Research Article

Common Fixed Points for Maps on Topological Vector Space Valued Cone Metric Spaces

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We introduced a notion of topological vector space valued cone metric space and obtained some common fixed point results. Our results generalize some recent results in the literature.

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1. Introduction

Huang and Zhang [1] generalized the notion of metric space by replacing the set of real numbers by ordered Banach space, defined a cone metric space, and established some fixed point theorems for contractive type mappings in a normal cone metric space. Subsequently, several other authors [2–5] studied the existence of common fixed point of mappings satisfying a contractive type condition in normal cone metric spaces. Afterwards, Rezapour and Hamlbarani [6] studied fixed point theorems of contractive type mappings by omitting the assumption of normality in cone metric spaces (see also [7–14]). In this paper we obtain common fixed points for a pair of self-mappings satisfying a generalized contractive type condition without the assumption of normality in a class of topological vector space valued cone metric spaces which is bigger than that introduced by Huang and Zhang [1].

Let \((E, \tau)\) be always a topological vector space and \(P\) a subset of \(E\). Then, \(P\) is called a cone whenever

(i) \(P\) is closed, nonempty and \(P \neq \{0\},\)
(ii) \(ax + by \in P\) for all \(x, y \in P\) and nonnegative real numbers \(a, b,\)
(iii) \(P \cap (-P) = \{0\}.)
For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to $P$ by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int } P$, where $\text{int } P$ denotes the interior of $P$.

**Definition 1.1.** Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

1. $(d_1) 0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
2. $(d_2) d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $(d_3) d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a topological vector space valued cone metric space.

Note that Huang and Zhang [1] notion of cone metric space is a special case of our notion of topological vector space valued cone metric space.

**Example 1.2.** Let $X = [0, 1]$, and let $E$ be the set of all real valued functions on $X$ which also have continuous derivatives on $X$, then $E$ is a vector space over $\mathbb{R}$ under the following operations:

$$ (f + g)(t) = f(t) + g(t), \quad (\alpha f)(t) = \alpha f(t), $$

(1.1)

for all $f, g \in E, \alpha \in \mathbb{R}$. Let $\tau$ be the strongest vector (locally convex) topology on $E$, then $(X, \tau)$ is a topological vector space which is not normable and is not even metrizable (see [15]). Define $d : X \times X \to E$ as follows:

$$ (d(x, y))(t) = |x - y|e^t, \quad P = \{x \in E : x(t) \geq 0 \forall t \in X\}. $$

Then $(X, d)$ is a topological vector space valued cone metric space.

Example 1.2 shows that this category of cone metric spaces is larger than that considered in [1–8].

**Definition 1.3.** Let $(X, d)$ be a topological vector space valued cone metric space, and let $x \in X$ and $\{x_n\}_{n \geq 1}$ be a sequence in $X$. Then

1. $(i) \{x_n\}_{n \geq 1}$ converges to $x$ whenever for every $c \in E$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.
2. $(ii) \{x_n\}_{n \geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$ there is a natural number $N$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.
3. $(iii) (X, d)$ is a complete topological vector space valued cone metric space if every Cauchy sequence is convergent.

**2. Fixed Point**

In this section, we shall give some results which generalize [6, Theorems 2.3, 2.6, 2.7, and 2.8] (and so [1, Theorems 1, 3, and 4]).
Theorem 2.1. Let \((X, d)\) be a complete topological vector space valued cone metric space and let the self-mappings \(S, T : X \to X\) satisfy

\[
d(Sx, Ty) \leq kd(x, y) + l(d(x, Ty) + d(y, Sx)),
\]

(2.1)

for all \(x, y \in X\), where \(k, l \in [0, 1)\) with \(k + 2l < 1\). Then \(S\) and \(T\) have a unique common fixed point.

Proof. For \(x_0 \in X\) and \(n \geq 0\), define \(x_{2n+1} = Sx_{2n}\) and \(x_{2n+2} = Tx_{2n+1}\). Then,

\[
d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})
\]

\[
\leq kd(x_{2n}, x_{2n+1}) + l[d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})]
\]

\[
= kd(x_{2n}, x_{2n+1}) + l[d(x_{2n}, Tx_{2n+1})]
\]

\[
\leq kd(x_{2n}, x_{2n+1}) + l[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})]
\]

\[
= [k + l]d(x_{2n}, x_{2n+1}) + ld(x_{2n+1}, x_{2n+2}).
\]

(2.2)

It implies that \(d(x_{2n+1}, x_{2n+2}) \leq [(k + l)/(1 - l)]d(x_{2n}, x_{2n+1})\). Similarly,

\[
d(x_{2n+2}, x_{2n+3}) = d(Sx_{2n+2}, Tx_{2n+3})
\]

\[
\leq kd(x_{2n+2}, x_{2n+3}) + l[d(x_{2n+2}, Tx_{2n+3}) + d(x_{2n+3}, Sx_{2n+2})]
\]

\[
\leq kd(x_{2n+2}, x_{2n+3}) + l[d(x_{2n+2}, x_{2n+3}) + d(x_{2n+3}, x_{2n+2})]
\]

\[
= [k + l]d(x_{2n+2}, x_{2n+3}) + ld(x_{2n+2}, x_{2n+3}).
\]

(2.3)

Hence, \(d(x_{2n+2}, x_{2n+3}) \leq [(k + l)/(1 - l)]d(x_{2n+1}, x_{2n+2})\). Thus,

\[
d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1),
\]

(2.4)

for all \(n \geq 0\), where \(\lambda = ((k + l)/(1 - l)) < 1\). Now, for \(n > m\), we have

\[
d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m)
\]

\[
\leq \left(\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m\right) d(x_0, x_1)
\]

\[
\leq \frac{\lambda^m}{1 - \lambda} d(x_0, x_1).
\]

(2.5)

Let \(0 < c\). Take a symmetric neighborhood \(V\) of \(0\) such that \(c + V \subseteq \text{int} P\). Also, choose a natural number \(N_1\) such that \((\lambda^m/(1 - \lambda))d(x_1, x_0) \in V\), for all \(m \geq N_1\). Then, \((\lambda^m/(1 - \lambda))d(x_1, x_0) \ll c\), for all \(m \geq N_1\). Thus,

\[
d(x_n, x_m) \leq \frac{\lambda^m}{1 - \lambda} d(x_1, x_0) \ll c,
\]

(2.6)
for all \( n > m \). Therefore, \( \{x_n\}_{n=1}^\infty \) is a Cauchy sequence in \((X,d)\). Since \( X \) is complete, there exists \( u \in X \) such that \( x_n \to u \). Choose a natural number \( N_2 \) such that \( d(x_n,u) \ll \frac{c(1-l)}{2(1+l)} \) for all \( n \geq N_2 \). Thus,

\[
d(u,Tu) \leq d(u,x_{2n+1}) + d(x_{2n+1},Tu) \\
= d(u,x_{2n+1}) + d(Sx_{2n},Tu) \\
\leq d(u,x_{2n+1}) + kd(u,x_{2n}) + l[d(u,Sx_{2n}) + d(x_{2n},Tu)] \\
\leq d(u,x_{2n+1}) + kd(u,x_{2n}) + l[d(u,x_{2n+1}) + d(x_{2n},u) + d(u,Tu)] \\
= (1+l)d(u,x_{2n+1}) + (k+l)d(u,x_{2n}) + ld(u,Tu).
\]

So,

\[
d(u,Tu) \leq \frac{[1+l]}{1-l} d(u,x_{2n+1}) + \frac{[k+l]}{1-l} d(u,x_{2n}) \\
\leq \frac{[1+l]}{1-l} d(u,x_{2n+1}) + \frac{[1+l]}{1-l} d(u,x_{2n}) \\
= \frac{c}{2} + \frac{c}{2} = c,
\]

for all \( n \geq N_2 \). Therefore, \( d(u,Tu) \ll c/i \) for all \( i \geq 1 \). Hence, \((c/i) - d(u,Tu) \in P\) for all \( i \geq 1 \). Since \( P \) is closed, \(-d(u,Tu) \in P\) and so \( d(u,Tu) = 0 \). Hence, \( u \) is a fixed point of \( T \). Similarly, we can show that \( u = Su \). Now, we show that \( S \) and \( T \) have a unique fixed point. For this, assume that there exists another point \( u^* \) in \( X \) such that \( u^* = Tu^* = Su^* \). Then,

\[
d(u,u^*) = d(Su,Tu^*) \\
\leq kd(u,u^*) + l[d(u,Tu^*) + d(u^*,Su)] \\
\leq kd(u,u^*) + l[d(u,u^*) + d(u^*,u)] \\
\leq (k+2l)d(u,u^*).
\]

Since \( k+2l < 1 \), \( d(u,u^*) = 0 \) and so \( u = u^* \). \( \square \)

The following corollary generalizes [6, Theorems 2.3, 2.7, and 2.8] (and so [1, Theorems 1 and 4]).

**Corollary 2.2.** Let \((X,d)\) be a complete topological vector space valued cone metric space and let the self-mapping \( T : X \to X \) satisfy \( d(Tx,Ty) \leq ad(x,y) + bd(x,Ty) + cd(y,Tx) \) for all \( x,y \in X \), where \( a,b,c \in [0,1) \) with \( a+b+c < 1 \). Then \( T \) has a unique fixed point.
Theorem 2.3. Let \((X, d)\) be a complete topological vector space valued cone metric space and let the self-mappings \(S, T : X \to X\) satisfy

\[
d(Sx, Ty) \leq kd(x, y) + l(d(x, Sx) + d(y, Ty)),
\]

for all \(x, y \in X\), where \(k, l \in [0, 1)\) with \(k + 2l < 1\). Then \(S\) and \(T\) have a unique common fixed point.

Proof. For \(x_0 \in X\) and \(n \geq 0\), define \(x_{2n+1} = Sx_{2n}\) and \(x_{2n+2} = Tx_{2n+1}\). Then,

\[
d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}) \\
\leq kd(x_{2n}, x_{2n+1}) + l[d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1})] \\
= kd(x_{2n}, x_{2n+1}) + l[d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, x_{2n+2})] \\
= [k + l]d(x_{2n}, x_{2n+1}) + ld(x_{2n+1}, x_{2n+2}).
\]

It implies that \(d(x_{2n+1}, x_{2n+2}) \leq [(k + l)/(1 - l)]d(x_{2n}, x_{2n+1})\). Similarly,

\[
d(x_{2n+2}, x_{2n+3}) = d(Sx_{2n+2}, Tx_{2n+1}) \\
\leq kd(x_{2n+2}, x_{2n+1}) + l[d(x_{2n+2}, Sx_{2n+2}) + d(x_{2n+1}, Tx_{2n+1})] \\
= kd(x_{2n+2}, x_{2n+1}) + l[d(x_{2n+2}, x_{2n+3}) + d(x_{2n+1}, x_{2n+2})] \\
= [k + l]d(x_{2n+1}, x_{2n+2}) + ld(x_{2n+2}, x_{2n+3}).
\]

Hence, \(d(x_{2n+2}, x_{2n+3}) \leq [(k + l)/(1 - l)]d(x_{2n+1}, x_{2n+2})\). Thus,

\[
d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1),
\]

for all \(n \geq 0\), where \(\lambda = (k + l)/(1 - l) < 1\). Now, for \(n > m\) we have

\[
d(x_n, x_m) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\
\leq (\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m) d(x_0, x_1) \\
\leq \frac{\lambda^m}{1 - \lambda} d(x_0, x_1).
\]
Let \( 0 \ll c \). Take a symmetric neighborhood \( V \) of 0 such that \( c + V \subseteq \text{int} \, P \). Also, choose a natural number \( N_1 \) such that \((\lambda^m/(1 - \lambda))d(x_1, x_0) \in V\), for all \( m \geq N_1 \). Then, \((\lambda^m/(1 - \lambda))d(x_1, x_0) \ll c\), for all \( m \geq N_1 \). Thus,

\[
d(x_n, x_m) \leq \frac{\lambda^m}{1 - \lambda}d(x_1, x_0) \ll c, \tag{2.16}
\]

for all \( n > m \). Therefore, \( \{x_n\}_{n \geq 1} \) is a Cauchy sequence in \((X, d)\). Since \( X \) is complete, there exists \( u \in X \) such that \( x_n \to u \). Choose a natural number \( N_2 \) such that \( d(x_n, u) \ll [c(1 - l)/2(1 + l)] \) for all \( n \geq N_2 \). Thus,

\[
d(u, Tu) \leq d(u, x_{2n+1}) + d(x_{2n+1}, Tu)
= d(u, x_{2n+1}) + d(Sx_{2n}, Tu)
\leq d(u, x_{2n+1}) + kd(u, x_{2n}) + l[d(u, Tu) + d(x_{2n}, Sx_{2n})]
\leq d(u, x_{2n+1}) + kd(u, x_{2n}) + l[d(u, x_{2n+1}) + d(x_{2n}, u) + d(u, Tu)]
= (1 + l)d(u, x_{2n+1}) + (k + l)d(u, x_{2n}) + ld(u, Tu).
\]

So,

\[
d(u, Tu) \leq \left[1 + \frac{l}{1 - l}\right]d(u, x_{2n+1}) + \left[\frac{k + l}{1 - l}\right]d(u, x_{2n})
\leq \left[1 + \frac{l}{1 - l}\right]d(u, x_{2n+1}) + \frac{1 + l}{1 - l}d(u, x_{2n}) \tag{2.18}
\ll \frac{c}{2} + \frac{c}{2} = c,
\]

for all \( n \geq N_2 \). Therefore, \( d(u, Tu) \ll c/i \) for all \( i \geq 1 \). Hence, \( (c/i) - d(u, Tu) \in P \) for all \( i \geq 1 \). Since \( P \) is closed, \(-d(u, Tu) \in P \) and so \( d(u, Tu) = 0 \). Hence, \( u \) is a fixed point of \( T \). Similarly, we can show that \( u = Su \). Now, we show that \( S \) and \( T \) have a unique fixed point. For this, assume that there exists another point \( u^* \) in \( X \) such that \( u^* = Tu^* = Su^* \). Then,

\[
d(u, u^*) = d(Su, Tu^*)
\leq kd(u, u^*) + l[d(u, u^*) + d(u^*, u)]
= kd(u, u^*). \tag{2.19}
\]

Since \( k < 1, d(u, u^*) = 0 \) and so \( u = u^* \).

The following corollary generalizes [6, Theorem 2.6] (and so [1, Theorem 3]).
Corollary 2.4. Let \((X, d)\) be a complete topological vector space valued cone metric space and let the self-mapping \(T : X \to X\) satisfy \(d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty)\) for all \(x, y \in X\), where \(a, b, c \in [0, 1)\) with \(a + b + c < 1\). Then \(T\) has a unique fixed point.

Proof is similar to the proof of Corollary 2.2.

Example 2.5. Let \((X, d)\) be a topological vector space valued cone metric space of Example 1.2. Define \(S, T : X \to X\) as follows:

\[
S(t) = T(t) = \begin{cases} 
\frac{t}{3} & \text{if } x \neq 1, \\
1 & \text{if } x = 1.
\end{cases}
\] (2.20)

Then,

\[
|Sx - Ty|e^t \leq k|x - y|e^t + l[|x - Sx|e^t + |y - Ty|e^t],
\] (2.21)

if \(k = 1/6, \ l = 5/18\). Hence all conditions of Theorem 2.3 are satisfied.

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References


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