)} |z|^{\mu+1} \left(1 + \frac{1}{(\mu + 4)(\gamma + 1)} |z| E_{\eta, \beta} \right) \end{aligned} \quad (3.22)$$

for some $\eta \in \mathbb{R}$, $\mu > 0$, $z \in \mathcal{U}$, and $p \in \mathbb{N}$.

We next prove the distortion theorems involving fractional calculus and generalized convolution operator defined by (2.20).

Theorem 3.6. Suppose $0 \leq \mu < 1$, $z \in \mathcal{U}$, $p \in \mathbb{N}$, and $\eta \in \mathbb{R}$. Also, let $a \leq 1$, $b \leq p - \mu + 1$, and $c \geq p + 1$. If $f \in \mathcal{T}_{\delta, p, \gamma}(\beta)$, then

$$\begin{aligned} \left| D_z^\mu H(f)(z) \right| &\geq \frac{\Gamma(p + 1)}{\Gamma(p - \mu + 1)} |z|^{p-\mu} \left(1 - \frac{ab(p + 1)}{2c(1 - \mu + p)(\delta + p)} |z| E_{\eta, \beta} \right), \\ \left| D_z^\mu H(f)(z) \right| &\leq \frac{\Gamma(p + 1)}{\Gamma(p - \mu + 1)} |z|^{p-\mu} \left(1 + \frac{ab(p + 1)}{2c(1 - \mu + p)(\delta + p)} |z| E_{\eta, \beta} \right) \end{aligned} \quad (3.23)$$

for some $\eta \in \mathbb{R}$. Here, the operator $H(f)(z)$ is defined by (2.20).

Proof. By making use of (2.20), we have

$$\begin{aligned} D_z^\mu H(f)(z) &= D_z^\mu \left\{ z^p - \sum_{k=1}^{\infty} \frac{(a)_k (b)_k (1 + (p/\gamma))_k}{(p/\gamma)_k (c)_k (1)_k} a_{p+k} z^{p+k} \right\} \\ &= \frac{\Gamma(p + 1)}{\Gamma(p - \mu + 1)} z^{p-\mu} - \sum_{k=1}^{\infty} \frac{(a)_k (b)_k (1 + (p/\gamma))_k \Gamma(p + k + 1)}{(p/\gamma)_k (c)_k (1)_k \Gamma(p + k - \mu + 1)} a_{p+k} z^{p+k-\mu}. \end{aligned} \quad (3.24)$$

It is easy to verify that

$$1 + \frac{k\gamma}{p} = \frac{(1 + (p/\gamma))_k}{(p/\gamma)_k}. \quad (3.25)$$

This implies that

$$\frac{\Gamma(p - \mu + 1)}{\Gamma(p + 1)} z^{\mu-p} D_z^\mu H(f)(z) = 1 - \sum_{k=1}^{\infty} \lambda(k) \left(1 + \frac{k\gamma}{p} \right) a_{p+k} z^k, \quad (3.26)$$

where

$$\lambda(k) = \frac{(a)_k(b)_k\Gamma(p+k+1)\Gamma(p-\mu+1)}{(c)_k(1)_k\Gamma(p+k-\mu+1)\Gamma(p+1)}. \quad (3.27)$$

Since λ is a decreasing function of k , when $a \leq 1$, $c \geq p+1$ and $b \leq p-\mu+1$, we get

$$0 < \lambda(k) \leq \lambda(1) = \frac{ab(p+1)}{c(p-\mu+1)}. \quad (3.28)$$

From Lemma 2.1 and $\delta+p \geq 1$, we obtain

$$(\delta+p) \sum_{k=1}^{\infty} \left(1 + \frac{k\gamma}{p}\right) a_{p+k} \leq \sum_{k=1}^{\infty} \frac{(\delta+p)_k(1+(k\gamma/p))}{(1)_k} a_{p+k} \leq \frac{1}{2} E_{\eta\beta} \quad (3.29)$$

for some $\eta \in \mathbb{R}$. It follows from (3.26) and (3.28) that

$$\begin{aligned} \left| \frac{\Gamma(p-\mu+1)}{\Gamma(p+1)} z^{\mu-p} D_z^{\mu} H(f)(z) \right| &\geq 1 - \frac{ab(p+1)}{2c(p-\mu+1)(\delta+p)} |z| E_{\eta\beta}, \\ \left| \frac{\Gamma(p-\mu+1)}{\Gamma(p+1)} z^{\mu-p} D_z^{\mu} H(f)(z) \right| &\leq 1 + \frac{ab(p+1)}{2c(p-\mu+1)(\delta+p)} |z| E_{\eta\beta} \end{aligned} \quad (3.30)$$

for some $\eta \in \mathbb{R}$, which yield (3.23). □

We state an obvious variant of Theorem 3.6 as follows.

Corollary 3.7. *Let the function f defined by (1.2) be in the class $\mathcal{T}_{\delta,p,\gamma}(\beta)$. Also let $a \leq 1$, $b \leq p-\mu+1$, $c \geq p+1$, and $\delta+p > 1$. Then*

$$\begin{aligned} \left| D_z^{\mu} H(f)(z) \right| &\geq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} |z|^{p-\mu} \left(1 - \frac{1}{2} |z| E_{\eta\beta} \right), \\ \left| D_z^{\mu} H(f)(z) \right| &\leq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} |z|^{p-\mu} \left(1 + \frac{1}{2} |z| E_{\eta\beta} \right) \end{aligned} \quad (3.31)$$

for some $\eta \in \mathbb{R}$, $0 \leq \mu < 1$, $z \in \mathcal{U}$, and $p \in \mathbb{N}$.

The proof of Theorem 3.8 is much akin to that of Theorem 3.6, and so it is omitted here.

Theorem 3.8. Let $a \leq 1$, $b \leq p + \mu + 1$ and $c \geq p + 1$. Also, let $\mu > 0$, $p \in \mathbb{N}$, and $z \in \mathcal{U}$. If $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then

$$\begin{aligned} \left| D_z^{-\mu} H(f)(z) \right| &\geq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} |z|^{p+\mu} \left(1 - \frac{ab(p+1)}{2c(1+\mu+p)(\delta+p)} |z|^{E_{\eta,\beta}} \right), \\ \left| D_z^{-\mu} H(f)(z) \right| &\leq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} |z|^{p+\mu} \left(1 + \frac{ab(p+1)}{2c(1+\mu+p)(\delta+p)} |z|^{E_{\eta,\beta}} \right) \end{aligned} \quad (3.32)$$

for some $\eta \in \mathbb{R}$.

Next we prove the following.

Theorem 3.9. Let $a \leq 1$, $b \leq p - \mu + 1$, and $c \geq p + 1$. Also, let $0 \leq \mu < 1$, $p \in \mathbb{N}$, and $z \in \mathcal{U}$. If $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then

$$\begin{aligned} \left| D_z^{\mu} \left(D^{\delta+p-1} H(f)(z) \right) \right| &\geq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} |z|^{p-\mu} \left(1 - \frac{ab(p+1)}{2c(1-\mu+p)} |z|^{E_{\eta,\beta}} \right), \\ \left| D_z^{\mu} \left(D^{\delta+p-1} H(f)(z) \right) \right| &\leq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} |z|^{p-\mu} \left(1 + \frac{ab(p+1)}{2c(1-\mu+p)} |z|^{E_{\eta,\beta}} \right) \end{aligned} \quad (3.33)$$

for some $\eta \in \mathbb{R}$.

Proof. We have

$$D^{\delta+p-1} H(f)(z) = z^p - \sum_{k=1}^{\infty} A_k z^{p+k}, \quad (3.34)$$

where

$$A_k = \frac{(\delta+p)_k (a)_k (b)_k (1+(p/\gamma))_k}{(1)_k (p/\gamma)_k (c)_k (1)_k} a_{p+k}. \quad (3.35)$$

Therefore

$$D_z^{\mu} \left(D^{\delta+p-1} H(f)(z) \right) = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} z^{p-\mu} - \sum_{k=1}^{\infty} A_k \frac{\Gamma(p+k+1)}{\Gamma(p+k-\mu+1)} z^{p+k-\mu}. \quad (3.36)$$

So, from (3.25), we have

$$\frac{\Gamma(p-\mu+1)}{\Gamma(p+1)} z^{\mu-p} D_z^\mu \left(D^{\delta+p-1} H(f)(z) \right) = 1 - \sum_{k=1}^{\infty} \frac{(\delta+p)_k (p+k\gamma)}{p(1)_k} \lambda(k) a_{p+k} z^k, \quad (3.37)$$

where $\lambda(k)$ is defined by (3.27). Since λ is a decreasing function of k , when $a \leq 1$, $b \leq p-\mu+1$ and $c \geq p+1$, then

$$0 < \lambda(k) \leq \lambda(1) = \frac{ab(p+1)}{c(p-\mu+1)}. \quad (3.38)$$

From (3.37), (3.38), and Lemma 2.1, we find that

$$\begin{aligned} \left| \frac{\Gamma(p-\mu+1)}{\Gamma(p+1)} z^{\mu-p} D_z^\mu \left(D^{\delta+p-1} H(f)(z) \right) \right| &\geq 1 - \frac{ab(p+1)}{2c(1-\mu+p)} |z|_{E_{\eta,\beta}}, \\ \left| \frac{\Gamma(p-\mu+1)}{\Gamma(p+1)} z^{\mu-p} D_z^\mu \left(D^{\delta+p-1} H(f)(z) \right) \right| &\leq 1 + \frac{ab(p+1)}{2c(1-\mu+p)} |z|_{E_{\eta,\beta}} \end{aligned} \quad (3.39)$$

for some $\eta \in \mathbb{R}$. The above inequalities lead us to the desired inequalities (3.33). $\square$

The proof of the following theorem is similar to Theorem 3.9, and so it is omitted here.

Theorem 3.10. Let $a \leq 1$, $b \leq p+\mu+1$, and $c \geq p+1$. Also, let $\mu > 0$, $z \in \mathcal{U}$, and $p \in \mathbb{N}$. If $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then

$$\begin{aligned} \left| D_z^{-\mu} \left(D^{\delta+p-1} H(f)(z) \right) \right| &\geq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} |z|^{p+\mu} \left(1 - \frac{ab(p+1)}{2c(1+\mu+p)} |z|_{E_{\eta,\beta}} \right), \\ \left| D_z^{-\mu} \left(D^{\delta+p-1} H(f)(z) \right) \right| &\leq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} |z|^{p+\mu} \left(1 + \frac{ab(p+1)}{2c(1+\mu+p)} |z|_{E_{\eta,\beta}} \right) \end{aligned} \quad (3.40)$$

for some $\eta \in \mathbb{R}$.

Upon setting $\delta = 0$ and $p = 1$ in Theorems 3.6, 3.8, 3.9, and 3.10, we arrive at the following result.

Corollary 3.11. Let $a \leq 1$, $c \geq 2$, $\mu > 0$, $z \in \mathcal{U}$, and $p \in \mathbb{N}$. If $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then

$$\begin{aligned} \left| D_z^\mu H(f)(z) \right| &\geq \frac{1}{\Gamma(2-\mu)} |z|^{1-\mu} \left(1 - \frac{ab}{c(2-\mu)} |z|_{E_{\eta,\beta}} \right), \\ \left| D_z^\mu H(f)(z) \right| &\leq \frac{1}{\Gamma(2-\mu)} |z|^{1-\mu} \left(1 + \frac{ab}{c(2-\mu)} |z|_{E_{\eta,\beta}} \right) \end{aligned} \quad (3.41)$$

for some $\eta \in \mathbb{R}$ and $b \leq 2 - \mu$. Furthermore

$$\begin{aligned} \left| D_z^{-\mu} H(f)(z) \right| &\geq \frac{1}{\Gamma(2+\mu)} |z|^{1+\mu} \left(1 - \frac{ab}{c(2+\mu)} |z| E_{\eta, \beta} \right), \\ \left| D_z^{-\mu} H(f)(z) \right| &\leq \frac{1}{\Gamma(2+\mu)} |z|^{1+\mu} \left(1 + \frac{ab}{c(2+\mu)} |z| E_{\eta, \beta} \right) \end{aligned} \quad (3.42)$$

for some $\eta \in \mathbb{R}$ and $b \leq 2 + \mu$.

Remark 3.12. Under the hypothesis of Corollary 3.11, $D_z^\mu H(f)(z)$ and $D_z^{-\mu} H(f)(z)$ are included in disks with its center at origin and radii r and R , respectively, given by

$$\begin{aligned} r &= \frac{1}{\Gamma(2-\mu)} \left(1 + \frac{ab}{c(2-\mu)} E_{\eta, \beta} \right) \quad (0 \leq \mu < 1, \ b \leq 2 - \mu, \text{ for some } \eta \in \mathbb{R}), \\ R &= \frac{1}{\Gamma(2+\mu)} \left(1 + \frac{ab}{c(2+\mu)} E_{\eta, \beta} \right) \quad (\mu > 0, \ b \leq 2 + \mu, \text{ for some } \eta \in \mathbb{R}). \end{aligned} \quad (3.43)$$

4. Distortion Inequalities of Integral Operator

In this section, we obtain the distortion theorems involving the integral operator $J_{\delta, p}$ of functions in the class $\mathcal{T}_{\delta, p, \gamma}(\beta)$ and fractional calculus operator.

Theorem 4.1. Let $0 \leq \mu < 1/(\delta + p + 1)$, $z \in \mathcal{U}$, and $p \in \mathbb{N}$. If $f \in \mathcal{T}_{\delta, p, \gamma}(\beta)$, then

$$\begin{aligned} \left| D_z^\mu \left(z^{-\delta} (J_{\delta, p} f)(z) \right) \right| &\geq \frac{1}{\Gamma(2-\mu)} |z|^{-\mu} \left(1 - \frac{p}{2(\delta + p + 1)(p + \gamma)} |z| E_{\eta, \beta} \right), \\ \left| D_z^\mu \left(z^{-\delta} (J_{\delta, p} f)(z) \right) \right| &\leq \frac{1}{\Gamma(2-\mu)} |z|^{-\mu} \left(1 + \frac{p}{2(\delta + p + 1)(p + \gamma)} |z| E_{\eta, \beta} \right) \end{aligned} \quad (4.1)$$

for some $\eta \in \mathbb{R}$. The operator $(J_{\delta, p} f)(z)$ is defined by (2.21).

Proof. Using the definition (2.25), for function $f \in \mathcal{K}(p)$ of the form (1.2), we have

$$(J_{\delta, p} f)(z) = z^\delta - \sum_{k=1}^{\infty} \frac{\delta + p}{p + k + \delta} a_{p+k}. \quad (4.2)$$

So

$$D_z^\mu \left(z^{-\delta} (J_{\delta,p} f)(z) \right) = \frac{1}{\Gamma(2-\mu)} z^{-\mu} - \sum_{k=1}^{\infty} \frac{(\delta+p)\Gamma(k+1)}{(\delta+p+k)\Gamma(k-\mu+1)} a_{p+k} z^{k-\mu}. \quad (4.3)$$

Therefore, we obtain

$$\Gamma(2-\mu) z^\mu D_z^\mu \left(z^{-\delta} (J_{\delta,p} f)(z) \right) = 1 - \sum_{k=1}^{\infty} \phi(k) a_{p+k} z^k, \quad (4.4)$$

where

$$\phi(k) = \frac{(\delta+p)\Gamma(k+1)\Gamma(2-\mu)}{(\delta+p+k)\Gamma(k-\mu+1)}. \quad (4.5)$$

Since $\phi(k)$ is a decreasing function of k , when $\mu < 1/(\delta+p+1)$, then

$$0 < \phi(k) \leq \phi(1) = \frac{\delta+p}{\delta+p+1}. \quad (4.6)$$

By using (3.11), (4.4), and (4.6), we get

$$\begin{aligned} \left| \Gamma(2-\mu) z^\mu D_z^\mu \left(z^{-\delta} (J_{\delta,p} f)(z) \right) \right| &\geq 1 - \frac{p}{2(\delta+p+1)(p+\gamma)} |z| E_{\eta,\beta}, \\ \left| \Gamma(2-\mu) z^\mu D_z^\mu \left(z^{-\delta} (J_{\delta,p} f)(z) \right) \right| &\leq 1 + \frac{p}{2(\delta+p+1)(p+\gamma)} |z| E_{\eta,\beta} \end{aligned} \quad (4.7)$$

for some $\eta \in \mathbb{R}$, which prove the inequalities (4.1). $\square$

The proof of the following theorem is similar to Theorem 4.1, and so it is omitted here.

Theorem 4.2. Let $\mu > 0$, $p \in \mathbb{N}$, and $z \in \mathcal{U}$. If $f \in \mathcal{T}_{\delta,p,\gamma}(\beta)$, then

$$\begin{aligned} \left| D_z^{-\mu} \left(z^{-\delta} (J_{\delta,p} f)(z) \right) \right| &\geq \frac{1}{\Gamma(\mu+1)} |z|^\mu \left(1 - \frac{p}{2(\delta+p+1)(\mu+1)(p+\gamma)} |z| E_{\eta,\beta} \right), \\ \left| D_z^{-\mu} \left(z^{-\delta} (J_{\delta,p} f)(z) \right) \right| &\leq \frac{1}{\Gamma(\mu+1)} |z|^\mu \left(1 + \frac{p}{2(\delta+p+1)(\mu+1)(p+\gamma)} |z| E_{\eta,\beta} \right) \end{aligned} \quad (4.8)$$

for some $\eta \in \mathbb{R}$.

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